Affine Quantum Groups

Lecture 11-12

Schur-Weyl duality
semi-infinite construction

Schur-Weyl duality

- Classical

$$\begin{array}{ccc} GL(n) \curvearrowright & \underbrace{\mathbb{C}^n \otimes \mathbb{C}^n \otimes \dots \otimes \mathbb{C}^n}_{N} & \curvearrowleft S_N \\ U(\mathfrak{gl}_n) & & \mathbb{C}[S_N] \end{array}$$

- <u>Thm</u> These two actions commute

$$(\mathbb{C}^n)^{\otimes N} = \bigoplus \underset{\text{irrep / }\mathfrak{gl}_n}{V_\lambda} \otimes \underset{\text{irrep }S_N}{R_\lambda}$$

- Centralizer of $\mathrm{Im}(U(\mathfrak{gl}_n))$ in $\mathrm{End}((\mathbb{C}^n)^{\otimes N})$ is $\mathrm{Im}(\mathbb{C}[S_n])$

Equivalently, map $\mathbb{C}[S_n] \to \mathrm{End}_{\mathfrak{gl}_n}((\mathbb{C}^n)^{\otimes N})$ is surjective

q-Schur-Weyl duality

- $R: V \underset{\Delta}{\otimes} V \to V \underset{\Delta^{op}}{\otimes} V$ $\qquad$ $\tilde{R} = P_{12} R : V \underset{\Delta}{\otimes} V \to V \underset{\Delta}{\otimes} V$

$$\tilde{R}_{12} \tilde{R}_{23} \tilde{R}_{12} = \tilde{R}_{23} \tilde{R}_{12} \tilde{R}_{23}$$

- sl_2 $\qquad$ $\tilde{R}_{V\otimes V} = q^{-1/2} \begin{pmatrix} q & & & \\ & 0 & 1 & \\ & 1 & q-q^{-1} & \\ & & & q \end{pmatrix}$

sl_3 $\qquad$ $\tilde{R}_{V\otimes V} = q^{-1/3} \left(q \sum E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ji} \otimes E_{ij} + (q-q^{-1}) \sum_{i<j} E_{jj} \otimes E_{ii} \right)$

- For generic n, $V = \mathbb{C}^n$

$$q^{1/n} \tilde{R}_{V\otimes V} = q \sum E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ji} \otimes E_{ij} + (q-q^{-1}) \sum_{i<j} E_{jj} \otimes E_{ii}$$

- **Def** Hecke algebra H_N (for A_{N-1}, sl_N) is generated by $T_1, \dots, T_{N-1}$ with relations

  ⓐ Braid $T_i T_j = T_j T_i \quad |i-j|>1, \qquad T_i T_j T_i = T_j T_i T_j \quad |i-j|=1$

  ⓑ Quadratic $(T_i - q)(T_i + q^{-1}) = 0$

- **Thm** ⓐ $\exists \quad H_N \to \mathrm{End}_{\mathcal{U}_q(sl_n)}((\mathbb{C}^n)^{\otimes N})$

  ⓑ This map is surj for $q \neq \sqrt[\#]{1}$

Pf ⓐ $T_i \mapsto \tilde{R}_{i,i+1}$ $\Rightarrow$ intertwiners, braid relations

eigenvalues of $\tilde{R}_{i,i+1}$ — $q, -q^{-1}$

ⓑ For generic q follows from $q=1$ $\square$

- **Remark** $\exists$ surj. map $\mathbb{C}[Br_{S_N}] \to H_N$

$(W = S_N)$ $\forall w \in W \rightsquigarrow T_w \in Br_{S_N} \rightsquigarrow T_w \in H_W$

- For generic q $\quad H_N \simeq \mathbb{C}[S_N] = \bigoplus_{\lambda,\, |\lambda|=N} \mathrm{End}\, R_\lambda$

We have 2 1-dim reps

"Trivial" $\quad T_i \mapsto q \qquad$ "Sign" $\quad T_i \mapsto -q^{-1}$

q- symmetizer $\quad S_N^+ = \frac{1}{[N]^+!} \sum_{w \in S_N} q^{\ell(w)} T_w$

q- antisymmetrizer $\quad S_N^- = \frac{1}{[N]^-!} \sum_{w \in S_N} (-q)^{-\ell(w)} T_w$

- $[K] = \frac{q^K - q^{-K}}{q - q^{-1}}, \quad [K]^+ = \frac{q^{2K} - 1}{q^2 - 1}, \quad [K]^- = \frac{q^{-2K} - 1}{q^{-2} - 1}.$

