План. 1) Квантовая модель Годена

H_i

$[H_i, H_j] = 0$

2) Как ввести магнитное поле

3) Вопрос диагонализация. Пример sl_2. Анзац Бете.

1. Квантовая модель Годена

$\mathfrak{g}$ – полупростая алгебра Ли

$\{x_\alpha\}$ – базис в $\mathfrak{g}$

$\{x^\alpha\}$ – двойств. базис в $\mathfrak{g}$

$C = \sum_\alpha x_\alpha x^\alpha$ – эл-т Казимира

$$x_\alpha(z) = \sum_{i=1}^N \frac{x_\alpha^{(i)}}{z - z_i} \qquad x_\alpha^{(i)} \quad U(\mathfrak{g})^{\otimes N}$$

$$x_\alpha^{(i)} = 1 \otimes 1 \otimes \dots \underbrace{x_\alpha}_{i} \otimes \dots \otimes 1$$

$$L(z) = \sum_\alpha x_\alpha(z)\, x^\alpha(z)$$

Пример sl_2

$C = \frac{1}{2}h^2 + ef + fe$

$h(z) = \sum_{i=1}^N \frac{h^{(i)}}{z - z_i}$

$e(z) = \sum_{i=1}^N \frac{e^{(i)}}{z - z_i}$

$f(z) = \sum_{i=1}^N \frac{f^{(i)}}{z - z_i}$

$L(z) = \frac{1}{2} h(z)h(z) + e(z)f(z) + f(z)e(z)$

– производящая ф-ция Гамильтонианов

$$L(z) = \sum_\alpha \sum_{i=1}^N \frac{x_\alpha^{(i)}}{z - z_i} \sum_{j=1}^N \frac{(x^\alpha)^{(j)}}{z - z_j} =$$

$$= \sum_{i=1}^N \frac{\sum_\alpha x_\alpha^{(i)} (x^\alpha)^{(i)}}{(z - z_i)^2} - 2 \sum_{j=1}^N \frac{H_i}{z - z_i}$$

$$H_i = \sum_{j \neq i}^N \frac{\sum_\alpha x_\alpha^{(i)} (x^\alpha)^{(j)}}{z_i - z_j}$$ – гамильтониана Годена

$$H_i := \frac{1}{2}\operatorname*{Res}_{z=z_i} L(z)$$

Опр. $$\Omega = \sum_\alpha x_\alpha \otimes x^\alpha \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

$$\Omega^{(ij)} \in U(\mathfrak{g})^{\otimes N}$$

$$\Omega^{(ij)} = 1 \otimes \ldots \overbrace{\underset{i}{\otimes} \ldots \otimes 1 \underset{j}{\otimes}}^{\Omega} \ldots \otimes 1$$

$$H_i = \sum_{j \neq i}^{N} \frac{\Omega^{(ij)}}{z_i - z_j}$$

Лемма. $$[\Omega, \overbrace{x \otimes 1 + 1 \otimes x}^{\Delta x}] = 0 \qquad \forall x \in \mathfrak{g}$$

Док-во.

$$\Omega = \frac{1}{2}\left(\Delta C - C \otimes 1 - 1 \otimes C\right)$$

$$\Delta\Big(\sum_\alpha x_\alpha x^\alpha\Big) = \sum_\alpha \Big(2 x_\alpha \otimes x^\alpha + \underbrace{x_\alpha x^\alpha \otimes 1}_{= C \otimes 1} + \underbrace{1 \otimes x_\alpha x^\alpha}_{= 1 \otimes C}\Big)$$

$$[\Omega, \Delta x] = \frac{1}{2}[\Delta C - C \otimes 1 - 1 \otimes C,\ x \otimes 1 + 1 \otimes x] =$$

$$= \frac{1}{2}[\Delta C, \Delta x] = \frac{1}{2}\Delta[C, x] = 0$$

Опр. $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ – гомоморфизм алгебр

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

Предложение. $[H_i, H_k] = 0 \qquad i \neq k$

Док-во.

