Affine Quantum Groups

Lecture 10

q-characters. General case

$U_q(\hat{\mathfrak{g}})$

- For simplicity untwisted, simply-laced
- Drinfeld-Jimbo presentation

Def $U_q(\hat{\mathfrak{g}})$ — generated by $E_0, \dots, E_r$, $F_0, \dots, F_r$, $K_0, \dots, K_r$ (d) subject of rel.

- $K_i K_i^{-1} = K_i^{-1} K_i = 1$, $K_i K_j = K_j K_i$ — quadratic relations
- $K_i E_j K_i^{-1} = q^{a_{ij}} E_j$, $K_i F_j K_i^{-1} = q^{-a_{ij}} F_j$, $[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$
- $\sum_{k=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-a_{ij}-k} = 0$, $\sum_{k=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^k F_j F_i^{1-a_{ij}-k} = 0$ — Serre relations

- $(a_{ij})_{i,j=0}^{r} = C^a$ — affine Cartan matrix

for $C = (2)$ $C^a = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

New Drinfeld realization

- Th $U_q(\hat{\mathfrak{g}})$ has presentation with generators $X_i^+[n]$, $X_i^-[n]$, $n \in \mathbb{Z}$, $h_i[n]$, $h_i[-n]$ $n \in \mathbb{Z}_{>0}$, $K_i^{\pm 1}$, $K^{\pm 1}$, $i = 1 \dots r$
  and relations
  - $[X_i^+(z), X_j^-(w)] = \frac{\delta_{ij}}{q-q^{-1}} \left(\psi_i^+(z)\, \delta\left(\frac{Kw}{z}\right) - \psi_i^-(w)\, \delta\left(\frac{w}{Kz}\right)\right)$
  - $[h_i[r], h_j[s]] = \frac{[r a_{ij}]}{r} \frac{K^r - K^{-r}}{q-q^{-1}} \delta_{r+s}$ $\quad K = K_\delta$
  - $[h_r, X^+(w)] = \frac{[r a_{ij}]}{r} w^r X^+(w)$ $\quad [h_{-r}, X^+(w)] = \frac{[r a_{ij}]}{r} K^{-r} w^{-r} X^+(w)$

    $[h_r, X^-(w)] = -K^r \frac{[r a_{ij}]}{r} w^r X^-(w)$ $\quad [h_{-r}, X^-(w)] = -\frac{[r a_{ij}]}{r} w^{-r} X^-(w)$
  - $X_i^+(z) X_j^+(w)(z - q^{a_{ij}} w) + X_j^+(w) X_i^+(z)(w - q^{a_{ij}} z) = 0$

    $X_j^-(z) X_i^-(w)(z - q^{-a_{ij}} w) + X_i^-(w) X_j^-(z)(w - q^{-a_{ij}} z) = 0$
  - Serre relations

- **Rem** As for $\mathfrak{sl}_2$, but replace $2 \rightsquigarrow a_{ij}$

- **Corollary** Let $J \subset I$ $I = \Pi$ - set of simple roots $\mathfrak{g}$
  There is $U_q(\hat{\mathfrak{g}}_J) \hookrightarrow U_q(\mathfrak{g})$

  In particular $i \in I$ $U_q(\hat{\mathfrak{sl}}_2)_i \hookrightarrow U_q(\mathfrak{g})$

Finite dim. reps

- Type I reps $V = \bigoplus_{\lambda} V_{(\lambda)}$ s.t. $\forall \xi \in V_{(\lambda)}$
  $K_i \xi = q^{\lambda_i} \xi$

  $\Downarrow$

  For f.d. rep. $K = 1$

- Def $\xi \in V$ is (generalized) ℓ-weight vector
  if it is (generalized) eigen vector
  for $\psi_i^+(z)$, $\psi_i^-(z)$ $\forall i$

- Def $\xi \in V$ ℓ-h.w. vector if it is ℓ-weight
  vector and $X_i^+(z)\xi = 0$ $\forall i$

• **Thm** Let V - f.d. irrep

(a) $\exists!$ (up multiple) ξ - ℓ.h.w vector

(b) $\psi_i^{\pm}(z)\xi = \phi_i^{\pm}(z)\xi$ then

$$\phi_i^{+}(z) = \phi_i^{-}(z) = q^{-\ell_i}\frac{P_i(q^2 z)}{P_i(z)}$$

where $P_i(z) = z^{\ell_i} + \# z^{\ell_i - 1} + \ldots + \#$

(c) Let $\xi \in V$ be ℓ-weight vector. then

$$\psi_i^{\pm}(z)\xi = \phi_{i,\xi}^{\pm}(z)\xi$$

$$\phi_{i,\xi}^{+}(z) = \phi_{i,\xi}^{-}(z) = q^{-r_i}\frac{R_i(q^2 z)}{R_i(z)}$$

$$R_i = \frac{P_i}{Q_i} = \frac{z^{p_i} + \ldots + \#}{z^{q_i} + \ldots + \#}$$

$$r_i = p_i - q_i$$

Pf (a) as for $\widehat{sl}_2$

(b),(c) Restrict V to $U_q(\widehat{sl}_2)_i$.

