Affine Quantum Groups

Lecture 13

Bosonization. Vertex operators.

Highest weight representations

- $\hat{\mathfrak{g}}$ – affine Lie algebra. V – $\hat{\mathfrak{g}}$ rep

$\xi \in V$ h.w. vector if $E_i \xi = 0 \quad i = 0, .., r-1$

$H_i \xi = \lambda_i \xi \quad i = 0, .., r-1$

$\lambda_i \in \mathbb{C}$

- Verma module $V_{\lambda, k}$ is module gen by

$\xi_{\lambda, k}$ s.t. $E_i \xi_{\lambda, k} = 0, \quad i = 0, .., r$

$H_i \xi_{\lambda, k} = \lambda_i \xi_{\lambda, k} \quad i = 1 .. r$

$K \xi_{\lambda, k} = k \xi_{\lambda, k}$

$\mathcal{U}(\hat{n}_-)$ acts on $V_{\lambda, k}$ freely

Remark $V_{\lambda,k} = \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\hat{\mathfrak{b}})} \mathbb{C}_{\lambda,k}$

$V_{\lambda,k}$ has basis $F_{\beta_1}^{i_1} F_{\beta_2}^{i_2} \dots F_{\beta_N}^{i_N} \xi_{\lambda,k}$

where $\{F_{\beta_1}, F_{\beta_2}, \dots\}$ a basis in $\hat{\mathfrak{n}}_-$

- Def $L_{\lambda,k}$ - is irreducible quotient $M_{\lambda,k}$.
- Def $L_{\lambda,k}$ - is integrable if for $\forall i = 0, \dots r$

$L_{\lambda,k}$ is (infinite) sum of f.d. reps $sl_2 = \langle E_i, F_i, H_i \rangle$

- Thm $L_{\lambda,k}$ is integrable $\iff$ $\lambda \in P^+$, $k \in \mathbb{Z}_{\geq 0}$
  $k \geq (\lambda, \theta)$
- Level is value of k.

Level 1 integrable representations

- Now $\mathfrak{g} = \mathfrak{sl}_2$ $\quad L_{h,k}$ integrable $\Longleftrightarrow$ $0 \le h \le k$, $h, k \in \mathbb{Z}$
- Level 1 $\Rightarrow$ $L_{0,1}$ $\quad L_{1,1}$ two integrable reps
  fundamental $\quad \varpi_0 \quad \varpi_1$
- For $L_{0,1}$ singular vectors $F_1 \xi_{0,1} = 0 \quad F_0^2 \xi_{0,1} = 0$
  For $L_{1,1}$ singular vectors $F_1^2 \xi_{1,1} = 0 \quad F_0^2 \xi_{0,1} = 0$

Bosonization (Free field realization)

- Heisenberg algebra generated by $\hat{Q}, a_r \ r \in \mathbb{Z}$
  relations $[a_r, a_s] = 2r\delta_{r+s}, \ [a_0, \hat{Q}] = 2$

- Fock module

$\alpha \in \mathbb{C}^*$ $\quad \mathcal{F}_\alpha = \mathbb{C}[a_{-1}, a_{-2}, \ldots]|\alpha\rangle$

$a_r|\alpha\rangle = 0, \ r > 0 \qquad a_0|\alpha\rangle = \alpha|\alpha\rangle$

- $e^{\beta\hat{Q}} : \mathcal{F}_\alpha \longrightarrow \mathcal{F}_{\alpha+2\beta}$ $\qquad e^{\beta\hat{Q}}|\alpha\rangle = |\alpha+2\beta]$
  $e^{\beta\hat{Q}}$ commutes with $a_r, \ r \neq 0$

computation $a_0 e^{\beta\hat{Q}}\xi = 2\beta e^{\beta\hat{Q}}$

- $\mathfrak{sl}_2 = \langle X^+, X^-, H \rangle$ $\qquad$ $\widehat{\mathfrak{sl}} = \langle X^+[n], X^-[n], H[n] \rangle$

currents $\quad X^+(w) = \sum X^+[n] w^{-n-1}$ $\qquad$ $X^-(w) = \sum X^-[n] w^{-n-1}$

