Affine Quantum Groups

Lecture 8-9

F.d. Representations of $U_q(\widehat{sl}_2)$

Evaluation Representations

- $ev_u : \mathcal{U}_q(\hat{\mathfrak{sl}}_2) \longrightarrow \mathcal{U}_q(\mathfrak{sl}_2)$ $\quad E_1 \mapsto E \quad E_0 \mapsto uF$
  $\quad F_1 \mapsto F \quad F_0 \mapsto u^{-1}E$

- <u>Def</u> For V rep. of $\mathcal{U}_q(\mathfrak{sl}_2)$ let $V(u)$ be $ev_u : \mathcal{U}_q(\hat{\mathfrak{sl}}_2) \longrightarrow \mathcal{U}_q(\mathfrak{sl}_2) \longrightarrow End(V)$

- <u>Example</u> $\mathbb{C}^2(u)$

$$E_1 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F_1 \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad K_1 \mapsto \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

$$E_0 \mapsto \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} \qquad F_0 \mapsto \begin{pmatrix} 0 & u^{-1} \\ 0 & 0 \end{pmatrix} \qquad K_0 \mapsto \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$$

- Intertwiner $R : \mathbb{C}^2(u_1) \otimes_{\Delta} \mathbb{C}^2(u_2) \longrightarrow \mathbb{C}^2(u_1) \otimes_{\Delta^{op}} \mathbb{C}^2(u_2)$

(Basis

$V_+ \otimes V_+,\ V_+ \otimes V_-,\ V_- \otimes V_+,\ V_- \otimes V_-)$

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q(u_1-u_2)}{u_1-q^2u_2} & \frac{u_2(1-q^2)}{u_1-q^2u_2} & 0 \\ 0 & \frac{u_1(1-q^2)}{u_1-q^2u_2} & \frac{q(u_1-u_2)}{u_1-q^2u_2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R(u_1/u_2)R(u_2/u_1) = 1$$

$$\operatorname{Det} R(u_1/u_2) = \frac{q^2u_1 - u_2}{u_1 - q^2u_2}$$

- Problem $0 \to \mathbb{C} \to \mathbb{C}^2(u) \otimes \mathbb{C}^2(uq^2) \to \mathbb{C}^3(uq) \to 0$

$0 \to \mathbb{C}^3(uq) \to \mathbb{C}^2(uq^2) \otimes \mathbb{C}(u) \to \mathbb{C} \to 0$

otherwise $\mathbb{C}^2(u_1) \otimes \mathbb{C}^2(u_2)$ irreducible

- Remark Not s/s category
- Remark For $q=1$ $\mathbb{C}^2(u) \otimes \mathbb{C}^2(u) = \mathbb{C}^3(u) \oplus \mathbb{C}^1(u)$

Duality

- Def If V - f.d rep, V^* - dual space
  $$\rho^*(x) = \rho(S(x))^*$$
- Property : $\exists$ maps
  $$\exists\ V^* \otimes V \to \mathbb{C} \qquad \mathbb{C} \to V \otimes V^*$$
- In our case $(\mathbb{C}^2(u))^* = \mathbb{C}^2(uq^2)$

${}^{**}V$	${}^*V$	V	V^*	V^{**}	V^{***}
	$\mathbb{C}^2(uq^{-2})$	$\mathbb{C}^2(u)$	$\mathbb{C}^2(uq^2)$	$\mathbb{C}^2(uq^4)$	$\mathbb{C}^2(uq^6)$

• **Prop** For any evaluation representation $V(u)^{**} \simeq V(q^4 u)$

Pf For $U_q(\hat{sl}_2)$ – true

$$S^2(E_0) = S(-E_0K_0^{-1}) = -K_0(-E_0K_0^{-1}) = q^2E_0$$

$$S^2(F_1) = S(-K_1F_1) = K_1F_1K_1^{-1} = q^{-2}F_1$$

$$u = E_0/F_1 \longmapsto S^2(E_0)/S^2(F_1) = q^4u$$

$$u^{-1} = F_0/E_1 \longmapsto S^2(F_0)/S^2(E_1) = q^{-4}u^{-1} \qquad \square$$

• **Problem** $V(u)^* \simeq V(q^2u)$ for any evaluation rep $V(u)$

• **Problem** Any irred. (type I) rep. of $U_q(\hat{sl}_2)$ of dim 2 is isomorphic to $\mathbb{C}^2(u)$ for some u

