1. Recollections and highest weight categories

Let G be a connected semisimple Lie group over $\mathbb{C}$. We denote by $\mathfrak{g}$ the Lie algebra of G. We denote by $\mathcal{U}$ the universal enveloping algebra of $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{b} \subset \mathfrak{g}$ is a Borel subalgebra containing $\mathfrak{h}$. Let $T \subset B \subset G$ be the corresponding maximal torus and Borel subgroup. We will denote by W the Weyl group of (T, G) ($W = N_G(T)/T$).

Recall now that we denote by $\mathcal{O}$ the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. We have the decomposition $\mathcal{O} = \bigoplus \mathcal{O}_\lambda$. The main object of our interest will be a *regular integral* block $\mathcal{O}_\lambda$ which is equivalent to $\mathcal{O}_0$ (via translation functors).

Let us recall the main properties of the category $\mathcal{O}_0$. Irreducible objects of $\mathcal{O}_0$ are paramerized by W via $w \mapsto L(w \cdot 0)$, we denote by $P(w \cdot 0)$ a projective cover of $L(w \cdot 0)$. We also have Verma modules $\Delta(w \cdot 0)$ to be called *standard* objects. The following properties of $\mathcal{O}_0$ are well-known for us:

(i) We have the morphism $P(w \cdot 0) \twoheadrightarrow \Delta(w \cdot 0)$ such that the kernel of this morphism admits a filtration whose quotients are of the form $\Delta(w' \cdot 0), w' \cdot 0 > w \cdot 0$.

(ii) $\mathrm{Hom}(\Delta(w_1 \cdot 0), \Delta(w_2 \cdot 0)) \neq 0$ implies $w_1 \cdot 0 \leqslant w_2 \cdot 0$, $\leqslant$ in the dominance order.

(iii) $\mathrm{End}(\Delta(w \cdot 0)) = \mathbb{C}$.

Let us now give a general definition.

Definition 1.1. *Let $\mathscr{C}$ be the category of modules over a finite dimensional algebra A. Let Ξ be the parametrizing set for simples in $\mathscr{C}$. The highest weight structure on $\mathscr{C}$ is the pre-order $\leqslant$ on Ξ and a collection $\Delta(\lambda) \in \mathscr{C}$ of standard objects in $\mathscr{C}$ such that the conditions $(i), (ii), (iii)$ hold.*

Remark 1.2. One can recover algebra A up to a Morita equivalence by the following formula: $A = \mathrm{End}(P)^{\mathrm{opp}}$, $P = \bigoplus_{\lambda \in \Xi} P(\lambda)$.

Here is the list of properties of HW categories which are already known for us when $\mathscr{C} = \mathcal{O}_0$.

Proposition 1.3. *Let $\mathscr{C}$ be a HW category then the following holds.*

a) Fix $\lambda, \mu \in \Xi$ then $L(\lambda)$ occurs in $\Delta(\mu)$ only if $\lambda \leqslant \mu$. Moreover the multiplicity of $L(\lambda)$ in $\Delta(\lambda)$ is one, $\Delta(\lambda) \twoheadrightarrow L(\lambda)$ and $\mathrm{Hom}(\Delta(\lambda), L(\mu)) = \delta_{\lambda,\mu}$.

b) If $\mathrm{Ext}^i(\Delta(\lambda), \Delta(\mu)) \neq 0$ for some $i > 0$ then $\lambda < \mu$.

c) If $\mathrm{Ext}^i(\Delta(\lambda), L(\mu)) \neq 0$ for some $i > 0$ then $\lambda < \mu$.

d) Fix $\lambda \in \Xi$. Consider the Serre subcategory $\mathscr{C}_{\leqslant \lambda}$ (resp. $\mathscr{C}_{\not> \lambda}$) spanned by $L(\mu)$ with $\mu \leqslant \lambda$ (resp. $\mu \not> \lambda$). Then $\Delta(\lambda)$ is the projective cover of $L(\lambda)$ in $\mathscr{C}_{\leqslant \lambda}$ (resp. $\mathscr{C}_{\not> \lambda}$).

Example 1.4. Let us give an example when $\mathrm{Ext}^1(\Delta(\lambda), \Delta(\mu)) \neq 0$. Let $\mathscr{C} = \mathcal{O}_0(\mathfrak{sl}_2)$, $\lambda = -2, \mu = 0$. Then the object $P(-2)$ includes in the following nonsplit short exact sequence:

$$0 \to \Delta(0) \to P(-2) \to \Delta(-2) \to 0.$$

It follows that $\mathrm{Ext}^1(\Delta(-2), \Delta(0)) \neq 0$. We see that $\lambda = -2 < 0 = \mu$ so there is no contradiction with Proposition 1.3.

Remark 1.5. Let us sketch proofs of b), c), d). To prove b) one should use the induction by λ starting from maximal λ (in this case $\Delta(\lambda)$ is projective because the morphism $P(\lambda) \twoheadrightarrow \Delta(\lambda)$ must be injective and the statement is obvious) and to consider the short exact sequence

$$0 \to K \to P(\lambda) \to \Delta(\lambda) \to 0$$

which gives a long exact sequence

$$\to \operatorname{Ext}^{i-1}(K, \Delta(\mu)) \to \operatorname{Ext}^i(\Delta(\lambda), \Delta(\mu)) \to \operatorname{Ext}^i(P(\lambda), \Delta(\mu)) \to$$

now by induction hypothesis (using the fact that K is filtered with assosiated graded $\Delta(\lambda'), \lambda' > \lambda$) we see that $\operatorname{Ext}^{i-1}(K, \Delta(\mu)) = 0 = \operatorname{Ext}^i(P(\lambda), \Delta(\mu))$.

To prove c) one should use a) and b), induction on μ starting from a minimal (in this case $L(\mu) = \Delta(\mu)$ and we are done by b)) and the short exact sequence

$$0 \to Q \to \Delta(\mu) \to L(\mu) \to 0.$$

Part d) follows from c).

