Nilpotent orbits

Ivan Motorin

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For drawings

Adjoint orbits

Reminder. There is a Poisson structure on $\mathfrak{g}^*$ and coadjoint orbits are symplectic submanifolds for it.

Proposition 1.

Adjoint orbits in $\mathfrak{g}$ are even-dimensional.

Proof: there is an invariant non-degenerate symmetric form $K(\cdot, \cdot)$ on $\mathfrak{g}$, so we can identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via

$$i_K(X) = K(X, *) \quad \forall X \in \mathfrak{g}$$

Also for any $x \in G_{ad}$ we have

$$i_K(x \cdot X) = K(x \cdot X, *) = K(X, x^{-1} \cdot (*)) = Ad_x^* i_K(X)$$

Therefore adjoint orbits in $\mathfrak{g}$ are in one to one correspondence with coadjoint orbits in $\mathfrak{g}^*$. $\square$

Semisimple orbits

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then the following map is surjective

$$\tilde{\mu} : \mathfrak{h} \to \{\text{semisimple orbits}\} \qquad \tilde{\mu}(X) = \mathcal{O}_X$$

Lemma 1.

$$W \cong N_{G_{ad}}(\mathfrak{h})/C_{G_{ad}}(\mathfrak{h}),$$

where $N_{G_{ad}}(\mathfrak{h}) = \{x \in G_{ad} | x \cdot \mathfrak{h} = \mathfrak{h}\}$,
$C_{G_{ad}}(\mathfrak{h}) = \{x \in G_{ad} | x \cdot Y = Y \quad \forall Y \in \mathfrak{h}\}$.

Idea of proof: $\theta_i = \exp(ad_{X_i}) \cdot \exp(-ad_{Y_i}) \cdot \exp(ad_{X_i})$ generates reflection s_i in co-roots. One can show that $C_{G_{ad}}(\mathfrak{h}) = T$ (maximal torus in G_{ad} having a lie algebra $\mathfrak{h}$) then prove that the stabilizer of the dominant Weyl chamber must be T.

Semisimple orbits

Reminder If $\mathfrak{g}$ is reductive and X is semisimple then $Z_{\mathfrak{g}}(X)$ is reductive subalgebra.

Theorem 1.

The following map is bijective:

$$\mu : \mathfrak{h}/W \to \{\text{semisimple orbits}\} \qquad \mu([X]) = \mathcal{O}_X$$

Proof: the map is well defined (Lemma 1).
If $\mu([X_1]) = \mu([X_2])$ then $\exists x \in G_{ad}, x \cdot X_1 = X_2$, so Cartan subalgebras $\mathfrak{h}, x \cdot \mathfrak{h}$ have common element X_2. It means that $\mathfrak{h}, x \cdot \mathfrak{h} \subset Z_{\mathfrak{g}}(X_2)$ and $\mathfrak{h}, x \cdot \mathfrak{h}$ are conjugate by some y from $Z^{\circ}_{G_{ad}}(X_2)$. Therefore $y \cdot x \cdot \mathfrak{h} = \mathfrak{h}$ and $y \cdot x \cdot X_1 = y \cdot X_2 = X_2$ thus $[X_1] = [X_2]$. $\square$

Semisimple orbits

If Π is a set of simple roots of $\mathfrak{g}$ then by theorem 1 μ is bijective map from D_Π:

$$D_\Pi = \{x \in \mathfrak{h} | \forall \alpha \in \Pi \quad \Re(\alpha(x)) > 0 \text{ or } \Re(\alpha(x)) = 0, \Im(\alpha(x)) \geq 0\}$$

to semisimple orbits. In particular there are infinitely many semisimple orbits.

Orbit closure

Lemma 2.

Fix an orbit $\mathcal{O}$ and $X = X_s + X_n \in \mathfrak{g}$ such that $X \in \bar{\mathcal{O}}$ then $X_s \in \bar{\mathcal{O}}$.