ℓ- weights

- **Prop** V — f.d. rep of $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$
  then $K = K_0 K_1 \to \underline{\pm 1}$

Pf $[h[r], h[s]] = \frac{[2r]}{r} \frac{K^r - K^{-r}}{q - q^{-1}} \delta_{r+s} \Longrightarrow K^r - K^{-r} = 0$ ■

- Type I reps $V = \bigoplus_{\lambda = (\lambda_0, \lambda_1)} V_{(\lambda)}$, $\forall \xi \in V_{(\lambda)}$ $K_i = q^{\lambda_i} \xi$ $i = 0, 1$

  $\Longrightarrow$ $K = 1$ on type I. Assume this below

- **Def** ξ is ℓ-weight vector if
  ξ is eigenvector of $\psi^+(z)$ $\psi^-(z)$

- **Def** ξ is ℓ-highest weight vector
  if $X^+(z)\xi = 0$, ξ is eigenvector of $\psi^+(z)$, $\psi^-(z)$

Thm If V - irreducible f.d rep then $\exists$ (and ! up to multiple) ℓ-h.w. vector

$q \neq \sqrt[\#]{1}$

Pf $E_{1,1}, F_{1,1}, K_1$ — $U_q(sl_2)$ subalgebra

$V = \oplus V_{(m)}$ $\forall \xi \in V_{(m)}$ $K_1 \xi = q^m \xi$. Then $X^+(z)\xi \in V_{(m+2)}$

Let ℓ be max. s.t. $V_{(\ell)} \neq 0$. Let $\xi \in V_{(\ell)}$ then $X^+(z)\xi = 0$

Subspace $V_{(\ell)}$ is preserved by $\psi^{\pm}(z)$. Hence assume that ξ eigenvector of $\psi^+(z), \psi^-(z)$

$V = U_q(\hat{sl}_2)\xi = \langle X^- X^- \psi^- \dots \psi^{\pm} \dots \psi^{\pm} \dots X^+ \dots X^+ \xi \rangle$

PBW property $X^-[m_1] X^-[m_2] \dots X^-[m_k] \xi \in V_{(\ell - 2k)}$

hence $\dim V_{(\ell)} = 1$, hence ξ unique up multiple

If $\exists$ another ℓ-h.w. vector $\xi \in V_{(\ell-2m)}$ $m>0$

$\xi \notin \langle X^-[m_1] \cdots X^-[m_k]\xi' \rangle \Longrightarrow$ contradiction. $\square$

- <u>Problem</u> Let V_λ h.w. rep of $U_q(sl_2)$ with

$K\xi_\lambda = q^\lambda \xi_\lambda$, $E\xi_\lambda = 0$ Then for evaluation rep. $U_q(\widehat{sl_2})$

$$\psi^+(z)\xi_\lambda = \phi_\lambda(u,z)\xi_\lambda = \psi^-(z)\xi_\lambda \qquad \phi_\lambda(u,z) = q^\lambda \frac{z+uq^{-\lambda-2}}{z+uq^{\lambda-2}}$$

<u>Hint</u> ⓐ Sufficient to check only one formula, see Lemma Below.
ⓑ Some intermediate formulas

$$E_\delta \xi_\lambda = u\frac{q^{-\lambda-2}-q^{\lambda-2}}{q-q^{-1}}\xi_\lambda,$$ let $\xi_{\lambda-2} = F_1\xi_\lambda$ then

$$E_\delta \xi_{\lambda-2} = \frac{u(q^\lambda - q^{\lambda-2} - q^{\lambda-4} + q^{-\lambda-2})}{(q-q^{-1})}\xi_{\lambda-2}$$

• <u>Prop</u> Let V, V' – reps, $\xi \in V$, $\xi' \in V'$ ℓ-h.w. vectors

$$\psi^+(z)\xi = \psi^-(z)\xi = \phi(z)\xi, \quad \psi^+(z)\xi' = \psi^-(z)\xi' = \phi'(z)\xi',$$

Then $\psi^+(z)\xi\otimes\xi' = \psi^-(z)\xi\otimes\xi' = \phi(z)\phi'(z)(\xi\otimes\xi')$, $X^+(z)\xi\otimes\xi'$

<u>Remark</u> For coproduct Δ^D (for $K=1$)

$$\Delta^D X^+(z) = 1\otimes X'(z) + X^+(z)\otimes\psi^-(z)$$

$$\Delta^D \Psi^+(z) = \Psi^+(z)\otimes\psi^+(z) \qquad \Delta^D\Psi^-(z) = \Psi^-(z)\otimes\Psi(z)$$

this is trivial.