- Thm $\quad L_{0,1} \simeq \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_{2n}$ $\qquad$ $L_{1,1} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_{2n-1}$

with action defined by

$$H[n] \mapsto a_n$$

$$X^+(w) \mapsto e^{\hat{Q}} w^{a_0} \exp\Big(\sum_{r>0} \frac{1}{r} a_{-r} w^r\Big) \exp\Big(\sum_{r>0} \frac{-1}{r} a_r w^{-r}\Big)$$

$$X^-(w) \mapsto e^{-\hat{Q}} w^{-a_0} \exp\Big(\sum_{r>0} \frac{-1}{r} a_{-r} w^r\Big) \exp\Big(\sum_{r>0} \frac{1}{r} a_r w^{-r}\Big)$$

- Vertex operators $\quad \varphi^+, \varphi^- : L_{m,1} \to L_{1-m,1}$

$$\varphi^+(w) = e^{\hat{Q}/2} w^{a_0/2} \exp\Big(\sum_{r>0} \frac{1}{2r} a_{-r} w^r\Big) \exp\Big(\sum_{r>0} \frac{-1}{2r} a_r w^{-r}\Big)$$

$$\varphi^+(w) = e^{-\hat{Q}/2} w^{-a_0/2} \exp\Big(\sum_{r>0} \frac{-1}{2r} a_{-r} w^r\Big) \exp\Big(\sum_{r>0} \frac{1}{2r} a_r w^{-r}\Big)$$

Fermions

- Generators $\psi_i[r],\ \psi_i^*[r],\quad i=1,\dots,n,\quad r\in\mathbb{Z}$
  Relations $[\psi_i[r],\psi_j^*[s]]=\delta_{i,j}\,\delta_{r+s},\quad [\psi_i[r],\psi_j[r]]=[\psi_i^*[r],\psi_j^*[s]]=0$

  Currents $\psi_i(z)=\sum\psi_i[n]\,z^{-n-1}\qquad \psi_j^*(z)=\sum\psi_j^*[n]\,z^{-n}$

- Fock module $\psi_i[r]|\emptyset\rangle=\psi_i^*[r-1]\,|\emptyset\rangle=0\quad r\geq 1$

- Level 1 rep of $\widehat{\mathfrak{gl}}_m\qquad E_{ij}(z)\longmapsto\psi_i(z)\,\psi_j^*(z)$

Fermions — vertex operators

Affine quantum group

- Highest weight vectors
  Highest weight representations

  Verma modules $M_{\lambda, k}$
  Irreducible quotients $L_{\lambda, k}$

- Bosonization - ?

  Below $U_q(\hat{sl}_2)$ use New Drinfeld realization

New realization

- Generators $X^+[n],\ X^-[n]$ $n\in\mathbb{Z}$, $h_r,\ h_{-r}$ $r\in\mathbb{Z}_{>0}$ $K_0^{\pm1}, K_1^{\pm1}$

Use $K^{\pm}(z)=K_1^{\pm1}\exp\left(\pm(q-q^{-1})\sum_{r>0} h_{\pm r} z^{\mp r}\right)$ insted of $\psi^{\pm}(z)$

- Relations
  - $[X^+(z), X^-(w)] = \frac{1}{q-q^{-1}}\left(K^+(z)\,\delta\left(\frac{Kw}{z}\right) - K^-(w)\,\delta\left(\frac{w}{Kz}\right)\right)$
  - $[h_r, h_s] = \frac{[2r]}{r}\,\frac{K^r-K^{-r}}{q-q^{-1}}\,\delta_{r+s}$
  - $[h_r, X^+(w)] = \frac{[2r]}{r} w^r X^+(w)$ $\qquad [h_{-r}, X^+(w)] = \frac{[2r]}{r} K^{-r} w^{-r} X^+(w)$

    $[h_r, X^-(w)] = -K^r\frac{[2r]}{r} w^r X^-(w)$ $\qquad [h_{-r}, X^-(w)] = -\frac{[2r]}{r} w^{-r} X^-(w)$
  - $X^+(z)X^+(w)(z-q^2w) + X^+(w)X^+(z)(w-q^2z) = 0$

    $X^-(z)X^-(w)(z-q^{-2}w) + X^-(w)X^-(z)(w-q^{-2}z) = 0$

Bosonization

- Heisenberg $[a_r, a_s] = \frac{[2r][r]}{r} \delta_{r+s,0}$ $\quad [a_0, \hat{Q}] = 2$

want $L_{0,1} = \bigoplus \mathcal{F}_{2n}$ $\quad L_{1,1} = \bigoplus \mathcal{F}_{2n-1}$ where $\mathcal{F}_2$ - Fock module

