Subregular Slodowy

27 November 2020

Nilpotent orbits

$N = \dim \mathfrak{g} - \mathrm{rk}\, \mathfrak{g}$

$\mathfrak{g} = \mathfrak{sl}_n$

N	regular	(n)
	$\downarrow$	
$N-2$	subregular (subprincipal)	$(n-1, 1)$
	$\vdots$	
	$\downarrow$	
	minimal	$(2, 1, \ldots, 1)$
	$\downarrow$	
0	0	$(1, 1, \ldots, 1)$

Subregular Nilpotent

- $$e = e_{\alpha_1} + \dots + e_{\alpha_{n-1}} = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & & 1 & \\ & 0 & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

$$h = \begin{pmatrix} n-2 & & & & \\ & n-4 & & 0 & \\ & & \ddots & & \\ & 0 & & 2-n & \\ & & & & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & & & & \\ (n-2)1 & 0 & & 0 & \\ & (n-3)\cdot 2 & 0 & & \\ & & \ddots & & \\ & 0 & & 1(n-2) & 0 \\ & & & 0 & 0 \end{pmatrix}$$

- sl_2 $\quad e = 0$

sl_3 $\quad e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Slodowy slice

- $$e = e_{\alpha_1} + \dots + e_{\alpha_{n-1}} = \begin{pmatrix} 0 & 1 & & & 0 \\ & \ddots & \ddots & & \\ & & & 1 & \\ & 0 & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

$$h = \begin{pmatrix} n-2 & & & & 0 \\ & n-4 & & & \\ & & \ddots & & \\ & 0 & & 2-n & \\ & & & & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & & & & & 0 \\ (n-2)\cdot 1 & 0 & & & & \\ & (n-3)\cdot 2 & 0 & & & \\ & & & \ddots & & \\ & 0 & & 1(n-2) & 0 & \\ & & & & 0 & 0 \end{pmatrix}$$

- $$\mathfrak{sl}_n = \underbrace{\mathfrak{sl}_{n-1}}_{= \mathbb{C}^3 \oplus \dots \oplus \mathbb{C}^{2n-3}} \oplus \underbrace{\mathbb{C}^{n-1}}_{= \mathbb{C}^{n-1}} \oplus \underbrace{(\mathbb{C}^{n-1})^*}_{= \mathbb{C}^{n-1}} \oplus \underbrace{\mathbb{C}}_{= \mathbb{C}}$$ as $\mathfrak{sl}_{n-1}$ module

as $\mathfrak{sl}_2$ module.

- $$S_e = e + \operatorname{Ker} \operatorname{ad}_f = e + a_1 f + a_2 f^2 + \dots + a_{n-2} f^{n-2} + a\left(\frac{1}{n-1}\left(\sum E_{ii} - E_{nn}\right)\right) + b \underbrace{E_{n-1,n}}_{= e_{\alpha_n}} + c \underbrace{E_{n,1}}_{= e_{-\theta}}$$

Słodowy slice

- $$S_e = e + \ker \mathrm{ad}_f = e + a_1 f + a_2 f^2 + \dots + a_{n-2} f^{n-2} + a\left(\frac{1}{n-1}\left(\Sigma\left(E_{ii} - E_{nn}\right)\right)\right) + b \underset{e_{2n}}{\underset{\parallel}{E_{n-1,n}}} + c \underset{e_{-\theta}}{\underset{\parallel}{E_{n,1}}}$$

- sl_2 $$S_e = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}$$

$$\dim S_e = n+1$$

- sl_3 $$S_e = \left\{ \begin{pmatrix} a/2 & 1 & 0 \\ a_1 & a/2 & b \\ c & 0 & -a \end{pmatrix} \right\}$$

- sl_4 $$S_e = \left\{ \begin{pmatrix} a/3 & 1 & 0 & 0 \\ 2a_1 & a/3 & 1 & 0 \\ 4a_2 & 2a_1 & a/3 & b \\ c & 0 & 0 & -a \end{pmatrix} \right\}$$

Casimir functions

- $K_\ell = \operatorname{Tr} X^{\ell+1} \qquad \ell = 1, \ldots, n-1$

- $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \qquad K_1 = -(a^2+bc)$

- $\begin{pmatrix} a/2 & 1 & 0 \\ a_2 & a/2 & b \\ c & 0 & -a \end{pmatrix} \qquad K_1 = \frac{3}{2}a^2 + 2a_1$