Proof: notice that $\mathcal{O}_X \subseteq \bar{\mathcal{O}}$. $[X_s, X_n] = 0$ therefore X_n lies in reductive $Z_{\mathfrak{g}}(X_s)$. Also X_n is nilpotent in $\mathfrak{g}$ so its semisimple part must be zero thus $X_n \in [Z_{\mathfrak{g}}(X_s), Z_{\mathfrak{g}}(X_s)]$. By Jacobson-Morozov theorem there is $h \in [Z_{\mathfrak{g}}(X_s), Z_{\mathfrak{g}}(X_s)], [h, x_n] = 2x_n$, so if we act by exponent of ch for a large negative $c \in \mathbb{R}$ on X we will get that $X_s \in \bar{\mathcal{O}}$. $\square$

Remark. In particular if X is nilpotent then $\{0\} \subseteq \bar{\mathcal{O}}_X$.

Closed semisimple orbits

Fact. There is an isomorphism of algebras $\mathbb{C}[\mathfrak{g}]^{G_{ad}} \cong \mathbb{C}[\mathfrak{h}]^W$ (via restriction).

Proposition 2.

If $X, Y \in \mathfrak{h}$ and $X \notin W \cdot Y$ then $\exists f \in \mathbb{C}[\mathfrak{h}]^W, f(X) \neq f(Y)$.

Proof: for a $w \in W$ pick a $g_w \in \mathfrak{h}^*$ so that $g_w(Y) = 1, g_w(w \cdot X) = 0$. Let $g = 1 - \prod_{w \in W} g_w$. Then $f = \prod_{w \in W} g \circ w$ suits the requirement as $f(Y) = 0, f(X) = 1$. $\square$

Closed semisimple orbits

Theorem 2. (Borel, Harish-Chandra)

An element of reductive Lie algebra $\mathfrak{g}$ is semisimple if and only if $\mathcal{O}_X$ is closed.

Proof: suppose that X is semisimple. Let $Y \in \bar{\mathcal{O}}_X$ then $\mathcal{O}_Y \subseteq \bar{\mathcal{O}}_X$, $\bar{\mathcal{O}}_Y \subseteq \bar{\mathcal{O}}_X$ and $\mathcal{O}_{Y_s} \subseteq \bar{\mathcal{O}}_X$. If $f \in \mathbb{C}[\mathfrak{g}]^{G_{ad}}$ then it must be constant on $\mathcal{O}_X$ and $\bar{\mathcal{O}}_X$, so $f(X) = f(Y_s)$. By the fact and proposition 2: $\mathcal{O}_X = \mathcal{O}_{Y_s}$. We know that $\mathcal{O}_{Y_s} \subseteq \bar{\mathcal{O}}_Y$. So $\bar{\mathcal{O}}_Y = \bar{\mathcal{O}}_X$ and $\mathcal{O}_Y = \mathcal{O}_X$. Reverse statement follows from lemma 2. $\square$

Examples: $\mathfrak{sl}_2, \mathfrak{sl}_3$

Remark. $\mathcal{O}$ is nilpotent if and only if $\{0\} \subseteq \bar{\mathcal{O}}$. In particular $\mathcal{N}$ must be the zero set of all $f \in \mathbb{C}[\mathfrak{g}]^{G_{ad}}$, such that $f(0) = 0$.

Lemma 3.

Let $\mathfrak{n}$ be the nilradical of a Borel subalgebra of $\mathfrak{g}$. Then $G_{ad} \cdot \mathfrak{n} = \mathcal{N}$. In particular, if a subset $\mathcal{X}$ of $\mathfrak{n}$ is dense in latter then $G_{ad} \cdot \mathcal{X}$ is dense in $\mathcal{N}$.

Proof: Any nilpotent element forms a nilpotent subalgebra of $\mathfrak{g}$ so it can be conjugated to a subalgebra of $\mathfrak{n}$. $\square$

Lemma 4.

Let H, X, Y be a $\mathfrak{sl}_2$ triple in $\mathfrak{g}$ which forms ad_H-eigenspace decomposition $\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i$. Then $dim(\mathcal{O}_X) = dim(\mathfrak{g}) - dim(\mathfrak{g}_0) - dim(\mathfrak{g}_1)$.