Corollary 1.6. *Let M be a standardly filtered object, $[M] = \sum[\Delta(\lambda_i)]$ for some $\lambda_1, \ldots \lambda_k \in \Xi$. Assume also that if $\lambda_i < \lambda_j$ then $i > j$. Then there exists a filtration $0 = F^0M \subset F^1M \subset \ldots \subset F^{k-1}M \subset F^kM = M$ such that $F^i/F^{i-1} \simeq \Delta(\lambda_i)$.*

Proof. We prove by induction on the length of M. Let $G^\bullet M$ be some standard filtration. Let i be such that $G^iM/G^{i-1}M \simeq \Delta(\lambda_1)$. We obtain a short exact sequence

$$0 \to G^{i-1}M \to G^iM \to \Delta(\lambda_1) \to 0. \tag{1.1}$$

It follows from Proposition 1.3 and our assumptions that $\operatorname{Ext}^1(\Delta(\lambda_1), G^{i-1}M) = 0$, hence, the sequence 1.1 splits and we have $G^iM \simeq \Delta(\lambda_1) \oplus G^{i-1}M$. So we have an embedding $\Delta(\lambda_1) \subset G^iM$ such that $G^iM/\Delta(\lambda_1)$ is standardly filtered, hence, $M/\Delta(\lambda_1)$ is standardly filtered and we are done by the induction hypothesis. □

Let us now recall that in $\mathcal{O}_0$ there are also *costandard* objects $\nabla(\lambda)$ (contragradient Verma modules). In general situation they can be constructed in the following way.

Definition 1.7. *By the definition, $\nabla(\lambda)$ is the injective envelope of $L(\lambda)$ in $\mathscr{C}_{\leqslant\lambda}$ or in $\mathscr{C}_{\not>\lambda}$.*

Remark 1.8. For $\mathscr{C} = \mathcal{O}_0$ we have the contravariant functor $\bullet^\vee \colon \mathcal{O}_0 \to \mathcal{O}_0$ given by $M \mapsto M^\vee$ (graded dual) with the action $\mathfrak{g} \curvearrowright M^\vee$ via $x \cdot f(v) = f(-\tau(x)v)$, where $\tau \colon \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$ is the Cartan involution. This functor sends $\Delta(\lambda)$ to $\nabla(\lambda)$ and interchanges projectives and injectives (this follows from the fact that $\bullet^\vee$ induces an equivalence $\mathcal{O}_0 \xrightarrow{\sim} \mathcal{O}_0^{\mathrm{opp}}$).

Remark 1.9. The category $\mathscr{C}^{\mathrm{opp}}$ is HW with respect to Ξ with standard objects $\nabla(\lambda)$.

We already know that the following lemma holds for $\mathscr{C} = \mathcal{O}_0$.

Lemma 1.10. *Pick $\lambda, \mu \in \Xi$ then* $\dim \operatorname{Hom}(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda,\mu}$.

Remark 1.11. Let us sketch the proof of Lemma 1.10. Suppose that $\mu \not> \lambda$ and consider the category $\mathscr{C}_{\not>\lambda}$. Recall now that $\Delta(\lambda)$ is the projective covering of $L(\lambda)$ in this category and $\nabla(\mu) \in \mathscr{C}_{\not>\lambda}$ has a filtration with quotients $L(\mu')$, $\mu' < \mu$ and $L(\mu) \hookrightarrow \nabla(\mu)$. It follows that none of μ' equals λ (otherwise $\lambda = \mu' < \mu$) so if $\mu \neq \lambda$ then $\mathrm{Hom}(\Delta(\lambda), \nabla(\mu)) = 0$. For $\mu = \lambda$ we see that the composition $\Delta(\lambda) \twoheadrightarrow L(\lambda) \hookrightarrow \nabla(\lambda)$ gives us a desired morphism.

Suppose now that $\lambda \not> \mu$ then we consider the category $\mathscr{C}_{\not>\mu}$ and realise $\nabla(\mu)$ as an injective envelope in this category.

Remark 1.12. Let us point out that the BGG reciprocity holds for $\mathscr{C}$. Pick $\lambda, \mu \in \Xi$ then the multiplicity of $\Delta(\mu)$ in $P(\lambda)$ coincides with the multiplicity of $L(\lambda)$ in $\nabla(\mu)$:

$$[\Delta(\mu) : P(\lambda)] = \dim \mathrm{Hom}(P(\lambda), \nabla(\mu)) = (L(\lambda) : \nabla(\mu))$$

where the first equality holds by Lemma 2.1. The dual statement says that $[\nabla(\mu), I(\lambda)] = (L(\lambda) : \Delta(\mu))$, where $I(\lambda)$ is the injective envelope of $L(\lambda)$.

2. Standardly and Costandardly filtered objects

Lemma 2.1. $\dim \mathrm{Hom}(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda,\mu}$ *and* $\mathrm{Ext}^i(\Delta(\lambda), \nabla(\mu)) = 0$ *for* $i > 0$.

Proof. We prove by induction on λ starting from maximal λ for which it's obvious because $\Delta(\lambda) = P(\lambda)$ in this case. Consider now the short exact sequence

$$0 \to K \to P(\lambda) \to \Delta(\lambda) \to 0$$

and apply $\mathrm{Hom}(-, \nabla(\mu))$. We obtain the long exact sequence

$$\to \mathrm{Ext}^{i-1}(K, \nabla(\mu)) \to \mathrm{Ext}^i(\Delta(\lambda), \nabla(\mu)) \to \mathrm{Ext}^i(P(\lambda), \nabla(\mu)) \to$$

and by the induction hypothesis (using the fact that K is filtered with assosiated graded $\Delta(\lambda'), \lambda' > \lambda$) we have $\mathrm{Ext}^{i-1}(K, \nabla(\mu)) = 0 = \mathrm{Ext}^i(P(\lambda), \nabla(\mu))$ for $i > 1$.