<u>Pf</u> • We have triangularity of Δ (for $K=1$)

$$\Delta(h[r]) = h[r]\otimes 1 + 1\otimes h[r] + \text{"low terms"}$$

"low terms" $\in \mathbb{C}\langle X^+[n], K_1\rangle \otimes \mathbb{C}\langle X^-[n], h[s], K_1\rangle$

each term contain at least one $X^+[n]$

Hence $\Delta(h[r])\xi\otimes\xi' = (h[r]\otimes 1 + 1\otimes h[r])\xi\otimes\xi'$ $\qquad r>0$

Similarly $\Delta(h[r])\xi\otimes\xi' = (h[r]\otimes 1 + 1\otimes h[r])\xi\otimes\xi' \quad r>0$

- We have

$$\Delta(X^+[n]) = 1\otimes X^+[n] + \mathbb{C}\langle X^+[m]\rangle \otimes \# \qquad n>0$$

Similarly $\Delta X^+[-n] = X^+[-n]\otimes 1 + 1\otimes X^-[n] + \mathbb{C}\langle X^+[-m]\rangle K^{\#}\otimes \#$

Hence $\Delta(X^+[n])\xi\otimes\xi' = 0 \qquad \forall n$ $\square$

- Example Recall for $V_\ell(u)$ ℓ-h.w $\phi_\ell(u,z) = q^\ell \dfrac{z+uq^{-\ell-2}}{z+uq^{\ell-2}}$

For $\mathbb{C}^2(u)$ ℓ-h.w $\phi_1(u) = q\dfrac{z+uq^{-3}}{z+uq^{-1}}$

$$\mathbb{C}^2(u)\otimes\mathbb{C}^2(uq^2) \rightsquigarrow \phi_1(u,z)\phi_1(uq^2,z) = q\frac{z+uq^{-3}}{z+uq^{-1}}\, q\frac{z+uq}{z+uq} = q^2\frac{z+uq^{-3}}{z+uq} = \underbrace{\phi_2(uq,z)}_{\mathbb{C}^3(u)}$$

Similarly for

$\mathbb{C}^2(u)\otimes\ldots\otimes\mathbb{C}(uq^{2(\ell-1)})$ has ℓ-h.w as $\phi_\ell(uq^{\ell-1}) \leftarrow \mathbb{C}^{\ell+1}(uq^{\ell-1})$

- Below we use $V_\ell(u) \rightsquigarrow V_\ell(a)$ for $\boxed{a = -uq^{\ell-2}}$

then ℓ-h.w $\phi_\ell(a,z) = q^\ell \dfrac{z - aq^{-2\ell}}{z-a}$

ℓ-h.v $V_1(a) \otimes V_1(aq^{-2}) \otimes \dots \otimes V_1(aq^{-2(\ell-1)}) = \ell$.h.w $V_\ell(a)$

ℓ.h.w $V_1(a_1) \otimes \dots \otimes V_1(a_\ell) = q^{-\ell} \dfrac{P(q^2 z)}{P(z)}$ where $P(z) = \prod(z-a_i)$

<u>Monic</u> polynomial $P(z)$ is called Drinfeld polynomial

- <u>Lemma</u> V- rep, ξ-ℓ.h.w. vector. Assume $\sum_{i=0}^n A_i X^-[i]\xi = 0$, for some n, $\{A_i\}$. Then $\psi^+(z)\xi = \psi^-(z)\xi = \phi(z)\xi$,

$\phi(z) = (\sum B_i z^{n-i}) / (\sum A_i z^{n-i})$ for some $\{B_i\}$

<u>Pf</u> $0 = X^+(z)(\sum A_i X^-[i]\xi) = \sum A_i z^{-i} (\psi^+(z) - \psi^-(z))(q-q^{-1})^{-1}\xi$

$\sum A_i z^{-i} \psi^+(z)\xi$ has terms $1, z^{-1}, z^{-2}, \ldots$

$\sum A_i z^{-i} \psi^-(z)\xi$ has terms $z^{-n}, z^{-n+1}, \ldots, z^{-1}, 1, z, \ldots$