Proof: $Z_{\mathfrak{g}}(X)$ is ad_H-stable, therefore $Z_{\mathfrak{g}}(X) = \oplus_{i \in \mathbb{Z}} Z_{\mathfrak{g}}(X) \cap \mathfrak{g}_i$. It follows that $Z_{\mathfrak{g}}(X)$ is the sum of the highest weight spaces, dimension of which is exactly $dim(\mathfrak{g}_0) + dim(\mathfrak{g}_1)$. $\square$

Denote $\mathfrak{q} = \oplus_{i \geq 0} \mathfrak{g}_i, \mathfrak{q}_{\mathfrak{i}} = \oplus_{j \geq i} \mathfrak{g}_j$. Let $\mathfrak{q} = Lie(Q)$, $Q \subset G_{ad}$ (closed, connected) and $\mathcal{P} = (G_{ad}^H)^\circ \cdot X$.

Lemma 5. (Kostant)

$$\mathcal{O}_X \cap \mathfrak{q}_2 = Q \cdot X = \mathcal{P} + \mathfrak{q}_3$$

In particular $\mathcal{O}_X \cap \mathfrak{q}_2$ is open and dense in $\mathfrak{q}_2$ (as $dim(\mathcal{P}) = dim(\mathfrak{g}_2)$ by Maltsev theorem).

Proof: $X + \mathfrak{q}_3 \subset Q \cdot X$ then $\mathcal{P} + \mathfrak{q}_3 \subset Q \cdot X$ as well. On the other hand $Q \cdot X \subseteq \mathcal{P} + \mathfrak{q}_3$ by definition. Therefore $Q \cdot X = \mathcal{P} + \mathfrak{q}_3$ and $Q \cdot X \cap \mathfrak{q}_2$ is open dense in $\mathfrak{q}_2$.
Now let $X' \in (G_{ad} \cdot X \cap \mathfrak{g}_2)$ then $dim(Z_{\mathfrak{g}}(X')) = dim(Z_{\mathfrak{g}}(X))$. Since $Z_{\mathfrak{g}}(X) \subset \mathfrak{q}$ we have
$dim(Z_{\mathfrak{q}}(X')) = dim(Z_{\mathfrak{g}}(X') \cap \mathfrak{q}) \leq dim(Z_{\mathfrak{g}}(X) \cap \mathfrak{q}) = dim(Z_{\mathfrak{q}}(X))$.
Therefore $dim(Q \cdot X') \geq dim(Q \cdot X)$ and $Q \cdot X' \cap Q \cdot X \neq \varnothing$. □

Principal orbit

Theorem 3.(de Siebenthal, Dynkin, Kostant)

In a semisimple $\mathfrak{g}$ there exists a unique orbit of maximal dimension $dim(\mathfrak{g}) - rank(\mathfrak{g})$ denoted by $\mathcal{O}_{prin}$ which is open dense in $\mathcal{N}$.

Proof: Let Π be a set of simple roots. We fix standard triples $\{H_\alpha, X_\alpha, Y_\alpha\}$ in $\mathfrak{g}$ corresponding to $\alpha \in \Pi$. We can find $H = \sum\limits_{\alpha \in \Pi} a_\alpha H_\alpha$ so that $\forall \alpha \in \Pi \quad \alpha(H) = 2$. $H, X = \sum\limits_{\alpha \in \Pi} X_\alpha, Y = \sum\limits_{\alpha \in \Pi} a_\alpha Y_\alpha$ is a standard triple. Clearly $\mathfrak{q}_2 = \mathfrak{n}$ thus $\mathcal{O}_X$ is open dense in $\mathcal{N}$ (and unique) by lemmas 3,5. We also notice that $\mathfrak{g}_0 = \mathfrak{h}, \mathfrak{g}_1 = 0$. $\square$

Typical semisimple orbit

Corollary. Because orbit $\mathcal{O}_X$ is an image of G_{ad} in $\mathfrak{g}$ and G_{ad} is irreducible, it follows that $\mathcal{N}$ is irreducible.

Remark. Dimension of $\mathcal{O}_{prin}$ coincides with dimension of typical semisimple orbit. Typical in a sense that semisimple X from $\mathcal{O}_X$ lies strictly inside of a Weyl chamber (X is regular).