It remains to show that $\mathrm{Ext}^1(\Delta(\lambda), \nabla(\mu)) = 0$. Suppose that $\mu \not> \lambda$. Consider the category $\mathscr{C}_{\not>\lambda}$. Object $\Delta(\lambda) \in \mathscr{C}_{\not>\lambda}$ is projective so $\mathrm{Ext}^1_{\mathscr{C}_{\not>\lambda}}(\Delta(\lambda), \nabla(\mu)) = 0$. But the category $\mathscr{C}_{\not>\lambda}$ is closed under extensions so $\mathrm{Ext}^1_{\mathscr{C}}(\Delta(\lambda), \nabla(\mu)) = 0$.

The analogous argument works if $\lambda \not> \mu$. ☐

Remark 2.2. Note that the fact that $\mathrm{Ext}^1(\Delta(\lambda), \nabla(\mu)) = 0$ in $\mathcal{O}_0$ was already proven by Nikita. He also considered two cases – $\mu \not> \lambda, \lambda \not> \mu$ and took dual spaces in the second case.

Proposition 2.3. *Object* $M \in \mathscr{C}$ *is standardly (resp. costandardly) filtered iff* $\mathrm{Ext}^i(M, \nabla(\lambda)) = 0$ *(resp.* $\mathrm{Ext}^i(\Delta(\lambda), M) = 0$*) for any* $i > 0$.

Proof. In one direction it follows from Lemma 2.1. We prove by induction by the length of M. Let λ be a minimal element of Ξ such that we have a surjection $\varphi\colon M \twoheadrightarrow L(\lambda)$. Let us prove that the map φ gives rise to a map $\tilde{\varphi}\colon M \to \Delta(\lambda)$. To do so consider the exact sequence

$$0 \to K \to \Delta(\lambda) \to L(\lambda) \to 0$$

and the corresponding long exact sequence:

$$\to \mathrm{Hom}(M, \Delta(\lambda)) \to \mathrm{Hom}(M, L(\lambda)) \to \mathrm{Ext}^1(M, K) \to$$

so to construct $\tilde{\varphi}$ it is enough to show that $\operatorname{Ext}^1(M,K)=0$. Recall that K can be filtered with quotients $L(\mu),\mu<\lambda$ so it is enough to show that $\operatorname{Ext}^1(M,L(\mu))=0$ for any $\mu<\lambda$. To show this let us consider the short exact sequence

$$0\to L(\mu)\to\nabla(\mu)\to N\to 0.$$

We have the corresponding long exact sequence:

$$\to\operatorname{Hom}(M,N)\to\operatorname{Ext}^1(M,L(\mu))\to\operatorname{Ext}^1(M,\nabla(\mu))\to$$

and $\operatorname{Hom}(M,N)=\operatorname{Ext}^1(M,\nabla(\mu))=0$ (here we use the fact that N can be filtered with quotients $L(\mu')$, $\mu'<\mu<\lambda$ for which $\operatorname{Hom}(M,L(\mu'))=0$). So it follows that $\operatorname{Ext}^1(M,L(\mu))=0$.

Let us now show that the map $\tilde{\varphi}\colon M\to\Delta(\lambda)$ is surjective. Let V be the image of $\tilde{\varphi}$. Suppose $V\neq\Delta(\lambda)$. Note that we have a surjection $V\twoheadrightarrow L(\lambda)$. It follows that the multiplicity of $L(\lambda)$ in $\Delta(\lambda)/V$ equals to zero. It follows that there exists $\lambda'\neq\lambda$ and a surjective morphism $\Delta(\lambda)/V\twoheadrightarrow L(\lambda')$, hence, we have a surjection $\Delta(\lambda)\twoheadrightarrow L(\lambda')$ and so a surjection $P(\lambda)\twoheadrightarrow L(\lambda')$ but this is impossible.

We obtain a short exact sequence

$$0\to M_0\to M\xrightarrow{\tilde{\varphi}}\Delta(\lambda)\to 0.$$

Pick $\mu\in\Xi$, we have the long exact sequence

$$\to\operatorname{Ext}^i(M,\nabla(\mu))\to\operatorname{Ext}^i(M_0,\nabla(\mu))\to\operatorname{Ext}^{i+1}(\Delta(\lambda),\nabla(\mu))\to$$

we see that $\operatorname{Ext}^i(M_0,\nabla(\mu))=0$ for any $i>0$ now the desired follows by the induction hypothesis. Note that the same proof works to show that if $\operatorname{Ext}^1(M,\nabla(\lambda))=0$ for any $\lambda\in\Xi$ then $M\in\mathscr{C}^{\Delta}$. □

Remark 2.4. Note that the surjectivity of $\tilde{\varphi}$ is obvious for $\mathscr{C}=\mathcal{O}_0$.

3. Tilting modules

Definition 3.1. *An object in $\mathscr{C}$ is called* tilting *if it is both standardly and costandardly filtered.*

Let us point out that by Proposition 2.3 for any tilting object T we have $\operatorname{Ext}^i(T,T)=0$ for $i>0$. Note also that if T is tilting and $T\simeq T_1\oplus T_2$ then both T_1 and T_2 are also tilting. It follows that each tilting is a direct sum of indecomposable tilting objects. We describe indecomposable tiltings in the following proposition.

Remark 3.2. Let us also point out that by Proposition 2.3 one can think about tilting objects as about *injective* objects in the category $\mathscr{C}^{\Delta}$ or as about *projective* objects in the category $\mathscr{C}^{\nabla}$. From this point of view the proposition bellow looks more natural.