$$[H_i, H_k] = \Big[\sum_{j \neq i} \frac{\Omega^{(ij)}}{z_i - z_j}, \sum_{\ell \neq k} \frac{\Omega^{(k\ell)}}{z_k - z_\ell}\Big]$$

i, j, k

$$\frac{1}{z_i-z_j}\,\frac{1}{z_k-z_j}\left[\Omega^{(ij)},\Omega^{(kj)}\right]+\frac{1}{z_i-z_j}\,\frac{1}{z_k-z_i}\left[\Omega^{(ij)},\Omega^{(ki)}\right]+$$

$$+\frac{1}{z_i-z_k}\,\frac{1}{z_k-z_j}\left[\Omega^{(ik)},\Omega^{(kj)}\right] =$$

$$\frac{1}{(z_i-z_k)(z_k-z_j)}=\frac{1}{(z_i-z_j)(z_k-z_j)}-\frac{1}{(z_i-z_j)(z_k-z_i)}$$

$$= \frac{1}{z_i-z_j}\,\frac{1}{z_k-z_j}\left(\left[\Omega^{(ij)},\Omega^{(kj)}\right]+\left[\Omega^{(ik)},\Omega^{(kj)}\right]\right)+$$

$$+\frac{1}{z_i-z_j}\,\frac{1}{z_k-z_i}\left(\left[\Omega^{(ij)},\Omega^{(ki)}\right]-\left[\underset{\Omega^{(ki)}}{\underset{\parallel}{\Omega^{(ik)}}},\Omega^{(kj)}\right]\right)$$

$$\left[\Omega^{(ij)}+\Omega^{(ik)},\Omega^{(kj)}\right]=0$$

$$\left[\sum x_\alpha^{(i)}\left((x^\alpha)^{(j)}+(x^\alpha)^{(k)}\right),\Omega^{(kj)}\right]=0$$

по лемме $\left[(x^\alpha)^{(j)}+(x^\alpha)^{(k)},\Omega^{(kj)}\right]=0$ ■

Следствие.

$$[L(z),L(u)]=0$$

$$[L(z),H_i]=0 \qquad \underset{u=z_i}{\mathrm{Res}}\,L(u)$$

$$L(z) = \sum_{\alpha} \sum_{i=1}^{N} \frac{x_\alpha^{(i)}}{z-z_i} \sum_{j=1}^{N} \frac{(x^\alpha)^{(j)}}{z-z_j} =$$

$$= \sum_{i=1}^{N} \frac{\sum_\alpha x_\alpha^{(i)} (x^\alpha)^{(i)}}{(z-z_i)^2} - 2 \sum_{j=1}^{N} \frac{H_i}{z-z_i} =$$

$$= \sum_{i=1}^{N} \frac{C^{(i)}}{(z-z_i)^2} - 2 \sum_{j=1}^{N} \frac{H_i}{z-z_i}.$$

Свойства.

1) $\sum_{i=1}^{N} H_i = 0$

2) $[H_j, \overset{(N)}{\Delta} X] = 0$ – следствие леммы

$$H_j \in \left(U(\mathfrak{g})^{\otimes N} \right)^{\mathfrak{g}}$$

$$\overset{(N)}{\Delta} X = X^{(1)} + X^{(2)} + \ldots X^{(N)}$$

2. Как ввести магнитное поле?

$$\tilde{X}(z) = \sum_{i=1}^{N} \frac{X^{(i)}}{z-z_i} + \varphi(x) \qquad \varphi \in \mathfrak{g}^*, \quad x \in \mathfrak{g}$$

$$\tilde{L}(z) = \sum_\alpha \tilde{X}_\alpha(z) \tilde{X}^\alpha(z) =$$

$$= L(z) + 2 \sum_{i=1}^{N} \frac{\sum_\alpha X_\alpha^{(i)} \varphi(x^\alpha)}{z-z_i} + \sum_\alpha \varphi(x_\alpha) \varphi(x^\alpha)$$

$$\tilde{H}_i = -\frac{1}{2} \operatorname{Res}_{z=z_i} \tilde{L}(z) = H_i - \sum_\alpha X_\alpha^{(i)} \varphi(x^\alpha)$$

$$\underline{y = \sum_\alpha x_\alpha \varphi(x^\alpha) \in \mathfrak{g}}$$

$$\tilde{H}_i = H_i - y^{(i)}$$

<u>Утв.</u> $[\tilde{H}_i, \tilde{H}_j] = 0$; $[\tilde{L}(z), \tilde{L}(u)] = 0$

Док-во

$$[H_i - y^{(i)}, H_j - y^{(j)}] = \underbrace{[H_i, H_j]}_{=0} - [y^{(i)}, H_j] - - [H_i, y^{(j)}] + \underbrace{[y^{(i)}, y^{(j)}]}_{=0}$$