- Level 1 $\quad K \mapsto q \quad h_r \mapsto a_r$

- $[a_r, X^+(w)] = \frac{[2r]}{r} w^r X^+(w)$, $\quad [a_{-r}, X^+(w)] = q^{-r} \frac{[2r]}{r} w^{-r} X^+(w)$

<u>Lemma</u> If $[a, a^+] \in \mathbb{C}$ then $[a, e^{\beta a^+}] = [a, \beta a^+] e^{\beta a^+}$

Using Lemma we get

$$X^+(w) = e^{\hat{Q}} w^{a_0+1} \exp\left(\sum \frac{1}{[r]} a_{-r} w^r\right) \exp\left(\sum \frac{q^{-r}}{[-r]} a_r w^{-r}\right)$$

$$[a_r, X^-(w)] = q^r \frac{[2r]}{-r} w^r X^-(w), \quad [a_{-r}, X^-(w)] = \frac{[2r]}{-r} w^{-r} X^-(w)$$

$$X^-(w) = e^{-\hat{Q}} w^{-a_0+1} \exp\Big(\sum \frac{q^r}{[-r]} a_{-r} w^r\Big) \exp\Big(\sum \frac{1}{[r]} a_r w^{-r}\Big)$$

Relations

<u>Lemma</u> If $[a, a^+] \in \mathbb{C}$ then $e^{\alpha a} e^{\beta a^+} = e^{\alpha\beta [a,a^+]} e^{\beta a^+} e^{\alpha a}$

$$X^+(z) X^+(w) = e^{2\hat{Q}} (zw)^{a_0+1} \exp\Big(\frac{1}{[r]} a_{-r} (w^r + z^r)\Big) \exp\Big(\frac{q^{-r}}{[-r]} a_r (w^{-r} + z^{-r})\Big) \cdot$$

$$\exp\big([(\log z) a_0, \hat{Q}]\big) \exp\Big(\sum_{r>0} \Big[\frac{q^{-r}}{[-r]} a_r z^{-r}, \frac{1}{[r]} a_{-r} w^r\Big]\Big)$$

first row ___ $= :X^+(z) X^+(w): \; z^2 \exp\Big(\sum_{r>0} \frac{q^{-r}}{[-r][r]} \frac{[2r][r]}{r} \Big(\frac{w}{z}\Big)^r\Big)$

$$= :X^+(z) X^+(w): \; z^2 \exp\Big(\sum_{r>0} \frac{1+q^{-2r}}{-r} \Big(\frac{w}{z}\Big)^r\Big) = :X^+(z) X^+(w): (z-w)(z-q^{-2}w)$$

Hence $X^+(z) X^+(w) (z - q^2 w) + X^+(w) X^+(z) (w - q^2 z) =$

$$= :X^+(z) X^+(w): \big((z-q^2 w)(z-w)(z-q^{-2}w) + (w-q^2 z)(w-z)(w-q^{-2}z) \big) = 0$$

Thm Using formulas above we have

$$L_{0,1} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_{2n} \qquad L_{1,1} = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_{2n-1}$$

- Problem (a) Check $X^- X^-$ relation.

(b)* Check $[X^+, X^-]$ relation

(c) For $L_{0,1}$ check $X^-[0]|0\rangle = 0$, $X^+[-1]^2|0\rangle = 0$

Vertex operators

- $\Phi(u) : L_{m,1} \longrightarrow L_{1-m,1} \otimes V(u)$

  $\Phi^*(u) : L_{m,1} \otimes V(u) \longrightarrow L_{1-m,1}$

  $\Psi(u) : L_{m,1} \longrightarrow V(u) \otimes L_{1-m,1}$

  $\Psi^*(u) : V(u) \otimes L_{m,1} \longrightarrow L_{1-m,1}$

$V(u) = \mathbb{C}^2(u)$

evaluation rep.