Proposition 3.3. *For each $\lambda\in\Xi$ there exists an indecomposable tilting object $T(\lambda)$ uniquely determined by the following property: $T(\lambda)\in\mathscr{C}_{\leqslant\lambda}$, $[\Delta(\lambda):T(\lambda)]=1=[\nabla(\lambda):T(\lambda)]$ and we have $\Delta(\lambda)\hookrightarrow T(\lambda)\twoheadrightarrow\nabla(\lambda)$.*

Proof. Fix $\lambda\in\Xi$ and order linearly elements of $\{\mu\in\Xi\,|\,\mu\leqslant\lambda\}$ refining the original poset structure on Ξ. Say $\lambda=\lambda_1>\lambda_2>\ldots>\lambda_k$. Let us construct the object $T^i(\lambda),i=1,\ldots,k$ inductively as follows. Set $T^1(\lambda)=\Delta(\lambda)$. Further, if $T^{i-1}(\lambda)$ is already defined let $T^i(\lambda)$ be the extension of $\mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1}(\lambda)) \otimes \Delta(\lambda_i)$ by $T^{i-1}(\lambda)$ corresponding to the unit endomorphism of $\mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1}(\lambda))$ i.e. we have the following short exact sequence

$$0 \to T^{i-1}(\lambda) \to T^i(\lambda) \to \mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1}) \otimes \Delta(\lambda_i) \to 0. \tag{3.1}$$

We claim that $T(\lambda) = T^k(\lambda)$ is tilting and satisfies all the desired properties. Note that by the definitions T^i are standardly filtered (we will sometimes drop λ from the notation).

Let us first of all show that $\mathrm{Ext}^1(\Delta(\lambda_j), T^i(\lambda)) = 0$ for any $j \leqslant i$. We prove it by induction on i. For $i = 1$ it follows from Proposition 1.3. Let us now prove the induction step. Let us apply $\mathrm{Hom}(\Delta(\lambda_j), -)$ to the sequence 3.1:

$$\to \mathrm{Ext}^k(\Delta(\lambda_j), T^{i-1}) \to \mathrm{Ext}^k(\Delta(\lambda_j), T^i) \to \mathrm{Ext}^k(\Delta(\lambda_i), T^{i-1}) \otimes \mathrm{Ext}^k(\Delta(\lambda_j), \Delta(\lambda_i)) \to$$

we see that for $j < i$ by the induction hypothesis and Proposition 1.3 we have $\mathrm{Ext}^k(\Delta(\lambda), T^i(\lambda)) = 0$.

Assume now that $j = i$. We still have $\mathrm{Ext}^1(\Delta(\lambda_i), \Delta(\lambda_i)) = 0$.

We apply a functor $\mathrm{Hom}(\Delta(\lambda_i), -)$ and obtain the following exact sequence

$$\to \mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1}(\lambda)) \otimes \mathrm{End}(\Delta(\lambda_i)) \xrightarrow{\psi} \mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1}(\lambda)) \to$$
$$\to \mathrm{Ext}^1(\Delta(\lambda_i), T^i(\lambda)) \to 0.$$

Note that $\mathrm{End}(\Delta(\lambda_i)) = \mathbb{C}$. It is easy to see from the definition of $T^i(\lambda)$ that $\psi = \mathrm{Id}$ so ψ is an isomorphism, hence, $\mathrm{Ext}^1(\Delta(\lambda_i), T^i(\lambda)) = 0$.

Here is the other approach which is more clear for me. Let us apply now a functor $\mathrm{Hom}(\mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1}(\lambda)) \otimes \Delta(\lambda_i), -)$. and obtain the following sequence (we drop λ from the notation)

$$\to \mathrm{End}(\mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1}) \otimes \Delta(\lambda_i)) \xrightarrow{\psi} \mathrm{Ext}^1(\mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1}) \otimes \Delta(\lambda_i), T^{i-1}) \xrightarrow{\phi}$$
$$\xrightarrow{\phi} \mathrm{Ext}^1(\mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1}) \otimes \Delta(\lambda_i), T^i) \to 0.$$

which can be rewrited in the following way:

$$\to \mathrm{End}(\mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1}) \otimes \Delta(\lambda_i)) \xrightarrow{\psi} \mathrm{End}(\mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1}) \otimes \Delta(\lambda_i)) \xrightarrow{\phi}$$
$$\xrightarrow{\phi} \mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1})^* \otimes \mathrm{Ext}^1(\Delta(\lambda_i), T^i) \to 0.$$

It follows from the construction of $T^i(\lambda)$ that the morphism ψ sends Id to Id and for any x, y, $\psi(x \circ y) = \psi(x) \circ y$ so ψ is an isomorphism (see Remark 3.5). It follows that $\mathrm{Ext}^1(\Delta(\lambda_i), T^i(\lambda)) = 0$.

We have shown that $\mathrm{Ext}^1(\Delta(\mu), T(\lambda)) = 0$ for any $\mu \leqslant \lambda$. Note now that $T(\lambda) \in \mathscr{C}_{\leqslant\lambda}$ so it follows from Proposition 1.3 that $\mathrm{Ext}^1(\Delta(\lambda'), T(\lambda)) = 0$ for any λ' not less then λ. It follows that $T(\lambda)$ is tilting. It also follows from the construction that $[\Delta(\lambda) : T(\lambda)] = 1$, $\Delta(\lambda) \hookrightarrow T(\lambda), T(\lambda) \in \mathscr{C}_{\leqslant\lambda}$.

Let us show now that the objects $T(\lambda)$ are indecomposable. To do so let us prove that $T^i(\lambda)$ is indecomposable for any i. We prove it by induction on i. Case $i = 1$ is clear. We have an exact sequence

$$0 \to T^{i-1}(\lambda) \to T^i(\lambda) \to \Delta(\lambda_i)^{l_i} \to 0,$$

where $l_i = \dim(\mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1}(\lambda)))$.