- **Prop** ⓐ $T_i S_N^+ = q S_N^+, \quad (S_N^+)^2 = S_N^+$

ⓑ $T_i S_N^- = -q^{-1} S_N^-, \quad (S_N^-)^2 = S_N^-$

Affine setting

- $U_q(\widehat{sl}_n)$ $\qquad \mathbb{C}^n = \langle e_0, \dots, e_{n-1} \rangle$

  $\mathbb{C}^n[y^{\pm 1}] = \langle e_h \mid h \in \mathbb{Z} \rangle$ - evaluation rep. with formal parameter

  $e_h = y^{-1} e_{h+n}$

- $E_i e_h = \delta_{i \equiv h+1} e_{h+1}$ $\qquad F_i e_h = \delta_{i \equiv h} e_{h-1}$, $\qquad K_i e_h = q^{\delta_{i\equiv h} - \delta_{i \equiv h+1}} e_h$

Problem This is rep of $U_q(\widehat{sl}_n)$

- **Remark** Order is reversed $e_{n-1}, e_{n-2}, \dots, e_0$

$E_1 : \; e_0 \mapsto e_1$

$$\begin{array}{c} e_{n-1} \\ \vdots \\ e_1 \\ e_0 \end{array} \qquad E_1 \to \begin{pmatrix} 0 & 0 & & \\ & \ddots & & \\ & & \vdots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

$E_0 : \quad e_{n-1} \mapsto e_n = y e_0$

$$\begin{array}{c} e_{n-1} \\ \vdots \\ e_1 \\ e_0 \end{array} \qquad E_0 \mapsto \begin{pmatrix} 0 & \ddots & & 0 \\ & & & \\ 0 & & & \\ y & 0 & & 0 \end{pmatrix}$$

- **Remark** On $\mathbb{C}^n[Y^{\pm 1}]$ we have commuting actions
  $U_q(\hat{\mathfrak{sl}}_n) \curvearrowright \mathbb{C}^n[Y^{\pm 1}] \curvearrowleft \mathbb{C}[Y^{\pm 1}]$

- **Question** Find $\mathrm{End}_{U_q(\hat{\mathfrak{sl}}_n)}\left((\mathbb{C}^n[Y^{\pm 1}])^{\otimes N}\right)$.

We use identification $= (\mathbb{C}^n)^{\otimes N}[Y_1^{\pm 1}, \dots, Y_N^{\pm 1}]$

Affine Hecke algebra

- **Def** $\mathcal{H}_N^{ae} = \mathbb{C}[Br_N^{ae}] \Big/ (T_i - q)(T_i + q^{-1}) = 0$

More precisely $T_0, \dots, T_r, \quad \tau \in \Omega$ Coxeter presentation.

ⓐ braid relations

ⓑ quadratic relations ⓒ $\tau T_i = T_{\tau(i)} \tau$

- **Def** (Bernshtein presentation - loop presentation)

For $\mathfrak{gl}_N$ $\mathcal{H}_N^{ae}$ generated by $T_1, \dots, T_{N-1}, Y_1^{\pm 1}, \dots, Y_N^{\pm 1}$

Subject of

ⓐ $T_1 \dots T_{N-1}$ generate $\mathcal{H}_N$ (Braid + quadratic)

ⓑ $Y_i Y_j = Y_j Y_i$ ⓒ $T_i Y_j = Y_j T_i \quad j \neq i, i+1$

$$T_i Y_i T_i = Y_{i+1}$$

- <u>Remark</u> Recall $W^{al} \begin{cases} \text{Coxeter definition} \\ W \ltimes P \end{cases}$

- <u>Thm</u> These two presentations are isomorphic.

Representation

- $(\mathbb{C}^n)^{\otimes N}$ $-$ H_n module, where $T_i \mapsto \tilde{R}_{i,i+1}$

  $\tilde{R} = q \sum E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ji} \otimes E_{ij} + (q-q^{-1}) \sum_{i<j} E_{ii} \otimes E_{jj}$ / order is reversed

- $(\mathbb{C}^n)^{\otimes N}[Y_1^{\pm 1}, \dots, Y_N^{\pm 1}] = H_N^a \otimes_{H_N} (\mathbb{C}^n)^{\otimes N}$ $-$ induced module

  In other words T_i acts on $(\mathbb{C}^n)^{\otimes N}$ as $\tilde{R}_{i,i+1}$

- <u>Example</u> $N=2$ $\quad T=T_1$ $\quad 0 \le h, g \le n-1$

  $T(e_{h+n} \otimes e_g) = T Y_1 e_h \otimes e_g = Y_2 T_1^{-1} e_h \otimes e_g = Y_2 (\tilde{R} - (q-q^{-1})) e_h \otimes e_g$