$$= -\left[y^{(i)}, \sum_{k\neq j} \frac{\Omega^{(jk)}}{z_j - z_k}\right] - \left[\sum_{\ell\neq i} \frac{\Omega^{(i\ell)}}{z_i - z_\ell}, y^{(j)}\right] =$$

$$= -\left[y^{(i)}, \frac{\Omega^{(ji)}}{z_j - z_i}\right] - \left[\frac{\Omega^{(ij)}}{z_i - z_j}, y^{(j)}\right] =$$

$$= \frac{1}{z_i - z_j}\left([y^{(i)}, \Omega^{(ji)}] - [\Omega^{(ij)}, y^{(j)}]\right) =$$

$$= \frac{1}{z_i - z_j}\left([y^{(i)} + y^{(j)}, \Omega^{(ij)}]\right) = 0$$

лемма

② Магнитное поле как предел.

$1, \dots N, N+1$

$$H_j = \sum_{k\neq j}^{N} \frac{\Omega^{(jk)}}{z_j - z_k} + \frac{\Omega^{(j,N+1)}}{z_j - z_{N+1}}$$

$$\frac{\Omega^{(j,N+1)}}{z_j - z_{N+1}} = \frac{\sum x_\alpha^{(j)} \underbrace{(x^\alpha)^{(N+1)}}_{\varphi(x^\alpha)}}{z_j - z_{N+1}}$$

Предел $z_{N+1} \to \infty$

$$\frac{x_\alpha^{(N+1)}}{z_{N+1}} \to \tilde{x}_\alpha^{(N+1)}; \quad \left[\frac{x_\alpha^{(N+1)}}{z_{N+1}}, \frac{x_\beta^{(N+1)}}{z_{N+1}}\right] = \frac{1}{(z_{N+1})^2} \sum C_{\alpha\beta}^{\gamma} x_\gamma^{(N+1)} \to 0$$

$$\left[\tilde{x}_\alpha^{(N+1)}, \tilde{x}_\beta^{(N+1)}\right] = 0$$

$\{\tilde{x}_\alpha^{(N+1)}\}$ — коммутативная алгебра

$S(\mathfrak{g})$ $\qquad U(\mathfrak{g}) \xrightarrow{\text{классический предел}} S(\mathfrak{g})$

$S(\mathfrak{g}) \ni \tilde{x}_\alpha \rightsquigarrow \varphi(\tilde{x}_\alpha) \qquad \varphi \in \mathfrak{g}^*$

Пример. $\mathfrak{g} = sl_2 \qquad e v_\lambda = 0 \qquad h v_\lambda = \lambda v_\lambda$

$$v_0 = v_\lambda; \quad v_1 = \frac{f}{\sqrt{\lambda}} v_\lambda, \quad v_2 = \frac{f^2}{\sqrt{(\lambda-1)\lambda 2}} v_\lambda, \quad v_3 = \frac{f^3}{\sqrt{\lambda(\lambda-1)(\lambda-2)3!}} \dots$$

смотрим при больших λ и малых k (v_k)

$$f v_k \sim \sqrt{\lambda}\, v_{k+1}; \quad e v_k \sim \sqrt{\lambda}\, v_{k-1}$$

$$h v_k \sim \lambda v_k$$

Предел: $\lambda \sim \gamma z_{N+1} \qquad z_{N+1} \to \infty$

$$\frac{e}{z_{N+1}}, \frac{f}{z_{N+1}} \to 0 \qquad \frac{h}{z_{N+1}} \to \gamma$$

$\varphi(h)=\gamma \qquad \varphi(e)=\varphi(f)=0$

$y=\gamma h$ – добавка „магнитное поле"

③ Пример sl_2. Анзац Бете.

sl_2

$C=\frac{1}{2}h^2+ef+fe$

$h(z)=\sum_{i=1}^{N}\frac{h^{(i)}}{z-z_i}$

$e(z)=\sum_{i=1}^{N}\frac{e^{(i)}}{z-z_i}$

$f(z)=\sum_{i=1}^{N}\frac{f^{(i)}}{z-z_i}$

$$L(z)=\frac{1}{2}h(z)^2+e(z)f(z)+f(z)e(z)$$

Задача. Искать собств. вектора $L(z)$.