Assume now that $T^i(\lambda) \simeq A \oplus B$. Note that A, B are standardly filtered with associated graded $\Delta(\mu)$ such that $\mu \geqslant \lambda_i$, hence, $\mathrm{Ext}^1(\Delta(\mu), \Delta(\lambda_i)) = 0$ for any such μ. It follows (see Corollary 1.6) that we have surjections $A \twoheadrightarrow \Delta(\lambda_i)^{r_1}, B \twoheadrightarrow \Delta(\lambda_i)^{r_2}$ with $r_1 + r_2 = l_i$. It follows from the induction hypothesis that $A \simeq \Delta(\lambda_i)^{r_1}$ (otherwise $T^{i-1}(\lambda)$ is not indecomposable). We obtain an exact sequence

$$0 \to T^{i-1} \to B \to \Delta(\lambda_i)^{r_2} \to 0$$

and apply $\mathrm{Hom}(\Delta(\lambda_i), -)$. We get

$$\mathbb{C}^{r_2} \to \mathbb{C}^{l_1} \to \mathrm{Ext}^1(\Delta(\lambda_i), B) = 0$$

where the last equality holds because $0 = \mathrm{Ext}^1(\Delta(\lambda_i), T^i(\lambda)) = \mathrm{Ext}^1(\Delta(\lambda_i), \Delta(\lambda_i)^{r_1}) \oplus \mathrm{Ext}^1(\Delta(\lambda_i), B) = \mathrm{Ext}^1(\Delta(\lambda_i), B)$. It follows that $r_2 = l_1$, hence, $A = 0$.

Let us now show that if T is an indecomposable tilting object then $T \simeq T(\lambda)$ for some $\lambda \in \Xi$. To do so let $\lambda \in \Xi$ be a maximal such that the multiplicity of $\Delta(\lambda)$ in T is nonzero. It follows from Proposition 1.3 (see also Corollary 1.6) that we have an embedding $\iota\colon \Delta(\lambda) \hookrightarrow T$. Consider now the exact sequence

$$0 \to \Delta(\lambda) \to T(\lambda) \to N \to 0$$

we obtain an exact sequence

$$0 \to \mathrm{Hom}(N, T) \to \mathrm{Hom}(T(\lambda), T) \to \mathrm{Hom}(\Delta(\lambda), T) \to \mathrm{Ext}^1(N, T) \to$$

but $\mathrm{Ext}^1(N, T) = 0$ because $N \in \mathscr{C}^{\Delta}$ and $T \in \mathscr{C}^{\nabla}$ so the embedding ι extends to a map $\tilde{\iota}\colon T(\lambda) \to T$. By the same reasons an embedding $j\colon \Delta(\lambda) \hookrightarrow T(\lambda)$ extends to a map $\tilde{j}\colon T \to T(\lambda)$. We see that $\tilde{\iota} \circ \tilde{j}$ and $\tilde{j} \circ \tilde{\iota}$ are Id on $\Delta(\lambda)$ so they are not nilpotent hence they are isomorphisms (we use the structure theorem for endomorphism rings of indecomposable objects, Misha explained it to us).

Let us finally show that for $\lambda \in \Xi$ the multiplicity of $\nabla(\lambda)$ in $T(\lambda)$ equals to one and $T(\lambda) \twoheadrightarrow \nabla(\lambda)$. Consider the category $\mathscr{C}_{\leqslant\lambda}$. Let $T^{\vee}(\lambda) \in \mathscr{C}_{\leqslant\lambda}$ be the indecomposable tilting such that the multiplicity of $\nabla(\lambda)$ in $T^{\vee}(\lambda)$ equals to one and $T^{\vee}(\lambda) \twoheadrightarrow \nabla(\lambda)$ (use the HW category $\mathscr{C}^{\mathrm{opp}}$ to construct it). Our goal is to show that $T^{\vee}(\lambda) = T(\lambda)$. Otherwise $T^{\vee}(\lambda) = T(\mu)$ for some $\mu < \lambda$. Note that $T(\mu) \in \mathscr{C}_{\leqslant\mu}$, hence, $\nabla(\lambda)$ appears in $T(\mu)$ with multiplicity zero. Contradiction. □

Remark 3.4. Let us point out that the category $\mathscr{C}^{\mathrm{opp}}$ is HW with respect to the poset Ξ and the object $T(\lambda)$ considered as an object of $\mathscr{C}^{\mathrm{opp}}$ is an indecomposable tilting with label λ (it follows from Proposition 3.3). For $\mathscr{C} = \mathcal{O}_0$ we have an equivalence $\bullet^{\vee}\colon \mathcal{O}_0 \to \mathcal{O}_0^{\mathrm{opp}}$ so $T(\lambda) = T^{\vee}(\lambda)$.

Remark 3.5. Recall that if we have a short exact sequence

$$0 \to A \to B \to C \to 0$$

then the element $b \in \mathrm{Ext}^1(C,A)$ which corresponds to B can be constructed as follows: apply $\mathrm{Hom}(C,-)$ and obtain the long exact sequence

$$\to \mathrm{Hom}(C,C) \xrightarrow{\psi} \mathrm{Ext}^1(C,A) \to$$

then b is the image of $\mathrm{Id} \in \mathrm{Hom}(C,C)$. Let us also point out that $\mathrm{End}(C) = \mathrm{Ext}^0(C,C)$ acts on $\mathrm{Hom}(C,C), \mathrm{Ext}^1(C,A)$ on the right and the map ψ is a homomorphism of $\mathrm{End}(C)$-modules.

Example 3.6. Let us consider the example $\mathscr{C} = \mathcal{O}_0(\mathfrak{sl}_2)$. We see that $T(-2) = \Delta(-2) = L(-2) = \nabla(-2)$. It is also easy to see that $T(0) = P(-2)$.

