Affine quantum groups

28.11.2021

Problems for the course Skoltech, fall 2021. There are mistakes here, if you find some please write to mbersht@gmail.com

Definitions and hints are in slides and references.

1 Quantum groups

Problem 1.1 (till 04.10.2021). *Let* $\mathfrak{g} = \mathfrak{sl}_3$, $V = \mathbb{C}^3$ *a) Compute first and second term in* E, F *of* $\overline{\mathcal{R}}$ *using pairing. b) Compute* $R_{V,V}$ *and* R_{V,V^*}. *c) Find eigenvalues of* $\widetilde{R}_{V,V}$.

Problem 1.2. *Let* $\mathfrak{g} = \mathfrak{sl}_3$, *Compute first and second term in* E, F *of* $\overline{\mathcal{R}}$ *using factorization formula.*

Problem 1.3. *For* $\mathfrak{g} = \mathfrak{sl}_3$ *and* $V = \mathbb{C}^3$ *compute* C_V, C_{V^*}.

Problem 1.4. *a) For* $\mathfrak{g} = \mathfrak{sl}_3$ *relate* l_{ij} *generators in RTT realization and Cartan-Weyl elements. b)* The same for* $\mathfrak{g} = \mathfrak{sl}_{n+1}$

Problem 1.5. *Show that* $\langle RL_1^+L_2^+ - L_2^+L_1^+R, - \rangle = 0$.

2 Affine algebras, affine Weyl group

Problem 2.1 (till 11.10.2021). *Show that generators* $E_0, \dots, E_r, H_0, \dots, H_r, F_0, \dots, F_r$ *satisfies relations of Kac-Moody algebra.*

Problem 2.2. *a) If* $s_i\lambda = \lambda$ *(equivaliently* $(\alpha_i^\vee, \lambda) = 0$*) then* $T_iY^\lambda = Y^\lambda T_i$.
b) If* $(\alpha_i^\vee, \lambda) = 1$, *then* $T_iY^\lambda T_i = Y^{s_i(\lambda)}$.

Problem 2.3. *For* $\Phi = A_n$ *the elements* $Y_i = T_{i-1}^{-1} \cdot \dots \cdot T_1^{-1}\tau T_{n-1} \cdot \dots T_i$. *Here* $\tau \in \Omega$ *is such that* $\tau T_i \tau^{-1} = T_{i+1}$, $\tau T_n \tau^{-1} = T_0$.
a) For $n = 1, 2$ *find formulas for* Y^{ϖ_i} *and show above.*
b) Show above for any* n.

Problem 2.4. ** Show that* $P \in W^{ae}$ *is the subgroup of elements with finitely many conjugates.*

3 Representations of affine algebra

4 Poincaré-Birkhoff-Witt basis for $U_q(\widehat{sl}_2)$

Problem 4.1 (till 01.11.2021). *Show that*

$$E_{\alpha+(n+1)\delta}E_{\alpha+m\delta} - q^2 E_{\alpha+n\delta}E_{\alpha+(m+1)\delta} + E_{\alpha+m\delta}E_{\alpha+(m+1)\delta} - q^2 E_{\alpha+m\delta}E_{\alpha+(n+1)\delta} = 0$$

Problem 4.2. ** Show that*

$$e^+(z)e^+(w)(z-q^2w) + e^+(w)e^+(z)(w-q^2z) = (1-q^2)(ze^+(w)^2 - we^+(z)^2)$$

5 New Drinfeld realization for $U_q(\widehat{sl}_2)$

Problem 5.1. *Finish the proof of*

$$[X^+(z), X^-(w)] = \frac{1}{q-q^{-1}}\Big(\psi^+(z)\delta(\frac{Kw}{z}) - \psi^-(z)\delta(\frac{w}{Kz})\Big)$$

Problem 5.2 (till 01.11.2021). *Show that for $r > 0$*

$$[h_r, X^+(w)] = \frac{[2r]_q}{r} w^r X^\pm(w), \quad [h_r, X^+(w)] = -K^r \frac{[2r]_q}{r} w^r X^\pm(w)..$$

Problem 5.3. *Show that*

$$\tau\Phi(h_r) = h_{-r}, \quad \tau\Phi(\Psi^+(z)) = K\Psi^-(z^{-1}), \quad \tau\Phi(\Psi^-(z)) = K^{-1}\Psi^+(z^{-1}).$$

Problem 5.4. *Show the existence of evaluation homorphism*

$$\mathrm{ev}_u \colon U_q(\widehat{\mathfrak{sl}}_2) \to U_q(\mathfrak{sl}_2), \quad E_1 \mapsto E,\ F_1 \mapsto F,\ E_0 \mapsto uF,\ F_0 \mapsto u^{-1}E.$$

Problem 5.5. *Find formula for the action of loop generators $X^+[n], X^-[n], h_r, h_{-r}$ for evaluation representation $\mathbb{C}^2(u)$*

Problem 5.6 (till 01.11.2021). *Check formula for R matrix acting on the tensor product $\mathbb{C}^2(u_1) \otimes \mathbb{C}^2(u_2)$*

6 Factorization of $\mathcal{R}$-matrix

Problem 6.1. *Prof any one of the formulas for $\Delta(E_{-\alpha+n\delta})$, $\Delta(F_{\alpha-n\delta})$, $\Delta(F_{-\alpha-n\delta})$.*