  $K_2 = -\frac{3}{4}a^3 + 3a a_1 + 3bc$

- $\begin{pmatrix} a/3 & 1 & 0 & 0 \\ 2a_1 & a/3 & 1 & 0 \\ 4a_2 & 2a_1 & a/3 & b \\ c & 0 & 0 & -a \end{pmatrix} \qquad K_1 = \frac{4}{3}a^2 + 8a_1$

  $K_2 = -\frac{8}{9}a^3 + 8a a_1 + 12a_2$

  $K_3 = \frac{28}{27}a^4 + \frac{16}{3}a^2 a_1 + 32a_1^2 + 16 a a_2 + 4bc$

Family of deformations

- deg — Kazhdan grading

  $\deg K_\ell = 2\ell+2$, $\deg a_i = 2i+2$, $\deg a = 2$, $\deg b = \deg c = n$

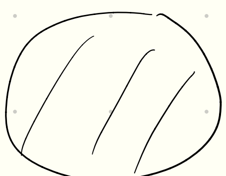
- $\mathfrak{sl}_2$ $\deg a = \deg b = \deg c = 2$

  $2, 4, \dots, 2(n-1) \quad n \quad n$

  $4 \quad 2(n-1) \quad 2n$

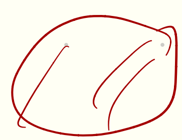
- Relations $K_\ell = \# a_\ell + P(a, a_1, \dots, a_{\ell-1})$, $1 \le \ell \le n-2$ $K_{n-1} = bc + P(a, a_1, \dots, a_{n-2})$

- $S_e \xrightarrow{\varphi} \mathfrak{g}/\!/G = \mathbb{C}^{n-1}$ $X \mapsto (K_2(X), \dots, K_n(X))$

   $\longrightarrow \overset{\psi}{pt}$

  φ — surj

  surface $\{bc = P(a) = a^n + \# a^{n-2} + \dots + \#\}$

- RK For $K_1 = \dots = K_{\ell-1} = 0$  $= \{bc = a^n\}$ — A_{n-1} singularity

Other types

- $\mathbb{C}[S_e]$ generated by $x_1, \dots, x_{r+2}$ $\quad \deg x_i = w_i$
  $\mathbb{C}[\mathfrak{g}]^G$ generated by $K_1, \dots, K_r,$ $\quad \deg K_i = d_i = 2m_i + 2$

- <u>Fact</u> $\varphi : S_e \to \mathfrak{g}/\!/G$ $\quad \mathrm{rk}\, d\varphi_0 = r-1$ hence $d_i = w_i$ for $1 \le i \le r-1$

	d_1	d_2	d_3			d_{r-2}	d_{r-1}	d_r	w_r	w_{r+1}	w_{r+2}
A_r	4	6	8	…		$2r-2$	$2r$	$2r+2$	2	$r+1$	$r+1$
B_r	4	8	12			$4r-8$	$4r-4$	$4r$	2	$2r$	$2r$
C_r	4	8	12			$4r-8$	$4r-4$	$4r$	4	$2r-2$	$2r$
D_r	4	8	12			$4r-8$	$2r$	$4r-4$	4	$2r-4$	$2r-2$
E_6	4	10	12			16	18	24	6	8	12
E_7	4	12	16		20	24	28	36	8	12	18
E_8	4	16	24	28	36	40	48	60	12	20	30
F_4	4	12					16	24	6	8	12
G_2	4							12	4	4	6

Other types

- Slodowy *Simple singularities and simple algebraic groups* Ch 7,8

- <u>Fact</u> a) $K_1, \dots, K_{r-1}, x_r, x_{r+1}, x_{r+2}$ coordinates on S_e

b) Surface given by $F(x_r, x_{r+1}, x_{r+2}) = 0$

$\deg F = d_r$, $\deg x_i = w_i$

c) F has isolated singularity at 0.

