Теорема 1 $\mathcal{U}(\hat{\mathfrak{g}})_k = \mathcal{U}(\hat{\mathfrak{g}})/(K-k)$ имеет нетривиальный центр только при $k = k_c$

$$Z_{\hat{\mathfrak{g}}} \simeq Z\left(\mathcal{U}(\hat{\mathfrak{g}})_{k_c}\right)$$

Теорема 2 $Z_{\hat{\mathfrak{g}}} \simeq \mathrm{Fun}\left(\mathrm{Op}_{{}^LG}(D^\times)\right)$ напомню позже

Модули Вакимото

конечный аналог

$$\begin{array}{c} \mathcal{O}(n) \\ \downarrow \\ SL_2 \circlearrowright \mathbb{P}^1 \end{array} \qquad SL_2 \circlearrowright \Gamma(\mathbb{P}^1, \mathcal{O}(n))$$

$$SL_2 \circlearrowright \Gamma(\mathbb{A}^1, \mathcal{O}(n)) \cong \mathbb{C}[z]$$

$$e \to \frac{\partial}{\partial z}$$

$$h \to -2z\frac{\partial}{\partial z} + n$$

$$f \to -z^2\frac{\partial}{\partial z} + nz$$

Модуль Вакимото

Опр Heis^{a,a^*} — алг. порожд. $a[l], a^*[l]$, $l \in \mathbb{Z}$

$$[a[l], a[m]] = 0 = [a^*[l], a^*[m]]$$

$$[a[m], a^*[l]] = \delta_{m+l,0}$$

Опр Heis^b — алг порожд $b[l]$ $l \in \mathbb{Z}$

$$[b[l], b[m]] = 2l\,\delta_{l+m,0}\,\varkappa$$

$\varkappa = (k-k_c)$

Рассм. $\mathrm{Heis}^{a,a^*} \otimes \mathrm{Heis}^b \circlearrowright \mathrm{Fock}^{a,a^*} \otimes \mathrm{Fock}^b \ni |\emptyset\rangle \otimes |\lambda\rangle$

$a[l]|\emptyset\rangle = 0 \quad l \geq 0$

$a^*[l]|\emptyset\rangle = 0 \quad l > 0$

$b[l]|\lambda\rangle = 0 \quad l > 0$

$b[0]|\lambda\rangle = \lambda|\lambda\rangle$

$e(z) = \sum_j e[\ell] z^{-\ell-1}$; $f(z)$, $h(z)$, $a(z)$, ...

$e \to \frac{\partial}{\partial z}$ $\qquad e(z) \longrightarrow a(z)$

$h \to -2z\frac{\partial}{\partial z} + \lambda \longrightarrow h(z) \longrightarrow -2 :a^*(z)a(z): + b(z)$

$f \to -z^2\frac{\partial}{\partial z} + \lambda z$ $\qquad f(z) \longrightarrow -:a^*(z)^2 a(z): + a^*(z)b(z) + k\partial_z a^*(z)$

Получ. предст. $M_{\lambda,k}$ — модуль Вакимото

Общий случай $\qquad \mathcal{O}(\lambda)$ $\qquad \lambda$ - вес

Конечный $\qquad \downarrow \leftarrow$ эквив.

$\qquad G/B$

$G \circlearrowright \Gamma(G/B, \mathcal{O}(\lambda))$

$\mathcal{U}(\mathfrak{g}) \circlearrowright \Gamma(\mathcal{U}, \mathcal{O}(\lambda))$ $\qquad \mathcal{U}$ - открытая клетка в G/B

$\qquad \qquad \mathcal{U} \simeq \mathbb{A}^{\dim \mathcal{U}}$

$\downarrow$

модуль Вакимото $\mathcal{U}(\hat{\mathfrak{g}})_k \circlearrowright M_{\lambda,k}$, λ - вес

Опр $Heis^{a,a^*}$ — $a_\alpha[\ell]$, $a^*_\alpha[\ell]$ $\quad \alpha \in \Delta_+$, $\ell \in \mathbb{Z}$

$[a_\alpha[\ell], a_\beta[m]] = 0 = [a^*_\alpha[\ell], a^*_\beta[m]]$

$[a_\alpha[\ell], a^*_\beta[m]] = \delta_{\alpha,\beta} \cdot \delta_{\ell+m,0}$

Опр $Heis^b$ порожд $b_i[\ell]$ $\quad i$ - простой полож. корень, $\ell \in \mathbb{Z}$

$[b_i[\ell], b_j[m]] = \ell \cdot A_{ji} \cdot \varkappa \cdot \delta_{\ell+m}$ $\qquad \ell > 0$