Hence $\left(\sum A_i z^{-i}\right) \psi^+(z)\xi = \left(\sum A_i z^{-i}\right) \psi^-(z)\xi = \left(\sum B_i z^{-i}\right)\xi$ $\square$

- Remark This Lemma works for any f.d. rep V

Moreover it works for any rep with f.d. K_1 weight spaces, e.g. $V_\lambda(u)$

Main Theorem

- <u>Thm</u> (Chari Pressley) For any f.d. $U_q(\hat{sl}_2)$ rep ℓ-h.w $\phi = q^{-\ell} \dfrac{P(q^2 z)}{P(z)}$ where $P(z)$ is monic, $\deg P = \ell$.
The map $V \to P$ is bijection
Drinfeld polynomial

<u>Sketch of pf</u> ⓐ For any $P(z) = (z-a_1) \cdot\cdot\cdot (z-a_\ell)$ let V_P be irred. quotient in submodule generated by ℓ-h.w vector in $V_1(a_1) \otimes \cdots \otimes V_\ell(a_\ell)$ Then V_P is irred. with ℓ-h.w.
$\phi = q^{-\ell} P(q^2 z)/P(z)$

ⓑ For given ϕ there $\underline{\underline{\exists!}}$ irrep with ℓ-h.w ϕ

(c) Let V - f.d. irrep, ξ - ℓ-h.w. vector.
Let $K_1\xi = q^{\ell}\xi$. Then $\ell \in \mathbb{Z}_{\geq 0}$, $F_1^{\ell+1}\xi = 0$. Acting by $X^+[1]^{\ell}$ we get $\sum_{i=0}^{\ell} A_i X^-[i]\xi = 0$. Hence (by Lemma and $\psi^{\pm}(z) = K_1^{\pm} + O(z^{\mp 1})$) $\phi(\ell) = q^{-\ell}\prod_{i=1}^{\ell}(z-b_i)/\prod_{i=1}^{\ell}(z-a_i)$, $\prod b_i = q^{2\ell}\prod a_i$

(d) We can reorder a_i, b_i s.t product $V_{\lambda_1}(a_1)\otimes\ldots\otimes V_{\lambda_\ell}(b_\ell)$, where $q^{2\lambda_i} = a_i/b_i$ satisfy Thm. below, hence it is irred. By (b) $V \simeq V_{\lambda_1}(a_1)\otimes\ldots\otimes V_{\lambda_\ell}(b_\ell)$, $\Rightarrow$ $\forall \lambda_i \in \mathbb{Z}_{\geq 0}$ $\Rightarrow$ q.e.d $\square$

- <u>Remark</u> Another aproach to Thm:
Introduce $P_r \approx \frac{1}{[r]!^2} X^+[0]^r X^-[1]^r$ then $(\sum P_r z^r)\xi = P(z)\xi$

Irreducible tensor product

- <u>Thm</u> Consider tensor product $V = V_{\lambda_1}(a_1) \otimes \dots \otimes V_{\lambda_k}(a_k)$, let $b_i = a_i q^{-2\lambda_i}$. Assume that

if $a_i / b_j = q^{2\ell}$, $\ell \in \mathbb{Z}_{\geq 0}$, $i, j \geq k$ then $\lambda_k \in \mathbb{Z}_{\geq 0}$, $\lambda_k \leq \ell$

Hence V is irreducible

- <u>Def</u> <u>String</u> — finite geometric progression with ratio q^2, i.e. set like $\{b, bq^2, \dots, bq^{2k}\}$

Two strings S_1 S_2 are in <u>special position</u> if $S_1 \cup S_2$ - string, $S_1 \neq S_1 \cup S_2$, $S_2 \neq S_1 \cup S_2$

Otherwise strings S_1, S_2 are in <u>general position</u>.

Problem Any finite multiset in $\mathbb{C}^*$ can be uniquely presented as a union of strings pairwise in general position.

For $V_\ell(a)$ let string be $S_\ell(a) = \{\underbrace{aq^{2-2\ell}, \dots, aq^{-2}, a}_{\ell \text{ numbers}}\}$

(string — roots of Drinfeld polynomial)

Corol $V_{\ell_1}(a_1) \otimes \dots \otimes V_{\ell_k}(a_k)$ is irred. if strings are in general position

But

Rk Thm works for some order of factors but if $V \otimes W$ is irred. then $W \otimes V$ is also irred. since has the same ℓ-h.w and size.