For $\widehat{sl}_2$ we know

$\square$

- **Remark** If $K_i\xi = q^{m_i}\xi$ then $\deg R_i = m_i$

- **Thm** (Chari-Pressley) For any set of Drinfeld polynomials $P_1,\dots,P_r$ $\exists!$ irred. f.d. rep $V_{\vec{P}}$ with ℓ-h.w $\phi_i = q^{-\deg P_i} P(q^2 z)/P(z$

- Uniqueness — standard
- Existance — sufficient for fundamental reps:

  $\varpi_i \rightsquigarrow P_j = 1,\ j\neq i,\quad P_i = (z-a)$

- For type A — use evaluation homomorphism

q-characters

- Let $V = \bigoplus V_{(\phi)}$ where $\forall \xi \in V_{(\phi)}$
  $\exists m$ $\left(\psi_i^{\pm}(z) - \phi(z)\right)^m \xi = 0$

$$\phi_i = \frac{R_i(q^2 z)}{R_i(z)} \qquad R_i(z) = q^{\#} \frac{\prod_j (z - a_j^{(i)})}{\prod (z - b_j^{(i)})} \qquad Y_\phi = \prod_i \left(\prod_j Y_i(a_j^{(i)}) \prod_j Y_i^{-1}(b_j^{(i)}) \right)$$

q-character of V $\qquad \chi_q(V) = \sum_\phi \dim V_\phi \; Y_\phi$

$$\chi_q : \quad K_0(Rep_{f.d}(U_q(\hat{\mathfrak{g}}))) \to \mathbb{C}[Y] = \mathbb{C}[Y_i^{\pm 1}(a) \mid i \in I, a \in \mathbb{C}^*]$$

- <u>Prop</u> (a) χ_q is ring homomorphism

(b) $J \subset I$ then diagram is commutative

$$\begin{array}{ccc} K_0(Rep_{f.d}(U_q(\hat{\mathfrak{g}}))) & \to & \mathbb{C}[Y] \\ \downarrow & & \downarrow \\ K_0(Rep_{f.d}(U_q(\hat{\mathfrak{g}}_J))) & \to & \mathbb{C}[Y_J] \end{array}$$

In particular for any $i \in I$ we have restriction to $\hat{sl}(2)$

Ⓒ Diagram is commutative

$$\begin{array}{ccc} K_0(Rep_{f.d}(U_q(\hat{g}))) & \to & \mathbb{C}[Y] & Y_i(a) \\ \downarrow & & \downarrow & \downarrow \\ K_0(Rep_{f.d}(U_q(g))) & \to & \mathbb{C}[y_i^{\pm 1} \mid i \in I] & y_i \end{array}$$

Pf For ⓐ $0 \to V' \to V \to V'' \to 0 \Rightarrow \chi_q(V) = \chi_q(V') + \chi_q(V'')$ easy

$\chi_q(V \otimes V') = \chi_q(V) \otimes \chi_q(V')$ — follows from triangularity $\Delta(h_i[r])$

ⓑ, Ⓒ - easy $\square$

- Lemma Let $\xi \in V_{(\phi)}$, $\xi' \in V_{\phi'}$ s.t $\langle \xi', X_i^-[n] \xi \rangle = a^n \langle \xi', X_i^-[0] \xi \rangle$

Then $Y_{\phi'} = Y_\phi A^{-1}(a)$, where $A(a) = Y(a) Y(aq^2) \prod_{j,\, a_{ij}=-1} Y_j^{-1}(aq)$

Pf Direct computation

Remark Y_i – "fund. weight", A_i – "simple root"

- **Corol** For any V, V' we have $V \otimes V' = V' \otimes V$ in $K_0(Rep_{f.d}(U_q(\hat{sl}_2))$

- **Theorem** (Frenkel–Reshetikhin–Mukhin)

For any irred. f.d. rep V we have $\chi_q(V) = m(1 + \sum M)$ where m corresponds ℓ-h.w vector and $M = \prod A_i^{-1}(c)$

For Drinfeld polynomials $P_i = \prod (z - a_j^{(i)})$
$\chi_q(V_{\bar{P}}) = \prod_i \prod_j Y_i(a_i^{(j)})$ + lower terms
Hence $\chi_q(V_{\bar{P}})$ linearly independent, hence χ_q is embedding