- <u>Problem</u> Action of T_i is given by

$$T_i \longmapsto S_i^Y \tilde{R}_{i,i+1} + \frac{(q-q^{-1})}{Y_i / Y_{i+1} - 1} (S_i^Y - 1)$$

where $S_i^Y Y_i = Y_{i+1} S_i^Y$, $S_i^Y Y_{i+1} = Y_i S_i^Y$

$S_i^Y Y_j = Y_j S_i^Y$ $\quad j \neq i, i+1$

<u>Hint</u> $T_i|_{(\mathbb{C}^n)^{\otimes N}} = \tilde{R}_{i,i+1}$, remains to check relations T_i, Y_j

Action of T

- Let $g = h + nk + s$, $k \geq 0$, $s = 0, .., n-1$

Prop ⓐ $s=0$

$$T(e_h \otimes e_g) = q\, e_g \otimes e_h + (q - q^{-1}) \sum_{j=0}^{k-1} e_{h+nj} \otimes e_{g-nj} \qquad k \geq 0$$

$$T(e_g \otimes e_h) = q^{-1} e_h \otimes e_g - (q - q^{-1}) \sum_{j=1}^{k-1} e_{g-nj} \otimes e_{h+jn} \qquad k > 0$$

ⓑ $s > 0$

$$T(e_h \otimes e_g) = e_g \otimes e_h + (q - q^{-1}) \sum_{j=0}^{k} e_{h+nj} \otimes e_{g-nj}$$

$$T(e_g \otimes e_h) = e_h \otimes e_g - (q - q^{-1}) \sum_{j=1}^{k} e_{g-nj} \otimes e_{h+nj}$$

Here $N = 2$ $\quad T = T_1$

- $N > 2$ $\quad T_k(e_{g_1} \otimes .. \otimes e_{g_k} \otimes e_{g_{k+1}} \otimes .. \otimes e_{g_N}) =$

$$= e_{g_1} \otimes .. \otimes e_{g_{k-1}} \otimes T(e_{g_k} \otimes e_{g_{k+1}}) \otimes e_{g_{k+2}} \otimes .. \otimes e_{g_N}$$

Problem Check one of relations in ⓑ

- **Thm** $\mathcal{H}_N^{ae}$ commutes with $U_q(\hat{sl}_n)$

Problem Check commutativity E_i, and T on $e_h \otimes e_g$ for $g=h+nk+s$, $k \geq 0$, $s>0$, $N=2$ (or perform any other nontrivial check)

- **Remark** Another version of affine q-Schur-Weyl duality is functor (Chari-Pressley)
  $\{\text{f.d. reps } U_q(\hat{sl}_n)\} \to \{\text{f.d. reps } H_N^a\}$

- For $a_1, \dots, a_N \in \mathbb{C}^*$ let $M_{\bar{a}} = \mathcal{H}^a / \langle Y_i - a_i \rangle$. In other words, M_a is induced from 1d rep. of $\mathbb{C}[Y_1^{\pm 1}, \dots, Y_N^{\pm 1}] \subset H_N^a$. We have $\dim M_a = N!$

Problem* ⓐ For $N=2$, M_{a_1,a_2} is irred if and only if $a_1/a_2 \neq q^2, q^{-2}$.
ⓑ If $a_i/a_j \neq q^2$ $\forall\, 1 \leq i,j \leq N$ then $M_{\bar{a}}$ is irred.

Hint ⓑ use Cherednik intertwiners

Affine R-matrix

- Let $N=2$ $T=T_1$, $S^Y = S_1^Y$

We have $T \longmapsto S^Y \tilde{R} + \dfrac{(q-q^{1})}{Y_1/Y_2 - 1}(S^Y - 1)$

- Consider $\tilde{R}^a = T + \dfrac{(q-q^{-1})Y_2}{Y_1 - Y_2}$. Then $\tilde{R}^a \sim S^Y$ i.e

$\tilde{R}^a Y_1 = Y_2 \tilde{R}^a$, $\tilde{R}^a Y_2 = Y_1 \tilde{R}^a$

- Let $s^e(e_h \otimes e_g) = e_g \otimes e_h$, $0 \leq g, h \leq n-1$, $s^e Y_{1,2} = Y_{1,2} s^e$

Then $s = s^e s^Y$ permutes factors

$$R^a = s^e \tilde{R}^a = s\tilde{R} + \frac{(q-q^{-1})Y_2}{Y_1 - Y_2} s$$

In matrix notations for $n=2$

$$R^a = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} + \frac{(q-q^{-1})Y_2}{Y_1-Y_2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{qY_1-q^{-1}Y_2}{Y_1-Y_2} & 0 & 0 & 0 \\ 0 & 1 & \frac{(q-q^{-1})Y_2}{Y_1-Y_2} & 0 \\ 0 & \frac{(q-q^{-1})Y_1}{Y_1-Y_2} & 1 & 0 \\ 0 & 0 & 0 & \frac{qY_1-q^{-1}Y_2}{Y_1-Y_2} \end{pmatrix} \text{ — affine } R\text{-matrix}$$

Hence action of H^a is given by $Y_i^{\pm 1}$ and affine R matrices R^a.