- Chari, Pressley, Tarasov (for $Y(\mathfrak{sl}_2)$). We follow Molev.

- Pf of the Thm Induction. Step $K-1 \to K$.

$V = V_{\lambda_1}(a_1) \otimes V'$, V' - irreducible, $\xi' = \xi_{\lambda_2} \otimes \dots \otimes \xi_{\lambda_K}$

Lemma If ξ in V is ℓ-h.w vector then $\xi = \xi_{\lambda_1} \otimes \xi$

Pf $\xi = \sum_{j=0}^{p} X^-[0]^j \xi_{\lambda_1} \otimes \xi_j'$ ℓ-h.w vector

$\Delta X^+[n] = 1 \otimes X^+[n] + \mathbb{C}\langle X^+[m] \mid m \le n \rangle \otimes \dots$ Hence $\xi_p' = \xi'$ -

next term $\left(\sum_{n,m\ge 0} X^+[n] \otimes \psi^+[m] z^{-n-m}\right) X^-[0]^p \xi_{\lambda_1} \otimes \xi_p' + \left(1 \otimes \sum_{n\ge 0} X^+[n] z^{-n}\right) X^-[0]^{p-1} \xi_{\lambda_1} \otimes \xi_p'$

$$\prod_{i=2}^{K} \frac{z-b_i}{z-a_i} \frac{\#}{z-a_1 q^{-2p}} X^-[c]^{p-1} \xi_{\lambda_1} \otimes \xi_p' + \sum \frac{\#}{z-a_i} X^-[c]^{p-1} \xi_{\lambda_1} \otimes \xi_p'$$

due to assumption, the pole $z-a_1 q^{-2p}$ does <u>not</u> cancel $\square$

Hence the only submodule of V is generated by $\xi = \xi_{\lambda_1} \otimes \xi'$. If it is not V hence $\exists$ submodule in $V^{*\varkappa}$ that <u>does not</u> contain ξ^*

Here $\varkappa$ is <u>antiautorphism</u> s.t.

$\varkappa(E_i) = K_i F_i$, $\varkappa(K_i) = K_i$, $\varkappa(F_i) = E_i K_i^{-1}$

<u>Lemma</u> (a) Antiautomorphism $\varkappa$ is well defined

(b) $\Delta \circ \varkappa = (\varkappa \otimes \varkappa) \circ \Delta$ (c.f. antipode S, $S \neq \varkappa$!)

(c) $V_\lambda(a)^* = V_\lambda(q^{-6} b^{-1})$ (as before $b = a q^{-2\lambda}$)

(d) $V^* = V_{\lambda_1}(q^{-6} b_1^{-1}) \otimes V_{\lambda_2}(q^{-6} b_2^{-1}) \otimes \dots \otimes V_{\lambda_k}(q^{-6} b_k^{-1})$

Hence V^* satisfy assumption of Thm $\Rightarrow$ the only ℓ-h.w vector in V^* is ξ^* $\Rightarrow$ V, V^* irred $\square$

q-characters

- V - f.d. (type I) rep $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$

$V = \bigoplus V_{(\phi)}$, where $V_{(\phi)}$ (generalized) eigspaces

$\forall \xi \in V_{(\phi)}$, $\exists d$ $(\psi_+(z) - \phi(z))^d \xi = (\psi_-(z) - \phi(z))^d \xi = 0$

- If $K_1 \xi = q^m \xi \Rightarrow \lim_{z\to\infty} \phi(z) = q^\ell$, $\lim_{z\to 0} \phi(z) = q^{-\ell}$

Usually assume $\dim V_{(\phi)} = 1$

under this assumption, for $\xi \in V_{(\phi)}$, $\xi' \in V_{\phi'}$ and $X \in \mathcal{U}_q(\hat{\mathfrak{g}})$ the matrix element $X: \xi \mapsto \xi'$ is well defined. We denote it $\langle \xi', X\xi \rangle$.