$S_e \to \mathfrak{g}/\!/G \qquad x \mapsto K_i(x)$

 $\longrightarrow$ pt

surface

- From this F, d_r, w_r, w_{r+1}, w_{r+2}

$A_r \quad a^{r+1} + bc \quad (2r+2, 2, r+1, r+1)$

$D_r \quad a^{r-1} + ab^2 + c^2 \quad (4r-4; 4, 2r-4, 2r-2)$

$E_6 \quad a^4 + b^3 + c^2 \quad (24; 6, 8, 12) \qquad E_7 \quad a^3 b + b^3 + c^2 \quad (36, 8, 12, 18)$

$E_8 \quad a^5 + b^3 + c^2 \quad (60; 12, 20, 30)$ $\boxed{\text{versal deform. of simple sing}}$

$B_r = A_{2r-1}/\Gamma$

$C_r = D_{r+1}/\Gamma$

$F_4 = E_6/\Gamma$, $G_2 = D_4/\Gamma$

same sing, another deform

Poisson algebra

- $\mathbb{C}[S_e] = \left(S(\mathfrak{g})/_{m-\chi(m)}\right)^M, \quad m = 1 \oplus \bigoplus_{i \le -2} \mathfrak{g}(i)$

- $$\underbrace{sl_{n-1}}_{m} \oplus \mathbb{C}^{n-1} \oplus (\mathbb{C}^{n-1})^* \oplus \mathbb{C}$$

$$\begin{array}{cc} n-2 & n-2 \\ n-4 & n-4 \\ \hline 2-n & 2-n \end{array}$$

$$\dim m = \frac{(n-1)(n-2)}{2} + n-2 = \frac{(n+1)(n-2)}{2}$$

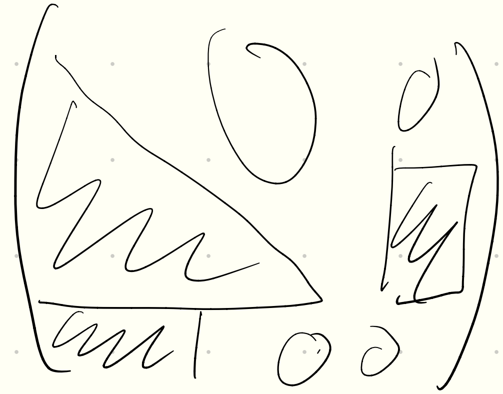
$$\dim \mathfrak{g} - 2\dim m = n+1$$

- sl_3 $\quad m = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \right\}$ $\quad \dim m = 2$

- sl_4 $\quad m = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & * \\ * & 0 & 0 & 0 \end{pmatrix} \right\}$ $\quad \dim m = 5$

Poisson algebra

- $\mathfrak{m} = \langle E_{ij} \mid 1 \le j < i \le n-1 \rangle$
  $\oplus \langle E_{n,j} \mid 1 \le j \le \frac{n-1}{2} \rangle \oplus \langle E_{n-j,n} \mid 1 \le j \le \frac{n-2}{2} \rangle$



- $\mathbb{C}[S_e] = \left(S(\mathfrak{g}) / \mathfrak{m} - \chi(\mathfrak{m}) \right)^M$, $\qquad x \in \mathfrak{g} \mapsto (x, \cdot) \in \mathbb{C}[S_e]$

- $\mathbb{C}[S_e] = \mathbb{C}[a, a_1, \dots, a_{n-2}, b, c]$
  $= \mathbb{C}[a, K_1, \dots, K_{n-2}, b, c]$

- $a = \frac{1}{n}\left(\sum E_{ii} - E_{nn} \right)$,

  $b = E_{n,n-1} + \dots$,

  $c = E_{1,n} + \dots$

$$\begin{pmatrix} a/_{n-1} & & & & & c \\ a_1 & & & & & 0 \\ \vdots & \ddots & & & & 0 \\ a_{n-2} & \cdots & a_1 & a/_{n-1} & & b \\ c & 0 & 0 & 0 & 0 & -a \end{pmatrix}$$

Example $\mathfrak{sl}_4$

- $$S_e = \begin{pmatrix} a/3 & 1 & 0 & 0 \\ 2a_1 & a/3 & 1 & 0 \\ 4a_2 & 2a_1 & a/3 & b \\ c & 0 & 0 & -a \end{pmatrix} \qquad m = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & * \\ * & 0 & 0 & 0 \end{pmatrix} \right\}$$

- $b = E_{4,3} - (E_{3,3} - E_{4,4})E_{4,2}$ $\quad c = E_{1,4} + (E_{4,4} - E_{1,1})E_{2,4}$, $\quad a = \frac{1}{4}(E_{1,1} + E_{2,2} + E_{3,3} - 3E_{4,4})$