- **Theorem** (Frenkel–Reshetikhin–Mukhin)

$\chi_q : K_0(Rep_{f.d.}(U_q(\hat{g})) \xrightarrow{\sim} \mathbb{C}[Y_j(a)|_{j \neq i}] \cap \mathbb{C}[Y_i(a)(1 + A_i^{-1}(a)]$

Example $\mathfrak{g} = \mathfrak{sl}_4$

① ϖ_1 $\quad \xi \quad K_1\xi = q\xi \quad K_2\xi = K_3\xi = \xi$

$P_1 = (z-a) \quad P_2 = P_3 = 1$

$$Y_1(a) \xrightarrow{A_1^{-1}(a)} Y_1(a)A_1^{-1}(a) = Y_1^{-1}(aq^2)Y_2(aq) \xrightarrow{A_2^{-1}(aq)} Y_1^{-1}(aq^2)Y_2(aq)A_2^{-1}(aq) = Y_2^{-1}(aq^3)Y_3(aq^2) \xrightarrow{A_3^{-1}(aq^2)} Y_3^{-1}(aq^4)$$

$$\chi_q\left(V_{\varpi_1}(a)\right) = Y_1(a) + Y_1^{-1}(aq^2)Y_2(aq) + Y_2^{-1}(aq^3)Y_3(aq^2) + Y_3^{-1}(aq^4)$$

Example $\mathfrak{g} = \mathfrak{sl}_4$

② ϖ_2

$P_1 = P_3 = 1$

$P_2 = z - a$

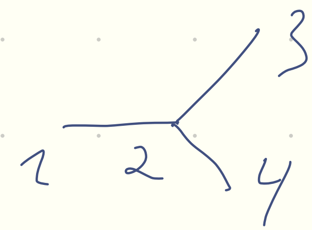
$$\begin{tikzcd}
& Y_2(a) \arrow[d, "A_2^{-1}(a)"] & \\
& Y_1(qa)\, Y_2^{-1}(q^2a)\, Y_3(qa) \arrow[dl, "A_1^{-1}(qa)"] \arrow[dr, "A_3^{-1}(qa)"] & \\
Y_1^{-1}(q^3a)\, Y_3(qa) \arrow[dr, "A_3^{-1}(qa)"'] & & Y_1(qa)\, Y_3^{-1}(q^3a) \arrow[dl, "A_1^{-1}(qa)", "1"'] \\
& Y_1^{-1}(aq^3)\, Y_2(aq^2)\, Y_3^{-1}(aq^3) \arrow[d, "A_2^{-1}(aq^2)"] & \\
& Y_2^{-1}(aq^4) &
\end{tikzcd}$$

Problems

- **<u>Problem</u>** For $\mathfrak{g} = \mathfrak{sl}_3$ find q-chacters and graphs of two different 8-dim reps
  (Two evaluations of adjoint rep)

- **<u>Problem</u>** For $\mathfrak{g} = \mathfrak{so}(8)$ find q-chacters and graphs of fundamental reps

D_4



<u>Hint</u> $V_{\varpi_1}(a)$, $V_{\varpi_3}(a)$, $V_{\varpi_4}(a)$ – are 8-dim as for $\mathfrak{so}(8)$

$V_{\varpi_2}(a)$ – is 29-dim, contrary 28-dim for $\mathfrak{so}(8)$

Screening operators

- <u>Remark</u> $Y(a)+Y^{-1}(aq^2)$ — similar to q-Virasoro

- Let $\tilde{\mathcal{Y}}_i = \bigoplus_{a\in\mathbb{C}^\times} \mathcal{Y}\, S_i(a)$, $\quad \mathcal{Y}_i = \tilde{\mathcal{Y}}_i \big/ \langle S_i(aq^2) - A_i(a) S_i(a) \rangle$

$S_i : \mathcal{Y} \to \mathcal{Y}_i$
screening operator

- $S_i(Y_j(a)) = \delta_{ij}\, Y_i(a)\, S_i(a)$
- Leibniz rule $S_i(BC) = C S_i(B) + B S_i(C)$

In particular $S_i(Y_j^{-1}(a)) = -\delta_{ij} Y_i^{-1}(a) S_i(a)$

- Reformulation of theorem above

<u>Thm</u> $\operatorname{Im} \chi_q = \bigcap_{i\in I} \operatorname{Ker} S_i$

<u>Problem</u> For $\mathfrak{g} = \mathfrak{sl}_2$ show that $\operatorname{Ker} S = \mathbb{C}[Y(a) + Y^{-1}(aq^2)]$

References

- Frenkel Reshetikhin The q-characters of representations quantum affine algebras and deformations of W algebras

- Frenkel Mukhin Combinatorics of q-characters of finite-dimensional representations of quantum affine algebras