- **Lemma** Let $\xi \in V_{(\phi)}$, $\xi' \in V_{\phi'}$ s.t $\langle \xi', X^-[n]\xi \rangle = a^n \langle \xi', X^-[0]\xi \rangle$

Then $\phi'(z) = \phi(z)\left(q^{-2}\dfrac{z - aq^{2}}{z - aq^{-2}}\right)$

Pf $\quad h[m]\,X^-[0]\xi = X^-[0]\,h[m]\xi - \dfrac{q^{2m} - q^{-2m}}{m(q - q^{-1})} X^-[m]\xi =$

$$= X^-[0]\,h[m]\xi + \frac{(aq^{-2})^m - (aq^{2})^m}{m(q-q^{-1})} X^-[0]\xi$$

$$q^{-2}\exp\left((q-q^{-1}) \sum \frac{(aq^{-2})^m - (aq^{2})^m}{m(q-q^{-1})} z^{-m}\right) = q^{-2}\frac{z - aq^{2}}{z - aq^{-2}} \qquad \square$$

Problem Under the assumptions of Lemma

(a) $\langle \xi, X^+[n]\xi' \rangle = a^n \langle \xi, X^+[0]\xi' \rangle$ $\qquad$ (b)* $\langle \xi, X^+[0]\xi' \rangle \langle \xi', X^-[0]\xi \rangle = \dfrac{\mathrm{Res}_{z=a}\,\phi(z)}{a(q-q^{-1})}$

- This lemma applies for evaluation reps $V_\ell(u)$

$$\langle \xi', X^-[1]\xi \rangle = \langle \xi', -K_1 E_0 \xi \rangle = -u \langle \xi', K_1 X^-[0]\xi \rangle$$

Hence $\langle \xi', X^-[n]\xi \rangle = (-u)^n \langle \xi', K_1^n X^-[0]\xi \rangle$

If $\xi = \xi_m$, then $\xi' = \xi_{m-2}$ and

$\langle \xi_{m-2}, X^-[n]\xi_m \rangle = (-uq^{m-2})^n \langle \xi_{m-2}, X^-[0]\xi_m \rangle = (aq^{m-e})^n \langle \xi_{m-2}, X^-[0]\xi_m \rangle$

- <u>Def</u> If $\phi(z) = q^{-\ell} \dfrac{R(q^2 z)}{R(z)}$, $\quad R(z) = \prod_{i=1}^{\ell_+}(z-a_i) \Big/ \prod_{i=1}^{\ell_-}(z-b_i)$

Then $Y_\phi = \prod_{i=1}^{\ell_+} Y(a_i) \prod_{i=1}^{\ell_-} Y^{-1}(b_i)$

<u>q character</u> of V: $\quad \chi_q(V) = \sum_\phi \dim V_\phi \, Y_\phi$

$\chi_q : K_0(Rep(U_q(\widehat{sl_2}))) \to \mathbb{C}[Y] = \mathbb{C}[Y^{\pm 1}(a)] \quad a \in \mathbb{C}^*$

- <u>Remarks</u>

(a) For f.d rep ℓ-h.w has the form $q^{-\ell} P(zq^2)/P(z)$

Hence the corresponding $Y_\phi = \prod Y(a_i)$

For $V_\ell(a)$ ℓ-h.w $\phi(z) = q^\ell \dfrac{z - q^{-2\ell}a}{z-a}$ hence $Y_\phi = Y(a)Y(aq^{-2})\dots Y(aq^{-2(\ell-1)})$

(b) If $\langle \xi', X^-[n]\xi\rangle = a^n \langle \xi', X^-[0]\xi\rangle$ then $Y_{\phi'} = Y_\phi A_a^{-1}$ where

$A(a) = Y(a)Y(aq^2)$. Y – "fund. weight", A – "simple root"

(c) For $V_\ell(a)$ we have

$$\chi_q(V_\ell(a)) = Y(a)Y(aq^{-2})\dots Y(aq^{-2(\ell-1)})(1 + A^{-1}(a)(1 + A^{-1}(aq^{-2})(1 + \dots A(aq^{-2(\ell-1)})\dots)$$

cf cluster mutations.