$V(u) = \langle \xi_+ , \xi_- \rangle$

- In components

$$\Phi(\zeta) = \Phi_+(\zeta) \otimes \xi_+ + \Phi_-(\zeta) \otimes \xi_-$$

$$\Phi_+^*(\zeta) = \Phi_+(\zeta \otimes \xi_+), \qquad \Phi_-^*(\zeta) = \Phi^*(\zeta \otimes \xi_-)$$

Similarly for Ψ, Ψ^*

New Drinfeld coproduct

- Recall that

$$\Delta^D K^+(z) = K^+(z K_{(2)}^{-1}) \otimes K^+(z) \qquad \Delta^D K^-(z) = K^-(z) \otimes K^-(K_{(1)}^{-1} z)$$

In terms of modes

$$\Delta^D h_r = h_r \otimes K^r + 1 \otimes h_r \qquad \Delta^D h_{-r} = h_{-r} \otimes 1 + K^{-r} \otimes h_{-r}$$

- Recall that

$$K^{\pm}(z) = \begin{pmatrix} q\frac{z-q^{-2}a}{(z-a)} & 0 \\ 0 & \frac{z-q^2 a}{q(z-a)} \end{pmatrix} = \begin{pmatrix} q\frac{z+uq^{-3}}{z+uq^{-1}} & 0 \\ 0 & \frac{z+uq}{q(z+uq^{-1})} \end{pmatrix}$$

where $a = -uq^{-1}$

In terms of h_r

$$h_r \xi_+ = \frac{[r]}{r}(-uq^{-2})^r \xi_+ = \frac{[r]}{r}(aq^{-1})^r \xi_+$$

$$h_r \xi_- = \frac{[r]}{-r}(-u)^r \xi_- = \frac{[r]}{-r}(aq)^r \xi_-$$

$r \in \mathbb{Z}$

- $\varphi^D : L_{i,1} \longmapsto L_{1-i,1} \otimes \mathbb{C}^2(u)$
- Intertwining property

$$\varphi^D_+(h_r \xi) \otimes \xi_+ = (h_r \otimes K^r + 1 \otimes h_r)\varphi^D_+(\xi) \otimes \xi_+ = \left(h_r \varphi^D_+ \xi + \frac{[r]}{r}(-uq^{-2})^r \varphi^D_+\right) \otimes \xi_+$$

$$\varphi^D_+(h_{-r} \xi) \otimes \xi_+ = (h_{-r} \otimes 1 + K^{-r} \otimes h_{-r})\varphi^D_+(\xi) \otimes \xi_+ = \left(h_{-r} \varphi^D_+ \xi + q^{-r}\frac{[r]}{r}(-uq^{-2})^{-r} \varphi^D_+\right) \otimes \xi_+$$

- We have

$$[h_r, \varphi^D_+] = -\frac{[r]}{r}(-uq^{-2})^r \varphi^D_+ \qquad [h_{-r}, \varphi^D_+] = -\frac{[r]}{r}(-uq^{-2})^{-r} q^{-r} \varphi^D_+$$

Hence

$$\varphi^D_+(u) \approx \underbrace{e^{-\hat{Q}/2}(-u)^{-\frac{a_0}{2}}}_{\text{from } q=1 \text{ formula}} \exp\left(\sum \frac{a_{-r}}{-[2r]}(-uq^{-2})^r\right) \exp\left(\sum \frac{a_r}{[2r]}(-uq^{-1})^{-r}\right)$$

$\approx$ means up to factors like q, q^{a_0}, $(-u)$

- Similarly

$$[h_r, \varphi_-^D] = \frac{[r]}{r}(-u)^r \varphi_-^D \qquad [h_{-r}, \varphi_-^D] = \frac{[r]}{r}(-u)^{-r} q^{-r} \varphi_-^D$$

$$\varphi_-^D(u) \approx e^{\hat{Q}/2} (-u)^{a_0/2} \exp\left(\sum \frac{a_{-r}}{[2r]} (-u)^r\right) \exp\left(\sum \frac{a_r}{[-2r]} (-uq)^{-r}\right)$$

- Thm $\exists!$ (up to scalar) intertwine defined by formulas above.

- Problem* check intertwining property of φ^D with $X^+(z)$.

- Problem Find a_r, a_{-r} dependance of ψ_+^{*D}, ψ_-^{*D}

Drinfeld-Jimbo coproduct

Element	Action on $\mathbb{C}^2(u)$	Coproduct
$X^+[0] = E_1$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\Delta X^+[0] = X^+[0] \otimes K_1 + 1 \otimes X^+[0]$
$X^+[-1] = -F_0 K_0$	$-u^{-1} E_1 K_1^{-1} = \begin{pmatrix} 0 & -u^{-1}q \\ 0 & 0 \end{pmatrix}$	$\Delta X^+[-1] = X^+[-1] \otimes K_0 + 1 \otimes X^+[-1]$
$X^-[0] = F_1$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\Delta X^-[0] = X^-[0] \otimes 1 + K_1^{-1} \otimes X^-[0]$
$X^-[1] = -K_0^{-1} E_0$	$-K_1 F_1 u = \begin{pmatrix} 0 & 0 \\ -uq^{-1} & 0 \end{pmatrix}$	$\Delta X^-[1] = X^-[1] \otimes 1 + K_0^{-1} \otimes X^-[1]$