- Check of $b \in (S(\mathfrak{g})/I)^m$
  $\{E_{2,1}, b\} = *E_{4,1} \in I$, $\{E_{3,2}, b\} = E_{4,2}(E_{3,2} - 1) \in I$
  $\{E_{4,1}, b\} = *E_{4,1} \in I$, $\{E_{3,4}, b\} = (E_{33} - E_{4,4})(E_{3,2} - 1) - E_{3,4}E_{4,2} \in I$

- Brackets $\{a, b\} = -b$, $\{a, c\} = c$,

$bc = p(a) \rightsquigarrow \{b, c\}c = p'(a)c \rightsquigarrow \{b, c\} = p'(a)$

Another good grading

- **Def** $\mathfrak{g} = \bigoplus \mathfrak{g}(i)$ is good grading if
  - $e \in \mathfrak{g}(2)$ • $ad_e : \mathfrak{g}(i) \to \mathfrak{g}(i+2)$ surj $i \geq -1$
  - $ad_e : \mathfrak{g}(i) \to \mathfrak{g}(i+2)$ inj for $i \leq -1$

- In our case deg $\mapsto$ deg

$\mathfrak{sl}(n-1)\oplus\mathbb{C}$	$\mathbb{C}^n$			$(\mathbb{C}^n)^*$		
	$n-2$		$2n-5$	$n-2$		1
	$\vdots$	$\mapsto$	$\vdots$		$\mapsto$	
	$2-n$		-1	$2-n$		$5-2n$
deg $=$ deg	deg $=$ deg $-n+3$			deg $=$ deg $+n-3$		

$m =$ 

simpler then before

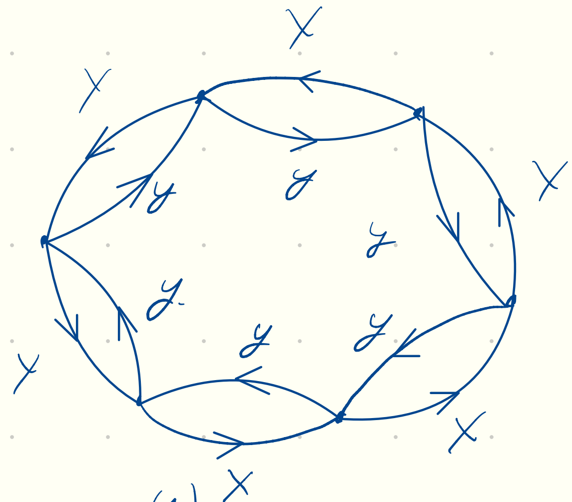
- $a = \frac{1}{n}\left(\sum E_{ii} - E_{nn}\right)$, $b = E_{n,n-1}$,

$c = E_{1,n} + (E_{n,n} - E_{1,1})E_{2,n} + (-E_{1,2} + (E_{n,n} - E_{1,1})(E_{n,n} - E_{2,2}))E_{3,n} + (-E_{1,3} - (E_{n,n} - E_{1,1})E_{2,3} + (E_{n,n} - E_{1,1})(E_{n,n} - E_{2,2})(E_{n,n} - E_{3,3}))E_{n,4} + \dots$

We see $\{a, b\} = -b$, $\{a, c\} = c$

Quiver desription

- $\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]$ $\quad \{y_i,x_j\}=\delta_{ij}$



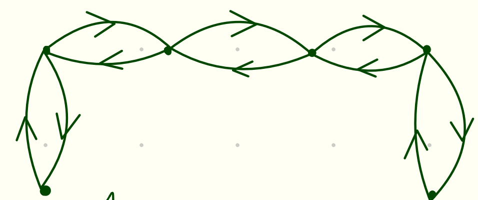
$A_5^{(1)}$ quiver

- $(\mathbb{C}_m)^{n-1}$: $\quad x_i\mapsto t_i x_i \quad \prod t_i=1$

  $y_i\mapsto t_i^{-1}y_i$

$x_iy_i-x_{i+1}y_{i+1}$ — momentum map

- Hamiltonian reduction $\mathbb{C}^{2n}$ by $\mathbb{C}_m^{n-1}$



A_4 quiver with framing

$$\left(\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]/(x_iy_i-\frac{1}{n}\sum x_jy_j-\hbar_i)\right)^{\mathbb{C}_m^{n-1}}$$

generated by $b=\prod y_i$, $c=\prod x_i$, $a=\frac{1}{n}\sum x_iy_i$

- $bc=(a-\hbar_1)(a-\hbar_2)\dots(a-\hbar_n)$ $\quad \{a,b\}=-b \quad \{a,c\}=c$
- $\mathbb{C}[x,y]^{\mathbb{Z}/n\mathbb{Z}}$ $\quad x\mapsto\omega x$, $y\mapsto\omega^{-1}y$ $\quad b=y^n$, $c=x^n$, $a=xy$