Example 3.7. More general $T(0) = P(w_0 \cdot 0) = P_{\min}$, $T(w_0 \cdot 0) = T^1(w_0 \cdot 0) = \Delta(w_0 \cdot 0) = L_{\min}$. Actually if we order linearly elements of $W \cdot 0$: $0 = \lambda_1 > \lambda_2 > \ldots > \lambda_k = w_0 \cdot 0$ then $T^1(0) = \Delta(0), T^k(0) = P_{\min}$. To see this we note that $P_{\min}$ is both injective and projective, hence, tilting, we also now that $P_{\min}$ is indecomposable and that $[P_{\min}] = \sum_{w \in W}[\Delta(w \cdot 0)]$. It follows that we have an embedding $\Delta(0) \hookrightarrow P_{\min}$ (because $\mathrm{Ext}^1(\Delta(0), \Delta(w \cdot 0)) = 0$ for any $w \in W$). Now it follows that $P_{\min} = T(0)$.

Remark 3.8. Let us point out that the objects $P_{\min}, L_{\min}$ are selfdual.

4. Ringel duality

Set $T := \bigoplus_{\lambda \in \Xi} T(\lambda)$. Set $A^\vee := \mathrm{End}(T)^{\mathrm{opp}}$. Recall that $\mathrm{RHom}(T,T)^{\mathrm{opp}} = A^\vee$. Recall also $A = \mathrm{End}(P)^{\mathrm{opp}}$. We set $\mathscr{C}^\vee := A^\vee$-mod. This category is called *Ringel dual* to $\mathscr{C}$.

Let us now recall a definition of the derived category $D(\mathscr{C})$. Objects of this category are classes of right bounded (bounded) complexes of objects from $\mathscr{C}$. Two objects are called equivalent if there exists a morphism between them which induces an isomorphism on cohomologies.

Remark 4.1. Let us point out that in a HW category any object has a finite projective resolution.

Proof. It is enought to show that $\mathscr{C}$ has finite homological dimension i.e. there exists d such that for any $\lambda \in \Xi$ and $i > d$ we have $\mathrm{Ext}^i(L(\lambda), M) = 0$ for any $M \in \mathscr{C}$ (actually ane can take $d = 2|\Xi|$). Let us firs of all prove by induction on λ starting from a maximal that for $i > |\{\lambda' \mid \lambda' > \lambda\}|$ we have $\mathrm{Ext}^i(\Delta(\lambda), M) = 0$ for any $M \in \mathscr{C}$. It immediately follows from the fact that for a maximal λ object $\Delta(\lambda)$ is projective and from the axact sequence

$$0 \to K \to P(\lambda) \to \Delta(\lambda) \to 0$$

which induces the long exact sequence

$$\to 0 \to \mathrm{Ext}^{i-1}(K, M) \to \mathrm{Ext}^i(\Delta(\lambda), M) \to 0 \to .$$

Now we can prove by induction on λ starting from the minimal one that for any $i > d + |\{\lambda' \mid \lambda' < \lambda\}|$ we have $\mathrm{Ext}^i(L(\lambda), M) = 0$ for any M. It can be easily derived from the exact sequence

$$0 \to Q \to \Delta(\lambda) \to L(\lambda) \to 0.$$

□

Theorem 4.2. *The functor* $\mathcal{T} := \mathrm{RHom}_A(T, -)\colon D(\mathscr{C}) \xrightarrow{\sim} D(\mathscr{C}^\vee)$ *is the derived equivalence.*

Proof. Let us construct the inverse functor. Note that T is an $A = \mathrm{End}(P)^{\mathrm{opp}}$- $\mathrm{End}(T)^{\mathrm{opp}} = A^\vee$ bimodule and the left adjoint to $\mathrm{Hom}_A(T, -)$ is a functor $T \otimes_{A^\vee} -$. We claim that the derived functor $T \overset{L}{\otimes}_{A^\vee} -\colon D(\mathscr{C}^\vee) \to D(\mathscr{C})$ is inverse to $\mathcal{T}$. Let us first of all show that the adjunction morphism $T \overset{L}{\otimes} (\mathrm{RHom}(T, -)) \xrightarrow{\mathrm{TH}} \mathrm{Id}$ is an isomorphism. We see that $T \mapsto \mathrm{RHom}(T, T) = A^\vee \mapsto T \otimes_{A^\vee} A^\vee = T$. Note now that if for $M \in D^b(\mathscr{C})$ the morphism $\mathrm{TH}(M)$ is an isomorphism and $M \simeq M_1 \oplus M_2$ then $\mathrm{TH}(M_1), \mathrm{TH}(M_2)$ are isomorphisms. It follows from the fact that $\mathrm{TH}(M) = \mathrm{TH}(M_1) \oplus \mathrm{TH}(M_2)$. So we see that for any $\lambda \in \Xi$ morphsim $\mathrm{TH}(T(\lambda))$ is an isomorphism. It is enough to show that for any $\lambda \in \Xi$ the morphism $\mathrm{TH}(P(\lambda))$ is an isomorphism (see Remark 4.3). Note that each $P(\lambda)$ is Δ-filtered so it is enough to show that for any $\lambda \in \Xi$ morphism $\mathrm{TH}(\Delta(\lambda))$ is an isomorphism because of the following general fact: if $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathscr{C}$ and TH being applied to two elements of the set $\{A, B, C\}$ is an isomorphism then it is also an isomorphism being applied to the third one (see Remark 4.3). It remains to prove that $\mathrm{TH}(\Delta(\lambda))$ is an isomorphism. We prove it by induction on λ starting from a minimal λ (in this case $\Delta(\lambda) = T(\lambda)$ and there is nothing to prove). Consider the exact sequence

$$0 \to \Delta(\lambda) \to T(\lambda) \to N \to 0.$$

It follows from Proposition 3.3 that $N \in \mathscr{C}_{<\lambda}$ and is standardly filtered so $\mathrm{TH}(T(\lambda)), \mathrm{TH}(N)$ are isomorphisms, hence, $\mathrm{TH}(\Delta(\lambda))$ is an isomorphism.