- Vertex operators

$$\varphi(u) : L_{m,1} \longrightarrow L_{1-m,1} \otimes V(u)$$

$$\varphi(\zeta) = \varphi_+(\zeta) \otimes \xi_+ + \varphi_-(\zeta) \otimes \xi_-$$

- Intertwining property with K_1

$$K_1 \varphi_+ K_1^{-1} = q^{-1} \varphi_+ \qquad K_1 \varphi_- K_1^{-1} = q \varphi_-$$

- Intertwining property with $X^+[0], X^+[-1], X^-[0], X^-[1]$

$$\varphi_+ X^+[0] = q X^+[0] \varphi_+ + \varphi_- \qquad \varphi_- X^+[0] = q^{-1} X^+[0] \varphi_-$$

$$\varphi_+ X^+[-1] = q^{-1} X^+[-1] \varphi_+ + (-uq^{-1})^{-1} \varphi_- \qquad \varphi_- X^+[-1] = q X^+[-1] \varphi_-$$

$$\varphi_+ X^-[0] = X^-[0] \varphi_+ \qquad \varphi_- X^-[0] = X^-[0] \varphi_- + K_1^{-1} \varphi_+$$

$$\varphi_+ X^-[1] = X^-[1] \varphi_+ \qquad \varphi_- X^-[1] = X^-[1] \varphi_- + K_0^{-1}(-uq^{-1}) \varphi_+$$

- Recall notation $a = -uq^{-1}$. From relations we get

$$a^{-1}(\varphi^+ X^+[0] - q X^+[0] \varphi^+) = (\varphi^+ X^+[-1] - q^{-1} X^+[-1] \varphi^+) \qquad (*)$$

- <u>Lemma</u> ⓐ $[X^+(w), \varphi^+] = 0$ ⓑ $[h_r, \varphi^+] = \frac{[r]}{-r}(aq^{-1})^r \varphi^+$

ⓒ $[h_{-r}, \varphi^+] = \frac{[r]}{-r} a^{-r} \varphi^+$

Pf • Commute $(*)$ with $X^-[0]$

$$a^{-1}\left(\varphi^+\frac{K_1-K_1^{-1}}{q-q^{-1}}-q\frac{K_1-K_1^{-1}}{q-q^{-1}}\varphi^+\right)=\left(\varphi^+KK_1^{-1}h_{-1}-q^{-1}KK_1^{-1}h_{-1}\varphi^+\right)$$

$$a^{-1}K_1^{-1}\varphi^+\frac{q-q^{-1}}{q-q^{-1}}=-q^{-1}KK_1[h_{-1},\varphi^+] \Rightarrow [h_{-1},\varphi^+]=-(a)^{-1}\varphi^+$$

• Commute $(*)$ with $X^-[1]$

$$a^{-1}\left(\varphi^+K^{-1}K_1h_1-qK^{-1}K_1h_1\varphi^+\right)=\left(\varphi^+\frac{K_1K^{-1}-K_1^{-1}K}{q-q^{-1}}-q^{-1}\frac{K_1K^{-1}-K_1^{-1}K}{q-q^{-1}}\varphi^+\right)$$

$$-a^{-1}qK^{-1}K_1[h_1,\varphi^+]=K_1K^{-1}\frac{q-q^{-1}}{q-q^{-1}}\varphi^+ \Rightarrow [h_1,\varphi^+]=(-aq^{-1})\varphi^+$$

• $[\varphi^+,X^-[0]]=0$ commute with h_{-1} $\Rightarrow$

$\#[\varphi^+,X^-[0]]+\#[\varphi^+,X^-[-1]]=0 \Rightarrow$

$[\varphi^+,X[-1]]=0$, induction $[\varphi^+,X[-n]]=0$

$[\varphi^+,X^-[1]]=0$ commute with h_1 $\Rightarrow [\varphi^+,X^+[2]]=0$

induction $[\varphi^+,X^+[n]]=0$

$$a^{-1}(\varphi^+ X^+[0] - q X^+[0]\varphi^+) - (\varphi^+ X^+[-1] - q^{-1} X^+[-1]\varphi^+) = 0 \qquad (*)$$