Problem 6.2. *Show that*

$$\Delta(E_{\alpha+n\delta}) = 1 \otimes E_{\alpha+n\delta} + E_{\alpha+n\delta} \otimes K_{\alpha+n\delta} + (q-q^{-1})\sum_{p=1}^{n} E_{\alpha+(n-p)\delta} \otimes K_{\alpha+(n-p)\delta}E_{p\delta}$$
$$+ \textit{very low terms}$$

where very low terms *are of the form $a \otimes b$, where a contains at least two terms of the form $E_{\alpha+p\delta}$.*

Problem 6.3. *a) Find Gauss decomposition for $R = R^-R^0R^+$ matrix acting on the tensor product $\mathbb{C}^2(u_1) \otimes \mathbb{C}^2(u_2)$.*

b) Show that $\mathcal{R}^-_{\mathbb{C}^2(u_1)\otimes\mathbb{C}^2(u_2)} = R^-$, $\mathcal{R}^+_{\mathbb{C}^2(u_1)\otimes\mathbb{C}^2(u_2)} = R^+$.

c) Show that $\mathcal{R}^0_{\mathbb{C}^2(u_1)\otimes\mathbb{C}^2(u_2)} = f(u_1/u_2)R^0$, for some function f.*

Problem 6.4. *Check that Drinfeld coproduct preserves some (a couple) of relations in new realization.*

7 RLL relaization

8 Finite dimensional representations of $U_q(\widehat{\mathfrak{sl}}_2)$

Problem 8.1. *Show that $\mathbb{C}^2(u_1) \otimes \mathbb{C}^2(u_2)$ is irreducible unless $u_1/u_2 = q^{\pm 2}$. In last case show that*

$$0 \to \mathbb{C} \to \mathbb{C}^2(u) \otimes \mathbb{C}^2(uq^2) \to \mathbb{C}^3(uq) \to 0$$
$$0 \to \mathbb{C}^3(uq) \to \mathbb{C}^2(uq^2) \otimes \mathbb{C}^2(u) \to \mathbb{C} \to 0$$

Problem 8.2. *$V(u)^* \simeq V(q^2u)$ for any finite dimensional evaluation representation $V(u)$.*

Problem 8.3. *Any irreducible representation of $U_q(\widehat{\mathfrak{sl}}_2)$ of dimension 2 is isomorphic to $\mathbb{C}^2(u)$ for some u.*

Problem 8.4. *Let V_λ be highest weight representation of $U_q(\mathfrak{sl}_2)$ with highest weight vector ξ_λ such that*

$$K\xi_\lambda = q^\lambda \xi_\lambda, \quad E\xi_\lambda = 0.$$

Then, for evaluation representation $V_\lambda(u)$of $U_q(\widehat{\mathfrak{sl}}_2)$ show that

$$\Psi^+(z)\xi_\lambda = \Psi^=(z)\xi_\lambda = \phi_\lambda(u,z)\xi_\lambda, \quad \phi_\lambda(u,z) = q^\lambda \frac{z + uq^{-\lambda-2}}{z + uq^{\lambda-2}}.$$

Problem 8.5. *Any finite multiset in $\mathbb{C}^*$ can be uniquely presented as a union of string pairwise in general position*

9 q-characters for $U_q(\widehat{\mathfrak{sl}}_2)$

Problem 9.1. *Assume that all l-weight spaces in V are one dimensional, $\zeta \in V_{(\phi)}$, $\zeta' \in V_{(\phi')}$, are such that $\langle \zeta' | X^-[n]\zeta \rangle = a^n \langle \zeta' | X^-[0]\zeta \rangle$.*
a) Show that $\langle \zeta | X^+[n]\zeta' \rangle = a^n \langle \zeta | X^+[0]\zeta' \rangle$.
b) Show that $\langle \zeta' | X^-[0]\zeta \rangle \langle \zeta | X^+[0]\zeta' \rangle = \frac{1}{a(q-q^{-1})} \operatorname{Res}_{z=a} \phi(z)$.*

Problem 9.2. *$\chi_q(V_{l_1}(a_1) \otimes V_{l_2}(a_2))$ contains more then one monomial $\prod Y(b_j)$ (i.e. without $Y^{-1}(b)$) if and only if the strings $S_{l_1}(a_1), S_{l_2}(a_2)$ are in general position.*

Problem 9.3. *For strings S_1, S_2 in special position let $S_3 = S_1 \cup C_2$, $S_4 = S_1 \cap S_2$, $\bar{S}_4$ is S_4 with two nearest neighbors and $S_3 \setminus \bar{S}_4 = S_5 \sqcup S_6$. Then we have equality in K_0*

$$V(S_1) \otimes V(S_2) = V(S_3) \otimes V(S_4) + V(S_5) \otimes V(S_6)$$

Problem 9.4. ** Let $S = \{a, aq^{-2}, \ldots, aq^{-2l}\}$. The classes $V(S')$ in $K_0(\mathrm{Rep}_{f.d.})$, for strings $S' \subset S$ satisfy relations of cluster algebra of type A_l, the only frozen variable is $V(S)$.*

10 q-characters in general

Problem 10.1. *For $\mathfrak{g} = \mathfrak{sl}_3$ find q-characters and graphs corresponding to two different 8 dimensional irreducible representations.*

Problem 10.2. *For $\mathfrak{g} = \mathfrak{so}_8$ find q-characters and graphs corresponding to fundamental representations.*

Problem 10.3. ** For $\mathfrak{g} = \mathfrak{sl}_2$ show that* $\operatorname{Ker} S = \mathbb{C}[Y(a) + Y^{-1}(aq^2)]$.

11 Schur-Weyl duality

12 Semi-infinite construction