A_{ji} — матрица Картана для $\mathfrak{g}$; $\quad \varkappa = k - k_c$

$Fock^{a,a^*} \otimes Fock^b_\lambda \ni |\emptyset\rangle \otimes |\lambda\rangle$

$a_\alpha[l]|\emptyset\rangle = 0 \quad l \geq 0 \qquad b_i[l]|\lambda\rangle = 0 \quad l > 0$

$a_\alpha[l]|\emptyset\rangle = 0 \quad l > 0 \qquad b_i[0]|\lambda\rangle = \lambda_i|\lambda\rangle$

Th (Вакимото, Фейгин, Френкель)

$\mathcal{U}(\hat{\mathfrak{g}})_k \curvearrowright Fock^{a,a^*} \otimes Fock^b$

$e_\alpha(z) \to \ldots$

$h_\alpha(z) \to \ldots$

$f_\alpha(z) \to \ldots$

полученный модуль $M_{\lambda,k}$

Предложение • $\mathcal{U}(\hat{\mathfrak{g}})_k \xhookrightarrow{i} End(M_{\lambda,k})$

• для $k = k_c$ имеем $i(Z_{\hat{\mathfrak{g}}}) = \operatorname{Im} i \cap Heis^b$

$M_{\lambda,k} \underset{\uparrow \atop Vect}{=} Fock^{a,a^*} \otimes Fock^b$

Замечание При $k = k_c$

$Heis^b$ – цент $Heis^{a,a^*} \otimes Heis^b$

Вертексная алгебра

Опр Вертекс. алгебра это набор

- V – вект. пр-во
- $|0\rangle \in V$
- $T: V \to V$
- $Y: V \to End(V)[[z^{\pm 1}]]$

удовл. аксиомам

- $Y(|0\rangle, z) = \mathrm{Id}_V$
- $[T, Y(v,z)] = \partial_z Y(v,z)$
- ...

Пример 1 Heis^b $\quad [b[\ell], b[m]] = \ell\,\delta_{\ell+m,0}$

$$V = \mathrm{Fock}^b_0$$

$$Y(b[-\ell_1]\cdots b[-\ell_r]\,|\emptyset\rangle, z) = :\partial^{\ell_1-1}b(z)\cdots\partial^{\ell_r-1}b(z):$$

Пример 2 $U(\hat{\mathfrak g})_k$;

вект. пр-во $\quad \mathbb{V}_{0,k} = \mathrm{Ind}^{\hat{\mathfrak g}}_{\mathfrak g[t]\oplus\mathbb{C}K}\, \mathbb{C}_{0,k}$

$$Y(X_1[-\ell_1]\cdots X_r[-\ell_r]\,|\emptyset\rangle, z) = :\partial^{\ell_1-1}X_1(z)\cdots\partial^{\ell_r-1}X_r(z):$$

" $:A(z)\,B(z): \; = A_+(z)\,B(z) + B(z)\,A_-(z)$ "

Предл Предст. Вакимото — морф. верт. алгебр (для $\varkappa=0$)

$$\begin{tikzcd}
U(\hat{\mathfrak g})_k[[z^{\pm1}]] \arrow[r, hook] & \mathrm{Heis}^{a,a^*}\otimes\mathrm{Heis}^b[[z^{\pm1}]] \\
\mathbb{V}_{0,k} \arrow[r, hook] \arrow[u] & \mathrm{Fock}^{a,a^*}\otimes\mathrm{Fock}^b \arrow[u]
\end{tikzcd}$$

коммут.

План

$$\begin{tikzcd}
Z_{\hat{\mathfrak g}}[[z^{\pm1}]] \arrow[r, hook] & U(\hat{\mathfrak g})_k[[z^{\pm1}]] \arrow[r, hook] & \mathrm{Heis}^{a,a^*}\otimes\mathrm{Heis}^b[[z^{\pm1}]] \\
\mathfrak{z}_{\hat{\mathfrak g}} \arrow[r] \arrow[u, hook] & \mathbb{V}_{0,k} \arrow[r, hook] \arrow[u] & \mathrm{Fock}^{a,a^*}\otimes\mathrm{Fock}^b \arrow[u]
\end{tikzcd}$$

$$\mathfrak{z}_{\hat{\mathfrak g}} \simeq \{ z|0\rangle \mid z\in Z_{\hat{\mathfrak g}} \}$$

$$\begin{tikzcd} Z_{\hat{\mathfrak{g}}}[[z^{\pm 1}]] \arrow[r, "i"] & \mathcal{M}_{\mathfrak{g}}^{b}[[z^{\pm 1}]] \\ \mathfrak{Z}_{\hat{\mathfrak{g}}} \arrow[u] \arrow[r, "\tilde{i}"] & \mathrm{Fock}^{b} \arrow[u] \end{tikzcd}$$