$[h,e]=2e$; $[h,f]=-2f$; $[e,f]=h$

Лемма 2.

$$(*)\quad [h(\nu),f(\mu)]=\frac{2}{\nu-\mu}\left(f(\mu)-f(\nu)\right)$$

$$(**)\quad [L(\nu),f(\mu)]=\frac{2}{\nu-\mu}\left(f(\mu)h(\nu)-f(\nu)h(\mu)\right)$$

Док-во $(*)$

$$\left[\sum_{i=1}^{N}\frac{h^{(i)}}{\nu-z_i},\ \sum_{j=1}^{N}\frac{f^{(j)}}{\mu-z_j}\right]=\sum_{i=1}^{N}\frac{[h^{(i)},f^{(i)}]}{(\nu-z_i)(\mu-z_i)}=$$

$$=\sum_{i=1}^{N}\frac{-2f^{(i)}}{(\nu-z_i)(\mu-z_i)}=\frac{2}{\nu-\mu}\left(\sum_{i=1}^{N}\frac{f^{(i)}}{\mu-z_i}-\sum_{i=1}^{N}\frac{f^{(i)}}{\nu-z_i}\right)$$

$$h^{(i)}|0\rangle = \lambda_i|0\rangle \qquad e^{(i)}|0\rangle = 0 \qquad V_{\lambda_1}\otimes V_{\lambda_2}\cdots\otimes V_{\lambda_N}$$

$$|K\rangle = f(\mu_1)f(\mu_2)\cdots f(\mu_K)|0\rangle$$

$$L(\nu)|K\rangle = L(\nu)f(\mu_1)f(\mu_2)\cdots f(\mu_K)|0\rangle =$$

$$= f(\mu_1)f(\mu_2)\cdots f(\mu_K)\,L(\nu)|0\rangle +$$

$$+\frac{2}{\nu-\mu_1}\Big(f(\mu_1)h(\nu)-f(\nu)h(\mu_1)\Big)f(\mu_2)\cdots f(\mu_K)|0\rangle +$$

$$+\frac{2}{\nu-\mu_2}f(\mu_1)\Big(f(\mu_2)h(\nu)-f(\nu)h(\mu_2)\Big)f(\mu_3)\cdots f(\mu_K)|0\rangle$$

$$+\quad\dots\quad+$$

$$\frac{2}{\nu-\mu_K}f(\mu_1)f(\mu_2)\cdots f(\mu_{K-1})\Big(f(\mu_K)h(\nu)-f(\nu)h(\mu_K)\Big)|0\rangle =$$

$$L(\nu)|0\rangle = \Big(\frac{1}{2}h(\nu)h(\nu) + e(\nu)f(\nu)+f(\nu)e(\nu)\Big)|0\rangle$$

$$= \Big(\frac{1}{2}h(\nu)h(\nu) + \sum_{i,j=1}^{N}\frac{e^{(i)}f^{(j)}}{(\nu-z_i)(\nu-z_j)}\Big)|0\rangle = \Big(\frac{1}{2}h(\nu)h(\nu) + \sum_{i,j=1}^{N}\frac{h^{(i)}}{(\nu-z_i)^2}\Big)|0\rangle = \frac{1}{2}\Big(\sum_{i=1}^{N}\frac{\lambda_i}{\nu-z_i}\Big)^2 + \sum_{i=1}^{N}\frac{\lambda_i}{(\nu-z_i)^2}$$

$$= \Bigg(\frac{1}{2}\Big(\sum_{i=1}^{N}\frac{\lambda_i}{\nu-z_i}\Big)^2 + \sum_{i=1}^{N}\frac{\lambda_i}{(\nu-z_i)^2}\Bigg)|K\rangle +$$

$$+\sum_{\alpha=1}^{K}\frac{2}{\nu-\mu_\alpha}f(\mu_1)f(\mu_2)\cdots f(\mu_K)h(\nu)|0\rangle$$

$$-\sum_{\alpha=1}^{K}\frac{2}{\nu-\mu_\alpha}f(\mu_1)\cdots \underbrace{f(\nu)}_{\alpha\text{-oe}}\cdots f(\mu_K)h(\mu_\alpha)|0\rangle$$