- <u>Remark</u> Formula $R^a = \# R + \# S$ is called Baxterization

q - wedge

- **Def** $\Lambda_q^N = \Lambda_q^N(\mathbb{C}^n[Y^{\pm 1}]) = S_-(\mathbb{C}^n)^{\otimes N}[Y_1^{\pm}, \dots, Y_N^{\pm}]$

q - deformed exterior power.

Here $S_- \in H_N$ - q antisimmetrizer $S^-_N = \frac{1}{[N]^{-1}!} \sum_{w \in S_N} (-q)^{-\ell(w)} T_w$

- **Def** $e_{i_1} \wedge e_{i_2} \wedge \dots e_{i_N} = S_-(e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_N})$

- **Prop** Let $g = h + nk + s$, $k \geq 0$, $s = 0, .., n-1$

ⓐ $s = 0$ $\quad e_g \wedge e_h = - e_h \wedge e_g$

ⓑ $K = 0, s > 0$ $\quad e_g \wedge e_h = -q\, e_h \wedge e_g$

ⓒ $K > 0, s > 0$

$$e_g \wedge e_h = -q\, e_h \wedge e_g - e_{g-nk} \wedge e_{h+nk} - q\, e_{h+nk} \wedge e_{g-nk}.$$

Remark We can either permute e_g and e_h or move them closer

Pf ⓐ $[2]\, e_g \wedge e_h = (1 - q^{-1} T)\, e_g \otimes e_g = \Big|$ use formula for action of $T \Big| =$

$$= e_g \otimes e_h - q^{-1} \Big(q^{-1} e_h \otimes e_g - (q - q^{-1}) \sum_{j=1}^{k-1} e_{g-nj} \otimes e_{h+jn} \Big)$$

$$\parallel \qquad \Rightarrow e_g \wedge e_h = - e_h \wedge e_g$$

$$[2]\, e_h \wedge e_g = (1 - q^{-1} T)\, e_h \otimes e_g$$

$$= \Big(e_h \otimes e_g - q^{-1} \Big(q\, e_g \otimes e_h + (q - q^{-1}) \sum_{j=0}^{k-1} e_{h+nj} \otimes e_{g-nj} \Big) \Big) \qquad \square$$

- **Problem** Prove ⓑ

- **Thm** Properties ⓐ, ⓑ, ⓒ can be used for vectors of the form $e_{i_1} \wedge \ldots \wedge e_g \wedge e_h \wedge \ldots \wedge e_{i_N}$

  $i_k = g,\ i_{k+1} = h$

Pf From ⓐ of prop $(1-qT)(e_g\otimes e_h+e_h\otimes e_g)=0$.

From properties S_- we have $S_-(1-q^{-1}T_k)=S_-$

Hence $S_-(e_{i_1}\wedge\ldots\wedge e_g\wedge e_h\wedge\ldots\wedge e_{i_N}+e_{i_1}\wedge\ldots\wedge e_h\wedge e_g\wedge\ldots\wedge e_{i_N})=$

$=([2]^-)^{-1}S_-(1-q^{-1}T_k)(e_{i_1}\wedge\ldots\wedge e_g\wedge e_h\wedge\ldots\wedge e_{i_N}+e_{i_1}\wedge\ldots\wedge e_h\wedge e_g\wedge\ldots\wedge e_{i_N})=0$ $\square$

- **Corol.** Elements $e_{i_1}\wedge\ldots\wedge e_{i_N}$ form basis $\Lambda_q^N(\mathbb{C}^n[Y^{\pm 1}])$
  $i_1<i_2<\ldots<i_N$

Pf From **Thm** they generate.
Linear independance from $q=1$ $\square$

- **Problem** Let $i_1,\ldots,i_N\in\mathbb{Z}$ s.t $\sum(i_k+m-k)>0$ and $i_k\le N-m$ for some $m\in\mathbb{Z}$. Then $e_{i_1}\wedge\ldots\wedge e_{i_N}=0$.