Now we need to show that the natural transformation from Id to $\mathrm{RHom}_A(T, T \overset{L}{\otimes}_{A^\vee} -)$ is an isomorphism. Category $D(\mathscr{C}^\vee) = D(\mathrm{End}(T)^{\mathrm{opp}})$ is generated by $\mathrm{End}(T)^{\mathrm{opp}} = A^\vee$ thus to finish the proof its enough to note that $\mathrm{RHom}_A(T, T \overset{L}{\otimes}_{A^\vee} \mathrm{End}(T)^{\mathrm{opp}}) = A^\vee$. ∎

Remark 4.3. Sequence $0 \to A \to B \to C \to 0$ gives a distinguished triangle $A \to B \to C \to A[1]$ in the derived category $D(\mathscr{C})$. Functor TH being a composition of derived functors maps distinguished triangles to distinguished triangles. We have a commutative diagram of distinguished triangles:

$$\begin{tikzcd}
\mathrm{TH}(A) \arrow[r] \arrow[d] & \mathrm{TH}(B) \arrow[r] \arrow[d] & \mathrm{TH}(C) \arrow[r] \arrow[d] & \mathrm{TH}(A[1]) \arrow[d] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1]
\end{tikzcd}$$

which induces the commutative diagram between long exact sequences

$$\begin{tikzcd}
{} \arrow[r] & H^i(\mathrm{TH}(A)) \arrow[r] \arrow[d] & H^i(\mathrm{TH}(B)) \arrow[r] \arrow[d] & H^i(\mathrm{TH}(C)) \arrow[r] \arrow[d] & H^{i+1}(\mathrm{TH}(A)) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H^i(A) \arrow[r] & H^i(B) \arrow[r] & H^i(C) \arrow[r] & H^{i+1}(A) \arrow[r] & {}
\end{tikzcd}$$

we see that if two of each three arrows are isomorphisms then the rest are also isomorphisms.

Let us also point out that we use the following fact – any object $C^\bullet$ of $D(\mathscr{C})$ is isomorphic to a complex of projective modules $P^\bullet$. So if we know that for any P^i the morphism $\mathrm{TH}(P^i)$ is an isomorphism then $\mathrm{TH}(P^\bullet)$ is also an isomorphism.

Proposition 4.4. *The category $\mathscr{C}^\vee$ is HW with respect to the poset Ξ^{opp} with standard objects $\mathrm{Hom}(T, \nabla(\lambda))$ and costandard objects $\mathrm{Hom}(T, \Delta(\lambda))$, projectives $\mathrm{Hom}(T, T(\lambda))$ and tiltings $\mathrm{Hom}(T, P(\lambda))$. The functor $\mathcal{T}$ sends $\Delta_{\mathscr{C}}(\lambda)$ to $\nabla_{\mathscr{C}^\vee}(\lambda)$, $T_{\mathscr{C}}(\lambda)$ to $P_{\mathscr{C}^\vee}(\lambda)$, $I_{\mathscr{C}}(\lambda)$ to $T_{\mathscr{C}^\vee}(\lambda)$. The functor $\mathcal{T}$ induces equivalences $\mathscr{C}^\Delta \xrightarrow{\sim} \mathscr{C}^{\vee,\nabla}, \mathscr{C}^\nabla \xrightarrow{\sim} \mathscr{C}^{\vee,\Delta}$.*

Proof. Let us show that $\mathrm{Hom}(T, T(\lambda))$ are indecomposable projective. They are projective as direct summands in $\mathrm{Hom}(T, T) \simeq \oplus_{\lambda\in\Xi} \mathrm{Hom}(T(\lambda), T)$, indecomposibility follows from Theorem 4.2. Now it follows that $\mathrm{Hom}(T, \nabla(\lambda))$ satisfy the property (i): we just should apply $\mathrm{Hom}(T, -)$ to the epimorphism $T(\lambda) \twoheadrightarrow \nabla(\lambda)$. Let us note now that by Theorem 4.2 and using the fact that $\mathrm{RHom}(T, \nabla(\lambda)) = \mathrm{Hom}(T, \nabla(\lambda))$ we see that $\mathrm{Hom}(\mathrm{Hom}(T, \nabla(\lambda)), \mathrm{Hom}(T, \nabla(\mu))) \simeq \mathrm{Hom}(\nabla(\lambda), \nabla(\mu))$ which is nonzero only if $\lambda \geqslant \mu$ and is one-dimensional if $\lambda = \mu$. The properties $(ii), (iii)$ follows.

Let us show that the functor $\mathcal{T}$ induces equivalences $\mathscr{C}^\Delta \xrightarrow{\sim} \mathscr{C}^{\vee,\nabla}, \mathscr{C}^\nabla \xrightarrow{\sim} \mathscr{C}^{\vee,\Delta}$. Note that $\mathrm{RHom}(T, -) = \mathrm{Hom}(T, -)$ on these categories, hence, $T \otimes - = T \overset{L}{\otimes} -$ on these categories (because they are left adjoint to $\mathrm{RHom}(T, -)$, $\mathrm{Hom}(T, -)$ respectively). It follows that $\mathrm{Hom}(T, -)$ and $T \otimes -$ are our mutually inverse equivalences. □

5. BGG category $\mathcal{O}$ and self dual HW categories

Definition 5.1. *We say that a* HW *category is* Ringel self dual *if there exists an equivalence $\mathscr{C} \simeq \mathscr{C}^\vee$.*

Lemma 5.2. *An object $M \in \mathcal{O}_0$ is tilting iff it is standardly (resp. costandardly) filtered and selfdual i.e. $M \simeq M^\vee$.*

Proof. Follows from Proposition 3.3 and the equivalence $\bullet^\vee : \mathcal{O}_0 \xrightarrow{\sim} \mathcal{O}_0^{\mathrm{opp}}$. □

Remark 5.3. Let us give another construction of $T(\lambda)$ which works only for $\mathscr{C} = \mathcal{O}_0$. We will construct them using translation functors.