Вопрос: найти образ $i(\mathfrak{Z}_{\hat{\mathfrak{g}}}) \subset \mathrm{Fock}^{b}$

Скрининги

$$\begin{tikzcd} \tilde{\mathcal{U}} = p^{-1}(\mathcal{U}) \arrow[r] \arrow[d] & G \arrow[d, "p"] \\ \mathcal{U} \arrow[r, hook] & G/B \end{tikzcd}$$

$$\Gamma(\mathcal{U}, \mathcal{O}(\lambda)) = \Gamma(\tilde{\mathcal{U}}, \mathcal{O})^{B, -\lambda}$$

$$\{ f \mid f(g b^{-1}) = \lambda(b) f(g) \}$$

$f_i \in \mathfrak{n}_-$ $\quad R_{f_i}$ – правое дейст f_i

$$R_{f_i} : \mathrm{Fun}(\tilde{\mathcal{U}})^{B, -\lambda} \longrightarrow \mathrm{Fun}(\tilde{\mathcal{U}})^{B, \alpha_i - \lambda}$$

$$\mathcal{U}(\mathfrak{g}) \curvearrowright \mathrm{Fun}(\tilde{\mathcal{U}})^{B, -\lambda}$$

гомоморфизм предст.

слева

Claim Имеются $\mathcal{U}(\hat{\mathfrak{g}})_\kappa$ сплет. операт.

$$S_i : M_{\lambda, \kappa} \longrightarrow M_{\lambda - \alpha_i, \kappa}$$

Предл. Обозн. $\bar S_i = S_i\big|_{\lambda=0}$

$$\bigcap \ker \bar S_i = \tilde\iota(\mathbb{V}_{0,k})$$

$$\cap \qquad \cap$$

$$M_{0,k}$$

Есть индуц опер $\bar V_i : Fock^b \longrightarrow Fock^b$

индуц $\bar S_i$

Теорема $\mathfrak{z}_{\hat{\mathfrak g}} \simeq \bigcap \ker \bar V_i$

Док-во • $\mathfrak{z}_{\hat{\mathfrak g}} \subset \bigcap \ker \bar V_i$

• знаем характеры □

Скрининги 2-го рода

$$\sum_{m\geq} \bar V^+[-m] z^{-m}$$

$$\parallel$$

$$\bar V_i(z) = \exp\left(\sum_{m>0} \frac{b_i[-m]}{m} z^m\right) \exp\left(-\sum_{m>0} \frac{b_i[m]}{m} z^{-m}\right)$$

$$\parallel$$

$$\sum \bar V_i[-m] z^m$$

Утв $\bar V_i = \bar V_i[1]$

$\varkappa = k - k_c \longrightarrow 0$ $\qquad b_i[m] = \varkappa \dfrac{\partial}{\partial b}$

$$V_i[1] = \varkappa \underbrace{\sum \bar V^+[-m] D_i[m]}_{} + o(\varkappa)$$

$$[D_i[m], b_j[-l]] = m A_{ji} \delta_{m,l}$$

$\bar V_i$ при $\varkappa = 0$

$$\mathfrak{z}_{\hat{\mathfrak{g}}} = \bigcap_i \ker \bar{V}_i^{(G)} = \bigcap_i \ker \tilde{V}_i^{(G)} = \mathrm{Fun}(\mathrm{Op}_{{}^LG}(D))$$

$$\mathrm{Op}_G(D) = \left\{ d + \begin{pmatrix} * & & \\ 1 & \ddots & \\ 0 & 1 & \ddots \end{pmatrix} \right\} \Big/ N_+$$

$$\mathrm{Fun}(\mathrm{Op}_G(D)) \subset \mathrm{Fun}\left(\left\{ d + \begin{pmatrix} & & 0 \\ 1 & & \\ 0 & 1 & \ddots \; 1 \end{pmatrix} \right\}\right)$$

$u \in \mathcal{L}[[t]]$

можно сделать калибр. преобраз. $x_i(t)\cdot e_i$

т.ч. $d + \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \longrightarrow d + \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots 1 \end{pmatrix}$

$$\begin{tikzcd}
Z_{\hat{\mathfrak{g}}}[[z^{\pm 1}]] \arrow[r, "\sim"] & \mathrm{Fun}(\mathrm{Op}_{{}^LG}(D^\times))[[z^{\pm 1}]] \\
\mathfrak{z}_{\hat{\mathfrak{g}}} \arrow[u] \arrow[r, "\sim"] & \mathrm{Fun}(\mathrm{Op}_{{}^LG}(D)) \arrow[u]
\end{tikzcd}$$

и просто

$S(\hat{\mathfrak{g}}_1) // N_+$