Hint $e_{i_1}\wedge\ldots\wedge e_{i_N}=\sum c_{i'}\, e_{i'_1}\wedge\ldots\wedge e_{i'_N}$ where $i'_1<i'_2<\ldots<i'_N$. Show that $\sum(i'_k+m-k)\ge 0$, and $i'_k\le N-m$. There is no such i'

Action of $U_q(\widehat{sl}_n)$

- Introduce notation

$$|\lambda\rangle_{N,m} = e_{-\lambda_1+m} \wedge e_{-\lambda_2+1+m} \wedge \dots \wedge e_{-\lambda_j+j-1+m} \wedge \dots \wedge e_{N-1+m} \in \Lambda^N_v \mathbb{C}^n[Y^{\pm1}]$$

for partition λ, $\ell(\lambda) < N$

<u>Remark</u> For any given m vectors $|\lambda\rangle_{N,m}$ do <u>not</u> form basis in $\Lambda^N_v(\mathbb{C}^n[Y^{\pm1}])$. But we get basis in limit $N \to \infty$.

- $U_q(\widehat{sl}_n)$ commutes with H_N. Hence $\Lambda^N_v(\mathbb{C}^n[Y^{\pm1}])$ is $U_q(\widehat{sl}_n)$-mod.

$$F_i(e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_N}) = F_i\, S_-(e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_N}) = S_-(F_i(e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_N}))$$

$$\Delta^{(N)} F_i = F_i \otimes 1 \otimes \dots \otimes 1 + K_i^{-1} \otimes F_i \otimes 1 \otimes \dots \otimes 1 + K_i^{-1} \otimes K_i^{-1} \otimes F_i \otimes 1 \otimes \dots \otimes 1 + \dots + K_i^{-1} \otimes K_i^{-1} \otimes \dots \otimes F_i$$

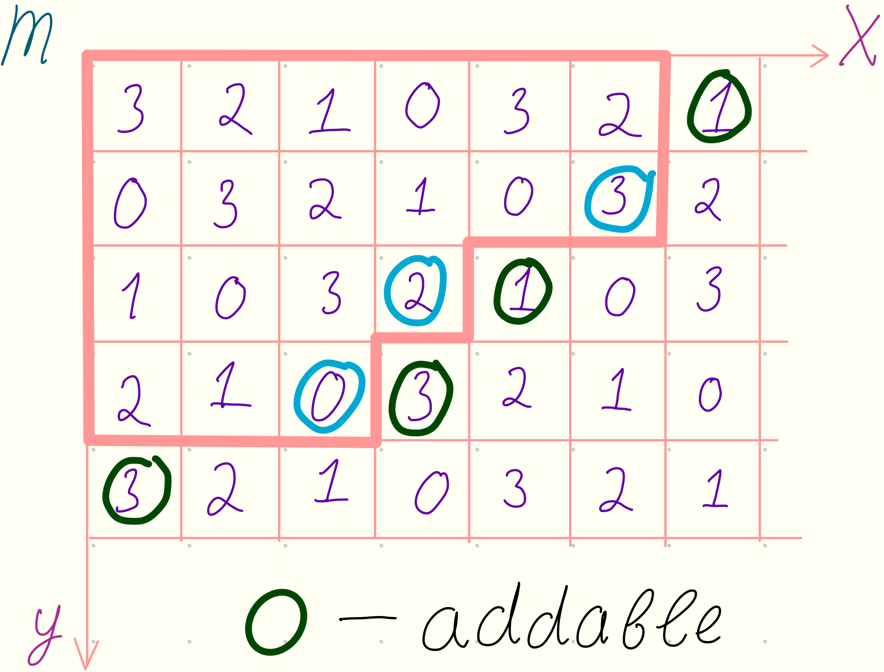
- $$F_i|\lambda\rangle = \sum_{\substack{j \\ -\lambda_j+(j-1)-m\equiv i}} q^{\sum_{j'=1}^{j-1}\delta_{-\lambda_{j'}+(j'-1)-m\equiv i}-\delta_{-\lambda_{j'}+j'-m\equiv i}} e_{-\lambda_1+m}\wedge\ldots\wedge e_{-\lambda_j+(j-1)+m+1}\wedge\ldots\wedge e_{N-1+m}$$

$$= \sum_{\substack{j \\ \lambda_j-(j-1)+m\equiv -i \\ \lambda_{j-1}>\lambda_j}} q^{\sum_{j'=1}^{j-1}\delta_{-\lambda_{j'}+j'+m\equiv i}-\delta_{-\lambda_{j'}+(j'-1)+m\equiv i}} |\lambda+1_j\rangle$$

add Box in j-th row

- It is convenient to color Boxes in resigues.