Commute $(*)$ with $X^-[n]$ $\quad n>0$

$$a^{-1}\left(\varphi^+ K^{-n}\frac{1}{q-q^{-1}} K_n^+ - q K^{-n}\frac{1}{q-q^{-1}}K_n^+\varphi^+\right) = \left(\varphi^+ \frac{1}{q-q^{-1}} K^{-n} K^+_{n-1} - q^{-1} K^{-n}\frac{1}{q-q^{-1}} K^+_{n-1}\varphi^+\right)$$

$$\varphi^+(K_n^+ - a K_{n-1}^+) = (q K_n^+ - a q^{-1} K_{n-1}^+)\varphi^+$$

$$\varphi^+ K^+(z)(z-a) = (qz - aq^{-1})K^+(z)\varphi^+$$

$$K(z)\varphi^+ K(z)^{-1} = \varphi^+ q^{-1}\frac{z-a}{z-aq^{-2}} \qquad [h_r, \varphi^+] = \frac{[r]}{-r}(aq^{-1})^r\varphi^+$$

Commute $(*)$ with $X^-[-n]$ $\quad n<0$

$$a^{-1}\left(\varphi^+ \frac{K^-_{-n}}{q^{-1}-q} - q\frac{K^-_{-n}}{q^{-1}-q}\varphi^+\right) = \varphi^+\frac{K K^-_{-n-1}}{q^{-1}-q} - q^{-1}\frac{K K^-_{-n-1}}{q^{-1}-q}\varphi^+$$

$$\varphi^+(K_{-n} - qaK_{-n-1}) = (qK^-_{-n} - aK^-_{-n-1})\varphi^+$$

$$\varphi^+K^-(z)(z-qa) = K^-(z)\varphi^+(qz-a) \qquad [h_{-r},\varphi^+] = \frac{[r]}{-r}a^{-r}\varphi^+$$

$$K^-(z)\varphi^+K^-(z)^{-1} = \varphi^+\frac{z-qa}{qz-a}$$

- Hence

$$\varphi_+(u) = e^{-\hat{d}/2}(-u)^{-a_0/2} \exp\Big(\sum \frac{a_{-r}}{-[2r]}(-uq^{-2})^r\Big) \exp\Big(\sum \frac{a_r}{[2r]}(-uq^{-1})^{-r}\Big)$$

In particular $\varphi_+ = \varphi_+^D$

- $\varphi_-(u) = \varphi_+(u) X^+[0] - q X^+[0] \varphi_+(u)$ note $\varphi_- \neq \varphi_-^D$

Integral formula $\varphi_-(u) = \oint \frac{dz}{z} : \varphi^+(u) X^+(z) : \frac{(1-q^2)z}{(u+zq)(u+q^3 z)}$

- Relations

$$\begin{tikzcd}
 & L_{1-m,1} \otimes V(u_1) \arrow[r] & L_{m,1} \otimes V(u_2) \otimes V(u_1) \arrow[d, "id \otimes R(u_2/u_1)"] \\
L_{m,1} \arrow[ur] \arrow[dr] & & \\
 & L_{1-m,1} \otimes V(u_2) \arrow[r] & L_{m,1} \otimes V(u_1) \otimes V(u_2)
\end{tikzcd}$$

Hence $R(u_2/u_1)\varphi(u_1)\varphi(u_2) = \varphi(u_2)\varphi(u_1)$

$R(u_2/u_1): V(u_2) \otimes V(u_1) \to V(u_1) \otimes V(u_2)$ intertwiner

$$R = f(u_2/u_1) \begin{pmatrix} 1 & & & \\ & \frac{q(u_2-u_1)}{u_2-q^2u_1} & \frac{u_1(1-q^2)}{u_1-q^2u_2} & \\ & \frac{u_2(1-q^2)}{u_1-q^2u_2} & \frac{q(u_2-u_1)}{u_2-q^2u_1} & \\ & & & 1 \end{pmatrix}$$

- Problem Find $f(w)$

<u>Remark</u> This $f(w)$ is normalization of universal R matrix

References

- Jimbo Miwa Algebraic Analysis of Solvable Lattice models Sec. 5, 6
- Ding Iohara Drinfeld comultiplication and vertex operators