Subregular orbit

Theorem 4. (Steinberg)

There exists a unique nilpotent orbit that is open and dense in $\bar{\mathcal{O}}_{prin} \setminus \mathcal{O}_{prin} = \mathcal{N} \setminus \mathcal{O}_{prin}$ denoted $\mathcal{O}_{subreg}$ and its dimension is $dim(\mathfrak{g}) - rank(\mathfrak{g}) - 2$.

Idea of proof: we can see from previous theorem that $X' = \sum_{\alpha \in \Pi} c_\alpha X_\alpha$ belongs to $\mathcal{P}$ iff all c_α are nonzero. We can define hyperplanes in $\mathfrak{g}_2$ defined as $\mathcal{D}_\alpha = \{X' \in \mathfrak{g}_2 | c_\alpha = 0\}$. The complement of $Q \cdot X$ in $\mathfrak{q}_2$ is union of $\mathcal{C}_\alpha := \mathcal{D}_\alpha + \mathfrak{q}_3$. We want to find an orbit $\mathcal{O}_\alpha$ whose intersection with $\mathcal{C}_\alpha$ is open dense in latter. After that we need to show that $\mathcal{O}_\alpha = \mathcal{O}_\beta = \mathcal{O}_{subreg}$.

Minimal orbit

We can easily observe that 0 is the orbit of minimal dimension ($Z(\mathfrak{g}) = 0$).

Theorem 5.

There exists a nonzero nilpotent orbit of minimal dimension $\mathcal{O}_{min} = \mathcal{O}_{X_\theta}$ which is contained in closure of any nonzero nilpotent orbit and X_θ is a nonzero vector in the highest root space.

There is a partial order structure on set of orbits which reads $\mathcal{O} \leq \mathcal{O}'$ iff $\bar{\mathcal{O}} \subseteq \bar{\mathcal{O}}'$. It is correct because Lie group orbits are open dense in their closure.

Minimal orbit

Proof: Suppose $X \in \mathcal{O}, X = \sum_{\alpha \in \Phi_+} c_\alpha X_\alpha$.
We can conjugate X so that $c_\theta \neq 0$. Namely, choose a maximal (w.r.t. standard partial order) β so that $c_\beta \neq 0$. If $\alpha \in \Pi, \alpha + \beta$ is a root then we can find $c \neq 0 \quad \exp(cZ_\alpha) \cdot X$ so that $c_{\alpha+\beta} \neq 0$.
Fix $H \in \mathfrak{h}$ so that $\forall \alpha \in \Pi \quad \alpha(H) = 2$. By conjugation of X with $\exp(rH)$ for large $r \in \mathbb{R}$ we can make its θ component arbitrary large in comparison with other components then we scale resulting nilpotent element back by $H' \in \mathfrak{h}$ from $\mathfrak{sl}_2$ triple corresponding to the new X. $\square$

Minimal orbit

Lemma 6.

$dim(\mathcal{O}_{min})$ equals one plus number of positive roots not orthogonal to θ.

Proof: X_θ and $Z_\mathfrak{g}(X_\theta)$ are $ad_\mathfrak{h}$-stable. Now any positive root element annihilates X_θ as well as an appropriate hyperplane in $\mathfrak{h}$. Negative root element $X_{-\beta}$ annihilates X_θ iff $(\theta, \beta) = 0$ (as $(\theta, \beta)X_\beta = [H_\theta, X_\beta]$). $\square$

Here is a table of orbit dimensions for classical Lie algebras:

group type	$\mathfrak{sl}_n$	$\mathfrak{sp}_{2n}$	$\mathfrak{so}_{2n}$	$\mathfrak{so}_{2n+1}$
$dim(\mathcal{O}_{min})$	2n-2	2n	4n-6	4n-4

Remark. These numbers coincide with $2h^\vee - 2$.

Proposition 3.1

If $\mathfrak{g} = \mathfrak{sl}_n$, then $\mathcal{O}_{prin} = \mathcal{O}_n, \mathcal{O}_{subreg} = \mathcal{O}_{n-1,1}, \mathcal{O}_{min} = \mathcal{O}_{2,1^{n-2}}$. Also $\mathcal{O}_\mu \subseteq \bar{\mathcal{O}}_\lambda$ iff $\lambda \gg \mu$.