Color of Box $\square=(x,y)$ $\quad c(\square)=y-x+m$



3	2	1	0	3	2	1
0	3	2	1	0	3	2
1	0	3	2	1	0	3
2	1	0	3	2	1	0
3	2	1	0	3	2	1

Example $m=3,\ n=4$

$\lambda=(6,6,4,3)$

○ — addable

○ — removable

Def $Add(\lambda)$ – Boxes which can be added to λ

$Rem(\lambda)$ – —— '' —— removed from λ

- Then $$F_i |\lambda\rangle_{N,m} = \sum_{\substack{\square=(\lambda_j+1,j)\in Add(\lambda) \\ c(\square)=i}} q^{\sum_{j'=1}^{j-1} \delta_{L_j+1\equiv i} - \delta_{L_j\equiv i}} |\lambda+1_j\rangle_{N,m}$$

where $L_j = j-\lambda_j-1+m$

- Similarly $$\Delta^{(N)}(E_i) = E_i\otimes K_i\otimes\ldots\otimes K_i + 1\otimes E_i\otimes K_i\otimes\ldots\otimes K_i + \ldots + 1\otimes\ldots\otimes 1\otimes E_i$$

$$E_i|\lambda\rangle_{N,m} = \sum_{\substack{\square=(\lambda_j,j)\in Rem(\lambda) \\ c(\square)=i}} q^{\sum_{j'=j+1}^{N} \delta_{L_j\equiv i} - \delta_{L_j+1\equiv i}} |\lambda-1_j\rangle_{N,m} +$$

$$+ \delta_{N+m\equiv i}\, e_{-\lambda_1+m}\wedge\ldots\wedge e_{-\lambda_{N-1}+N-2+m}\wedge e_{N+m}$$

Boundary term

- Similarly $$K_i|\lambda\rangle_{N,m} = q^{\sum_{j=1}^{N} \delta_{L_j\equiv i} - \delta_{L_j+1\equiv i}} |\lambda\rangle_{N,m}$$

- **Problem** Show that

ⓐ $K_i|\lambda\rangle_{N,m} = q^{|\lambda|_{i-1} - 2|\lambda|_i + |\lambda|_{i+1} + \delta_{m\equiv i} - \delta_{m+N\equiv i}} |\lambda\rangle_{N,m}$ where $|\lambda|_j = \#\{\square\in\lambda \mid c(\square)=j\}$

ⓑ $K_i|\lambda\rangle_{N,m} = q^{\#\{\square\in Add(\lambda) \mid C(\square)=i\} - \#\{\square\in Rem(\lambda) \mid C(\square)=i\} - \delta_{m+N\equiv i}} |\lambda\rangle_{N,m}$

Hint Induction by $|\lambda|$.

- Similarly one can show

$$F_i|\lambda\rangle_{N,m} = \sum_{\substack{\square=(\lambda_j+1,j)\in Add(\lambda)\\ C(\square)=i}} q^{\#\left\{\substack{\square'\in Add(\lambda)\\ C(\square')=i} \,\middle|\, \substack{\square' \text{ to the}\\ \text{left of } \square}\right\} - \#\left\{\substack{\square'\in Rem(\lambda)\\ c(\square')=i} \,\middle|\, \substack{\square' \text{ to the}\\ \text{left of } \square}\right\}} |\lambda+1_j\rangle$$

$$E_i|\lambda\rangle_{N,m} = q^{-\delta_{m+N\equiv i}} \sum_{\substack{\square=(\lambda_j+1,j)\in Rem(\lambda)\\ C(\square)=i}} q^{\#\left\{\substack{\square'\in Add(\lambda)\\ C(\square')=i} \,\middle|\, \substack{\square' \text{ to the}\\ \text{right of } \square}\right\} - \#\left\{\substack{\square'\in Rem(\lambda)\\ c(\square')=i} \,\middle|\, \substack{\square' \text{ to the}\\ \text{right of } \square}\right\}} |\lambda-1_j\rangle_{N,m}$$

$+$ Boundary term

Limit

- **Def** $\varphi^{(m)}_{N+M,N}: \Lambda^N_q(\mathbb{C}^n[Y^{\pm1}]) \longrightarrow \Lambda^{N+M}_q(\mathbb{C}^n[Y^{\pm1}])$