$$+\sum_{\alpha=1}^{K}\sum_{\beta>\alpha}^{K}\frac{2}{\nu-\mu_\alpha}\cdot\frac{2}{\nu-\mu_\beta}f(\mu_1)\cdots\Big(\underbrace{f(\mu_\beta)-f(\nu)}_{\beta\text{-oe}}\Big)\cdots f(\mu_K)|0\rangle$$

$$-\sum_{\alpha=1}^{K}\sum_{\beta>\alpha}^{K}\frac{2}{\nu-\mu_\alpha}\frac{2}{\mu_\alpha-\mu_\beta}f(\mu_1)\ldots\underbrace{f(\nu)}_{\alpha\text{-ое место}}\ldots\underbrace{\left(f(\mu_\beta)-f(\mu_\alpha)\right)}_{\beta\text{-ое}}\ldots f(\mu_K)|0\rangle$$

Собираем коэффициент при

$$f(\mu_1)f(\mu_2)\ldots\underbrace{f(\nu)}_{\alpha\text{-ое}}\ldots f(\mu_K)|0\rangle$$

$$-\frac{2}{\nu-\mu_\alpha}\cdot\sum_{i=1}^{N}\frac{\lambda_i}{\mu_\alpha-z_i}-\sum_{\beta<\alpha}\frac{2}{\nu-\mu_\beta}\frac{2}{\nu-\mu_\alpha}-$$

$$-\sum_{\beta>\alpha}\frac{2}{\nu-\mu_\alpha}\frac{2}{\mu_\alpha-\mu_\beta}+\sum_{\beta<\alpha}\frac{2}{\nu-\mu_\beta}\frac{2}{\mu_\beta-\mu_\alpha}=$$

$$=-\frac{2}{\nu-\mu_\alpha}\left(\sum_{i=1}^{N}\frac{\lambda_i}{\mu_\alpha-z_i}+\sum_{\beta>\alpha}\frac{2}{\mu_\alpha-\mu_\beta}\right)+$$

$$+\sum_{\beta<\alpha}\frac{2}{\nu-\mu_\beta}\underbrace{\left(\frac{2}{\mu_\beta-\mu_\alpha}-\frac{2}{\nu-\mu_\alpha}\right)}_{=\sum_{\beta<\alpha}\frac{2}{(\mu_\beta-\mu_\alpha)}\frac{2}{(\nu-\mu_\alpha)}}=$$

$$=-\frac{2}{\nu-\mu_\alpha}\left(\sum_{i=1}^{N}\frac{\lambda_i}{\mu_\alpha-z_i}+\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{K}\frac{2}{\mu_\alpha-\mu_\beta}\right)$$

Уравнение Бете $\alpha=1,\ldots K$

$$\boxed{\sum_{i=1}^{N}\frac{\lambda_i}{\mu_\alpha-z_i}+\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{K}{}'\frac{2}{\mu_\alpha-\mu_\beta}=0\qquad (!)}$$

Если $\mu_1, \dots \mu_k$ удовлетв. системе (!), то $|K\rangle = f(\mu_1) \dots f(\mu_k)|0\rangle$ — собственный для $L(z)$.

Собств. значение соответств. $|K\rangle$:

$$\ell(\nu) = \frac{1}{2}\left(\sum_{i=1}^{N} \frac{\lambda_i}{\nu - z_i} + \sum_{\alpha=1}^{K} \frac{2}{\nu - \mu_\alpha}\right)^2 + \sum_{i=1}^{N} \frac{\lambda_i}{(\nu - z_i)^2} - \sum_{\alpha=1}^{K} \frac{2}{(\nu - \mu_\alpha)^2}$$

$$L(\nu)|K\rangle = \ell(\nu)|K\rangle$$

$$H_j|K\rangle = h_j|K\rangle$$

$$h_j = -\frac{1}{2}\operatorname{Res}_{\nu = z_j} \ell(\nu) = -\frac{1}{2}\lambda_j\left(\sum_{i \neq j} \frac{\lambda_i}{z_j - z_i} + \sum_{\alpha=1}^{K} \frac{2}{z_j - \mu_\alpha}\right)$$

$$L(z) = \sum_{j=1}^{N} \frac{C^{(j)}}{(z - z_j)^2} - \sum_{j=1}^{N} \frac{2H_j}{(z - z_j)}$$

$$C^{(j)}|K\rangle = \frac{1}{2}\lambda_j^2 + \lambda_j$$