Proof (of the second statement): $\forall \lambda \in \mathcal{P}(n), \mathcal{O}_\lambda$ contains matrices X such that ranks of their powers are fixed: $rkX^i = a_{\lambda,i}$. $\bar{\mathcal{O}}_\lambda$ can contain only matrices with $a_{\mu,i} \leq a_{\lambda,i}$, because the set of matrices of $rk \leq k$ is closed. One can show that

$$rkX_\lambda^k = \sum_{\{i|\lambda_i \geq k\}} (\lambda_i - k)$$

and that $\lambda \gg \mu$ iff $\forall k$ $rkX_\lambda^k \geq rkX_\mu^k$. So the task comes down to making $a-1, b+1$ Jordan block out of a, b-block ($b \leq a-2$). It is possible. □

Classical types

Proposition 3.2

(i) If $\mathfrak{g} = \mathfrak{so}_{2n+1}$, then
$\mathcal{O}_{prin} = \mathcal{O}_{2n+1}, \mathcal{O}_{subreg} = \mathcal{O}_{2n-1,1^2}, \mathcal{O}_{min} = \mathcal{O}_{2^2,1^{2n-3}}$.
(ii) If $\mathfrak{g} = \mathfrak{sp}_{2n}$, then $\mathcal{O}_{prin} = \mathcal{O}_{2n}, \mathcal{O}_{subreg} = \mathcal{O}_{2n-2,2}, \mathcal{O}_{min} = \mathcal{O}_{2,1^{2n-2}}$.
(iii) If $\mathfrak{g} = \mathfrak{so}_{2n}$, then
$\mathcal{O}_{prin} = \mathcal{O}_{2n-1,1}, \mathcal{O}_{subreg} = \mathcal{O}_{2n-3,3}, \mathcal{O}_{min} = \mathcal{O}_{2^2,1^{2n-4}}$.

Exponents

Let $\lambda \in \mathcal{P}(N)$ we can construct new partition $\lambda^t = [\lambda_1^t, \dots, \lambda_N^t]$:

$$\lambda_j^t = |\{i | \lambda_i \geq j\}|$$

Define $d = [t_1, t_2, \dots, t_{ht(\theta)}]$, where $t_i = |\{\alpha \in \Phi_+ | ht(\alpha) = i\}|$. Then d^t is a partition which has $l = rank(\mathfrak{g})$ parts $\{m_1, \dots, m_l\}$. These numbers are called exponents of $\mathfrak{g}$.
Also $\mathfrak{g} \cong \bigoplus_i \mathbb{C}^{2m_i+1}$ as a principal $\mathfrak{sl}_2$-module.

Theorem 6. (no proof)

$S(\mathfrak{g}^*)^{G_{ad}}$ is a polynomial algebra generated by algebraically independent homogeneous polynomials $f_1, \dots, f_l$ with degrees $1 + m_1, \dots, 1 + m_l$.

Exponents

$\mathfrak{g}$	Exponents
$\mathfrak{sl}_n$	$1,2,\ldots,n-1$
$\mathfrak{so}_{2n+1}$	$1,3,5,\ldots,2n-1$
$\mathfrak{sp}_{2n}$	$1,3,5,\ldots,2n-1$
$\mathfrak{so}_{2n}$	$1,3,5,\ldots,2n-3,n-1$
E_6	1,4,5,7,8,11
E_7	1,5,7,9,11,13,17
E_8	1,7,11,13,17,19,23,29
F_4	1,5,7,11
G_2	1,5

Рис.: Exponents

Let X be a nilpotent element of semisimple $\mathfrak{g}$, $A := A(\mathcal{O}_X) := G_{ad}^X/(G_{ad}^X)^\circ$.

Theorem 7. (Springer) (no proof)

There is a bijection between irreducible representations of the Weyl group W of $\mathfrak{g}$ and the following data:

$$\{\text{Nilpotent orbits } \mathcal{O}_X \text{ and irreducible representations of } A(\mathcal{O}_X)\}$$

In the case of $\mathfrak{sl}_n$ $A(\mathcal{O}_X)$ is always trivial, so there are exactly $\mathcal{P}(n)$ irreducible representations of S_n.