  $w \longmapsto w \wedge (e_{N+m} \wedge \dots \wedge e_{N+M+m})$

Note, that $|\lambda\rangle_{N,m} \longmapsto |\lambda\rangle_{N+M,m}$

- **Prop** $\varphi^{(m)}_{N+M+K,N} = \varphi^{(m)}_{N+M+K,N+M} \circ \varphi^{(m)}_{N+M,N}$

we have system $\Lambda^1_q \xrightarrow{\varphi^{(m)}_{2,1}} \Lambda^2_q \xrightarrow{\varphi^{(m)}_{3,2}} \Lambda^3_q \xrightarrow{\varphi^{(m)}_{4,3}} \dots$

Def $\mathcal{F}_m = \Lambda_q^{\infty/2+m}(\mathbb{C}^n[Y^{\pm1}]) = \lim\limits_{N\to\infty} \Lambda^N_q(\mathbb{C}^n[Y^{\pm1}])$

$\varphi_{\infty,N}: \Lambda^N_q \longrightarrow \mathcal{F}_m$

- **Prop** Vectors $|\lambda\rangle_m = |\lambda\rangle_{\infty,m} = e_{-\lambda_1+m} \wedge e_{-\lambda_2+1+m} \wedge \dots \wedge e_{-\lambda_2+(j-1)+m} \wedge \dots$ form basis in $\mathcal{F}_m$

Pf Use vanishing in Problem above □

- Action of $\mathcal{U}_q(\widehat{sl}_n)$

Ⓕ Formulas for F_i : $F_i|\lambda\rangle_{N,m} = \sum_j q^{\#_j} |\lambda+1_j\rangle_{N,m}$
where $\#_j$ depends on $\lambda_{j'}$ for $j'<j$

Hence the same formulas define $F_i|\lambda\rangle_m$

Ⓚ Example $n=3$, $m=0$, $\lambda=(3)$

$$K_1(|\lambda\rangle_{2,0}) = K_1\, e_{-3}\wedge e_1 = e_{-3}\wedge e_1$$
$$K_1(|\lambda\rangle_{3,0}) = K_1\, e_{-3}\wedge e_1\wedge e_2 = e_{-3}\wedge e_1\wedge e_2$$
$$K_1(|\lambda\rangle_{4,0}) = K_1\, e_{-3}\wedge e_1\wedge e_2\wedge e_3 = q^{-1} e_{-3}\wedge e_1\wedge e_2\wedge e_3$$
$$K_1(|\lambda\rangle_{5,0}) = K_1\, e_{-3}\wedge e_1\wedge e_2\wedge e_3\wedge e_4 = e_{-3}\wedge e_1\wedge e_2\wedge e_3\wedge e_4$$

In the limit

$$\begin{array}{ccccccccc} e_{-3}\wedge & e_1\wedge & e_2\wedge & e_3\wedge & e_4\wedge & e_5\wedge & e_6\wedge & e_7\wedge & e_8\wedge \\ q^{-1} & q & 1 & q^{-1} & q & 1 & q^{-1} & q & 1 \end{array}$$

Hence K_i does <u>not</u> stabilize.

Define removing boundary terms $\lim_{N\to\infty} q^{\delta_{N+m\equiv i}} K_i$

$$K_i|\lambda\rangle_m = q^{\#\{\square\in\lambda \mid c(\square)=i\} - \#\{\square\in\lambda \mid c(\square)=i-1\} + \delta_{m\equiv i}} |\lambda\rangle_m$$

$$K_i|\lambda\rangle_m = q^{\#\{\square\in Add(\lambda) \mid c(\square)=i\} - \#\{\square\in Rem(\lambda) \mid c(\square)=i\}} |\lambda\rangle_m$$

Ⓔ Example $n=3,\ m=0,\ \lambda=(3)$

$$E_1(|\lambda\rangle_{2,0}) = E_{-1}(e_{-3}\wedge e_1) = q\, e_{-2}\wedge e_1$$

$$E_1(|\lambda\rangle_{3,0}) = E_{-1}\, e_{-3}\wedge e_1\wedge e_2 = q\, e_{-2}\wedge e_1\wedge e_2$$

$$E_1(|\lambda\rangle_{4,0}) = E_1\, e_{-3}\wedge e_1\wedge e_2\wedge e_3 = e_{-2}\wedge e_1\wedge e_2\wedge e_3 + e_{-3}\wedge e_1\wedge e_2\wedge e_4$$

$$E_1(|\lambda\rangle_{5,0}) = E_1\, e_{-3}\wedge e_1\wedge e_2\wedge e_3\wedge e_4 = q\, e_{-2}\wedge e_1\wedge e_2\wedge e_3\wedge e_4$$