Graph $\xrightarrow{A^{-1}(a)} \xrightarrow{A^{-1}(aq^{-2})} \cdots \xrightarrow{A^{-1}(aq^{-2(\ell-1)})}$

For example

$$\chi_q(V_1(a)) = Y(a) + Y^{-1}(aq^2)$$

$$\chi_q(V_2(a)) = Y(a)Y(aq^{-2}) + Y^{-1}(aq^2)Y(aq^{-2}) + Y^{-1}(aq^2)Y^{-1}(a)$$

$$\chi_q(V_3(a)) = Y(a)Y(aq^{-2})Y(aq^{-4}) + Y^{-1}(aq^2)Y(aq^{-2})Y(aq^{-4})$$
$$+ Y^{-1}(aq^2)Y^{-1}(a)Y(aq^{-4}) + Y^{-1}(aq^2)Y^{-1}(a)Y^{-1}(aq^{-2})$$

(d) q-character is not trace of $\psi^{\pm}(z)$

Example $V_1(u)$ $\quad \psi^{\pm}(z) = \begin{pmatrix} q\frac{z-q^{-2}a}{(z-a)} & 0 \\ 0 & \frac{z-q^{2}a}{q(z-a)} \end{pmatrix}$ $\quad \mathrm{tr}\,\psi^{\pm}(z) = q+q^{-1}$

we lost a!

(e) Condition $\psi^{+}(z)\xi = \phi(z)\xi$, where $\phi(z) = q^{-\ell}R(q^{2}z)/R(z)$,

$R(z) = \prod_{i=1}^{\ell_+}(z-a_i)\Big/\prod_{i=1}^{\ell_-}(z-b_i)$, $\quad \ell = \ell_+ - \ell_-$

Equivalent $\quad h[m]\xi = q^{-m}\frac{[m]}{m}\left(\sum a_i^m - \sum b_i^m\right)\xi$

(f) For $0 \to V' \to V \to V'' \to 0$ we have $\chi_q(V) = \chi_q(V') + \chi_q(V'')$

Hence χ_q is defined on $K_0(Rep_{f.d.}(U_q(\hat{sl}_2)))$

(g) q-character is refinement of character

The diagram is commutative

$$\begin{tikzcd} K_0(Rep(U_q(\hat{sl}_2))) \arrow[r, "\chi_q"] \arrow[d, "\text{Restriction}"'] & \mathbb{C}[Y] \arrow[d] \\ K_0(Rep(U_q(sl_2))) \arrow[r, "\chi"] & \mathbb{C}[y^{\pm 1}] \end{tikzcd} \qquad \begin{array}{c} Y(a) \\ \downarrow \\ y \end{array}$$

Multiplicativty

<u>Thm</u> $\chi_q(V\otimes V') = \chi_q(V)\otimes\chi_q(V')$

<u>Pf</u> Let $\xi_1, \dots, \xi_N$, be l-w. basis of V, ordered s.t.

- $i<j$, $K_1\xi_i = q^{m_i}\xi_i$, $K_1\xi_j = q^{m_j}\xi_j \Rightarrow m_i \geq m_j$.
- $(\psi_+(z)-\phi(z))\xi_j$ contain $\xi_i \Rightarrow i<j$

In particular ξ_1 - l. h.w vector

Let $\xi_1', \dots, \xi_{N'}'$ similar basis of V'.

Basis $\xi_1\otimes\xi_1', \xi_1\otimes\xi_2', \dots, \xi_1\otimes\xi_{N'}', \xi_2\otimes\xi_1', \xi_2\otimes\xi_2', \dots, \xi_2\otimes\xi_{N'}', \dots, \xi_N\otimes\xi_{N'}'$

Triangularity $\Delta(h[r]) = h[r] \otimes 1 + 1 \otimes h[r] +$ "low terms"

"low terms" $\in \mathbb{C}\langle X^+[n], K_1 \rangle \otimes \mathbb{C}\langle X^-[n], h[s], K_1 \rangle$

Hence $h[r]$ is triangular.

In particular $\zeta_1 \otimes \zeta_1'$ are l-w vectors

$$
\begin{array}{ccccc}
\zeta_1 \otimes \zeta_1' & \cdots \; \zeta_1 \otimes \zeta_N' & \zeta_2 \otimes \zeta_1' & \cdots & \zeta_2 \otimes \zeta_N' \\
\hline
\end{array}
$$

$$
\begin{pmatrix}
* & 0 & & 0 & \\
0 & * & & & \ddots \\
 & & * & 0 & 0 \\
 & & & & 0 \\
 & & & & *
\end{pmatrix}
$$

- Remark For new Drinfeld coproduct basis $\xi_i \otimes \xi_j'$ is ℓ-w basis.
Triangular matrix from ℓ-w bases $V \otimes_{\Delta} V' \longrightarrow V \otimes_{\Delta^D} V'$
geometrically Okounkov stable envelope matrix

- Corol Any f.d. rep of $U_q(\hat{sl}_2)$ has ℓ-w. basis with $\phi(z) = q^{-\ell} R(q^2 z)/R(z)$

Pf We know for evaluation reps $V_\ell(a)$
Hence for tensor products.
Any irrep is tensor product of eval. reps $\square$

- Corol For f.d. rep V_P with Drinfeld polynomial $P(z) = \prod_{i=1}^{\ell} (z - a_i)$ we have $\chi_q(V_P) = \prod Y(a_i)(1 + \sum M)$, where any M is monomial in A^{-1}. In particular any M has negative Y degree.