Quantization

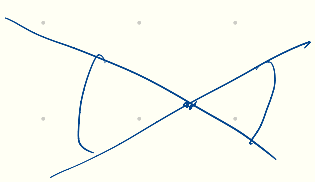
- $A = \mathbb{C}\langle x_1 \dots x_n, \partial_1 \dots, \partial_n \rangle / \partial_i x_j - x_j \partial_i = \delta_{ij}$

- $(\mathbb{G}_m)^{n-1} \curvearrowright A$ $\quad x_i \mapsto t_i x_i$, $\quad \partial_i \mapsto t_i^{-1} \partial_i$ $\quad x_i \partial_i - \frac{1}{n} \sum x_j \partial_j = \hbar_j$

$\hat{b} = \prod \partial_i \qquad \hat{c} = \prod x_i \qquad \hat{a} = \frac{1}{n} \sum x_i \partial_i$

$[\hat{a}, \hat{b}] = -\hat{b} \qquad [\hat{a}, \hat{c}] = \hat{c} \quad \dots \qquad bc = P_1(a)$, $\quad cb = P_2(a)$

- $\mathfrak{g} = \mathfrak{sl}_2 \qquad S_e = \mathfrak{g}^f \qquad W = U(\mathfrak{sl}_2) = \mathbb{C}\langle E, F, J \rangle^{/\!/ H/2}$



$[J, F] = -F$, $\quad [J, E] = E$

$EF = J^2 + J + K$, $\quad FE = J^2 - J + K$

$S_e \to \mathfrak{g}/\!/G$ in general

- $\varphi: S_e \to \mathfrak{g}/\!/G$ has open image

(If $\varphi(K_1, \dots, K_r)|_{S_e} = \text{const} \Rightarrow d\varphi|_{S_e} = 0$, $d\varphi|_{\mathfrak{g}\cdot e} = 0$,
contradicts transversality $\mathfrak{g}\cdot e + T_e S_e = \mathfrak{g}$).

Hence $\mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[S_e]$ is embedding

- Poisson corank of S_e is r, Poisson center of S_e is (finite extension of) $\mathbb{C}[\mathfrak{g}]^G$

$$\begin{tikzcd} \mathfrak{g} & e+\mathfrak{m}^\perp \arrow[l, hook'] \arrow[r] & S_e \arrow[dl] \\ & \mathfrak{g}/\!/G \arrow[from=ul, crossing over] & \end{tikzcd}$$

(over generic point $p \in \mathfrak{g}/\!/G$
$\varphi^{-1}(p)$ is hamiltonia reduction
of $\mathcal{O}_p$ by M, has codim r

$$\begin{tikzcd} \mathcal{O}_p & \mathcal{O}_p \cap (e+\mathfrak{m}^\perp) \arrow[l, hook'] \arrow[r] \arrow[d] & \varphi^{-1}(p) \arrow[dl] \\ & p & \end{tikzcd}$$

($S_e \to \mathfrak{g}//G$ in general)

- $\varphi : S_e \to \mathfrak{g}//G$ is surjective
($\mathrm{Im}\,\varphi$ contains open neighb. of 0, using G_m action
$\mathrm{Im}\,\varphi = \mathfrak{g}//G$)

- φ is flat, all fibers have same dimension

$$\begin{tikzcd} G \times S_e \arrow[r, "\alpha"] \arrow[dr, "\pi"'] & \mathfrak{g} \arrow[r, "\beta"] & \mathfrak{g}//G \\ & S_e \arrow[ur, "\varphi"'] & \end{tikzcd}$$

- $d\alpha$ is surj $\Rightarrow \alpha$ flat
- β flat
($\mathfrak{h} \to \mathfrak{h}/W$ is flat, $\mathbb{C}[\mathfrak{h}]$ is free over $\mathbb{C}[\mathfrak{h}]^W$. Hence $\mathbb{C}[\mathfrak{g}] \simeq \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}[\mathfrak{n}_- \oplus \mathfrak{n}_+]$ is free over $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{h}]^W$)

- $\alpha \circ \beta$ is flat, π is faithfully flat $\Rightarrow \varphi$ is flat

- φ faithfully flat