Hence E_i does <u>not</u> stabilize

Define removing boundary terms

$$E_i|\lambda\rangle_m = \sum_{\substack{\square=(\lambda_j+1,j)\in Rem(\lambda)\\ c(\square)=i}} q^{\#\left\{\substack{\square'\in Add(\lambda)\\ c(\square')=i}\,\middle|\,\substack{\square' \text{ to the}\\ \text{right of } \square}\right\} - \#\left\{\substack{\square'\in Rem(\lambda)\\ c(\square')=i}\,\middle|\,\substack{\square' \text{ to the}\\ \text{right of } \square}\right\}} |\lambda-1_j\rangle_m$$

In particular $E_1|3\rangle_0 = |2\rangle_0$

- <u>Thm</u> Formulas above define action $U_q(\hat{sl}_n)$ on $\mathcal{F}_m$.

<u>Idea of pf</u> This is a limit $\square$

- <u>Remarks</u> ⓐ Highest weight vector $|\phi\rangle_m \in \mathcal{F}_m$

$K_i|\phi\rangle_m = q^{\delta_{i\equiv m}}|\phi\rangle_m$ – fundamental h.w ϖ_i

$K = K_0K_1\cdots K_{n-1} \mapsto q$ Level 1

$F_i^{1+\delta_{i\equiv m}}|\phi\rangle_m = 0$ – <u>integrable</u> rep.

ⓑ Principal grading on $\mathcal{F}_m$: $\mathrm{pr.deg}\, |\lambda\rangle_m = |\lambda|$
Then $\mathrm{pr\,deg}\, E_i = -1$, $\mathrm{pr\,deg}\, K_i = 0$, $\mathrm{pr\,deg}\, F_i = 1$

ⓒ Vertex operator $\Phi : \mathbb{C}^n[y^{\pm 1}] \otimes \mathcal{F}_{m-1} \to \mathcal{F}_m$

$$\Phi(e_k \otimes w) = \Phi_k(w) = e_k \wedge w$$

Φ – intertwiner $\qquad \Phi(z) := \sum \Phi_k z^{-k}$

ⓓ As we will see $\mathcal{F}_m$ is not irreducible as $U_q(\widehat{\mathfrak{sl}}_n)$ module but irreducible as $U_q(\widehat{\mathfrak{gl}}_n)$ module.

Heisenberg algebra

- **Def** For $b_k \in \mathbb{Z}\setminus\{0\}$ let $b_k = \sum_{i=1}^{N} Y_i^k \in H_N^a$

- **Prop** b_k commutes with $H_N \subset H_N^a$, $\forall k, N$

Pf Sufficient to show b_k commutes with T_j $\forall j$
Sufficient to show $Y_j^k + Y_{j+1}^k$ commutes with T_j
Follows from $Y_j + Y_{j+1}$ and $Y_j Y_{j+1}$ commute with T_j ☐

- **Problem** $\forall k$ $\exists B_k$ acting on $\mathcal{F}_m$ s.t. $\lim_{N\to\infty} \varphi_{\infty,N} b_k |\lambda\rangle_{N,m} = B_k |\lambda\rangle_m$

- **Prop** ⓐ B_k commutes with $\mathcal{U}_q(\widehat{sl}_n)$
ⓑ $[B_k, \varphi_e] = \varphi_{e+k}$ ⓒ $\mathrm{pr\,deg}\, B_k = nk$

Pf Follows from similar properties of b_k ☐

- Clearly $[B_k, B_\ell]=0.$

<u>Thm</u> $[B_k, B_\ell] = \frac{k(1-q^{2nk})}{1-q^{2k}}\delta_{k+\ell,0}$

<u>Corol</u> $U_q(\hat{s\ell}_n)\otimes\mathbb{C}\langle B_k\rangle = U_q(\hat{g\ell}_n)$

- Sketch of the proof $[B_k, B_\ell]=\delta_{k+\ell,0}\,\mathrm{const}$

Let $\mathbb{B}=[B_k, B_\ell]$ Then ⓐ $\mathrm{prdeg}\,\mathbb{B}_{k+\ell}=n(k+\ell)$,
ⓑ $[\varphi_r, \mathbb{B}]=0 \quad \forall r$

Then $\mathbb{B}|\lambda\rangle_m = e_{-\lambda_1+m}\wedge e_{-\lambda_2+m-1}\wedge\ldots\wedge e_{N+m}\wedge\ldots\wedge e_{N+M+m-1}\wedge\mathbb{B}(|\emptyset\rangle_{N+M+m})$
for any big N, M. This vanishes for $k+\ell\neq 0$ and proportional to idenity for $k+\ell=0$ ☐

- <u>Problem</u> Compute $B_1 B_{-1}|\emptyset\rangle_m$.

References

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