Lemma 5.4. *An object $P \in \mathscr{C}$ is projective iff $\mathrm{Ext}^i(P, \Delta(\lambda)) = 0$ for any $\lambda \in \Xi$ and $i > 0$ iff $\mathrm{Ext}^1(P, \Delta(\lambda)) = 0$ for any $\lambda \in \Xi$.*

Proof. Suppose that $\mathrm{Ext}^i(P, \Delta(\lambda)) = 0$ for any $\lambda \in \Xi$ and $i > 0$. Let us prove by the induction on μ starting from a mininal μ that $\mathrm{Ext}^1(P, L(\mu)) = 0$ for any $i > 0$. Consider the exact sequence

$$0 \to K \to \Delta(\mu) \to L(\mu) \to 0.$$

We apply $\mathrm{Hom}(P, -)$ and by the induction hypothesis the desired follows.

Note now that if $\mathrm{Ext}^1(P, \Delta(\lambda)) = 0$ for any $\lambda \in \Xi$. Then $\mathrm{Ext}^i(P, \Delta(\lambda)) = 0$ for any $\lambda \in \Xi$ and $i > 0$. Because $\Delta(\lambda)$ has a projective resolution with standardly filtered kernels and cokernels . □

Lemma 5.5. *Category $\mathscr{C}$ is Ringel self dual if categories $\mathscr{C}^\Delta, \mathscr{C}^\nabla$ are equivalent.*

Proof. Suppose that there exists an equivalence $\Psi\colon \mathscr{C}^{\Delta} \simeq \mathscr{C}^{\nabla}$. Recall that an object P of an additive category $\mathcal{A}$ is projective if $\mathrm{Hom}(P,-)$ sends epimorphisms to epimorphisms. We claim that the projective indecomposable objects of $\mathscr{C}^{\Delta}$ are $P(\lambda), \lambda \in \Xi$. Otherwise there exists a projective object $M \in \mathscr{C}^{\Delta}$ whis is not projective in $\mathscr{C}$ i.e. by Proposition 1.3 there exists $\lambda \in \Xi$ such that $\mathrm{Ext}^1(M, \Delta(\lambda)) \neq 0$. We obtain a short exact sequence

$$0 \to \Delta(\lambda) \to Q \to M \to 0.$$

Note that $\Delta(\lambda), M \in \mathscr{C}^{\Delta}$, hence, $Q \in \mathscr{C}^{\Delta}$. We apply $\mathrm{Hom}(M,-)$ and obtain a long exact sequence

$$\to \mathrm{Hom}(M,Q) \xrightarrow{\varphi} \mathrm{Hom}(M,M) \to \mathrm{Ext}^1(M,\Delta(\lambda)) \to$$

and see that the morphism $\mathrm{Id} \in \mathrm{Hom}(M,M)$ does not lie in the image of φ.

Using Proposition 2.3 and the same arguments we see that projective indecomposable objects of $\mathscr{C}^{\nabla}$ are $T(\lambda), \lambda \in \Xi$. It follows that the equivalence Ψ sends $P = \bigoplus_{\lambda\in\Xi} P(\lambda)$ to $T = \bigoplus_{\lambda\in\Xi} T(\lambda)$ and induces the isomorphism $\mathrm{End}(P) \simeq \mathrm{End}(T)$ so $\mathscr{C} \simeq \mathscr{C}^{\vee}$. $\square$

Proposition 5.6. *The category $\mathcal{O}_0$ is Ringel selfdual i.e. there exists an equivalence $\Phi\colon \mathcal{O}_0 \simeq \mathcal{O}_0^{\vee}$. The equivalence Φ maps $\Delta(w\cdot 0)$ to $\nabla(w_0 w\cdot 0)$ and $P(w\cdot 0)$ to $T(w_0w\cdot 0)$ (here we identify $\mathcal{O}_0^{\vee,\Delta} \simeq \mathcal{O}_0^{\nabla}$).*

Proof. It follows from Lemma 5.5 that it is enough to construct an equivalence $\Phi\colon \mathcal{O}_0^{\Delta} \xrightarrow{\sim} \mathcal{O}_0^{\nabla}$. Recall the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$. We denote by $\mathcal{U}(\mathfrak{n}_-)$ the universal enveloping of $\mathfrak{n}_-$. Let $2\rho\colon \mathfrak{h} \to \mathbb{C}$ be the sum of positive roots of $\mathfrak{g}$. Set $S_{2\rho} := \mathcal{U}(\mathfrak{n}_-)^{\vee} \otimes_{\mathcal{U}(\mathfrak{n}_-)} \mathcal{U}$. Note that the space $S_{2\rho}$ has an $\mathcal{U}$-bimodule structure. The right action comes from the action of $\mathcal{U}$ on $\mathcal{U}$ via right multiplication. The left action comes from the isomorphism

$$S_{2\rho} = \mathcal{U}(\mathfrak{n}_-)^{\vee} \otimes_{\mathcal{U}(\mathfrak{n}_-)} \mathcal{U} \simeq \mathcal{U}(\mathfrak{n}_-)^{\vee} \otimes \mathcal{U}(\mathfrak{b}) \simeq \mathrm{Hom}(\mathcal{U}(\mathfrak{n}_-), \mathcal{U}(\mathfrak{b})) \simeq \mathrm{Hom}_{\mathfrak{b}}(\mathcal{U}, \mathbb{C}_{2\rho} \otimes \mathcal{U}(\mathfrak{b})),$$

where the last isomorphism.

We claim that the functors $S_{2\rho} \otimes_{\mathcal{U}} -, \mathrm{Hom}_{\mathcal{U}}(S,-)$ are mutually inverse functors between $\mathcal{O}_0^{\Delta}$ and $\mathcal{O}_0^{\nabla}$. $\square$

Corollary 5.7. *Composing equivalences $D(\mathcal{O}_0) \simeq D(\mathcal{O}_0^{\vee}), \mathcal{O}_0 \simeq \mathcal{O}_0^{\vee}$ we obtain a derived equivalence $T_{w_0}\colon D(\mathcal{O}_0) \xrightarrow{\sim} D(\mathcal{O}_0)$.*