- **Th** $K_0(\mathrm{Rep}_{f.d}\, U_q(\hat{sl}_2)) = \mathbb{C}[Y(a)+Y^{-1}(aq^2)] \subset \mathbb{C}[Y]$

Pf χ_q is ring homomorphism.

$K_0(\mathrm{Rep}_{f.d}\, U_q(\hat{sl}_2))$ generated by classes of $V_1(u)$
hence $\mathrm{Im}\,\chi_q$ generated by $\chi_q(V_1(a)) = Y(a)+Y^{-1}(aq^2)$

$K_0(\mathrm{Rep}_{f.d}\, U_q(\hat{sl}_2))$ has basis V_P, where $P(z)=\prod_{i=1}^{e}(z-a_i)$ — Drinfeld polynomial. $\chi_q(V_P) = \prod Y(a_i)$ + lower terms. $\prod Y(a_i)$ are linearly independent in $\mathbb{C}[Y]$ hence χ_q is embedding $\square$

- **Corol** For any V, V' we have $V\otimes V' = V'\otimes V$ in $K_0(\mathrm{Rep}_{f.d}(U_q(\hat{sl}_2))$

- **Def** Monomial is called <u>dominant</u> if it does not contain variables $Y_i^{-1}(a)$

Remark (a) If V - f.d. rep, $\xi \in V_\phi$ l.h.w. vector. Then the corresponding monomial Y_ϕ is dominant

(b) If V reducible, then $\chi_q(V)$ contains more then one dominant monom. Opposite is not true.

- Problem* Find irred. f.d. V s.t. $\chi_q(V)$ contains more then one dominant term.

Hint Take some $V(S_1)\otimes V(S_2)$ for S_1, S_2 in general position

- For any string S let $V(S)$ corresponding f.d. rep

E.g. $S=\{a\} \Rightarrow V(S)=V_1(a)$, $S=\{a, aq^{-2}\} \Rightarrow V(S)=V_2(a)$, $S=\phi \Rightarrow V(S)=\mathbb{C}$

Problem For strings S_1, S_2 in special position let $S_3=S_1\cup S_2$, $S_4=S_1\cap S_2$ $\overline{S_4}=S_4\cup\{$two nearest neighbors$\}$, $S_3\setminus\overline{S_4}=S_5\sqcup S_6$.

Then in K_0 we have $V(S_1)\otimes V(S_2)=V(S_3)\otimes V(S_4)+V(S_5)\otimes V(S_6)$

- **<u>Problem</u>*** Let $S=\{a, aq^{-2}, \dots, aq^{-2\ell}\}$. Then $V(S')$ for $S'\subset S$ in $K_0(\mathrm{Rep}_{f.d.})$ satisfies relations of cluster algebra A_ℓ, the only frozen variable $V_\ell(a)$.

<u>Hint</u> Variables for A_ℓ cluster algebra can be identified with diagonals of $\ell+3$-gon, frozen variables are sides of $\ell+3$-gon

- <u>Remark</u> (Leclerc-Hernandez) Let $\mathcal{C}_\ell$ be a full subcat. of $\mathrm{Rep}_{f.d.}$ whose objects satisfy
  - Every composition factor has the form $V(S_1)\otimes\dots\otimes V(S_k)$, $S_i\subset S$

Clearly $\mathcal{C}_\ell$ depend only on ℓ up to isomorphism.

Then $\mathcal{C}_\ell$ is monoidal category, $K_0(\mathcal{C}_\ell)$ - cluster algebra of type A_ℓ.

References

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- Etingof Semenyakin A brief introduction to quantum groups Sec 5

- Molev Yangians and classical Lie algebras

- Hernandez Leclèrc Quantum affine algebras and cluster algebras