Affine Quantum Groups

Lecture 6

Factorization of R-matrix

Drinfeld-Jimbo presentation

- $\boxed{U_q(\hat{sl}_2)}$ Generators $E_0, E_1, K_0, K_1, F_0, F_1$ Cartan matrix $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

Relations $[E_i, F_j] = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q - q^{-1}}$

$$K_i E_j = q^{a_{ij}} E_j K_i, \quad K_i F_j = q^{-a_{ij}} F_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i$$

$$E_i^3 E_j - (q^{-2}+1+q^2) E_i^2 E_j E_i + (q^{-2}+1+q^2) E_i E_j E_i^2 - E_j E_i^3 = 0$$

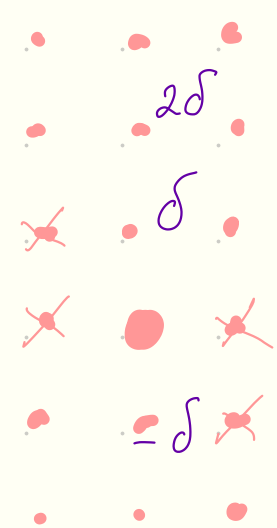
$$F_i^3 F_j - (q^{-2}+1+q^2) F_i^2 F_j F_i + (q^{-2}+1+q^2) F_i F_j F_i^2 - F_j F_i^3 = 0$$

$(i \neq j)$

- $\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta K_i = K_i \otimes K_i,$

$\Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i$

- $K = K_1 K_0$ — is central



- We know E_β for $\beta \in \Phi^+$ PBW theorem
  $F_{-\beta}$
- We know — new Drinfeld realization
- Goal — study coproduct
  want — universal R-matrix.
  many coproducts

Coproduct and Braid group

- For any root β define

$$\bar{R}_\beta = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E_\beta^n \otimes F_{-\beta}^n \qquad \bar{R}_\beta^{-1} = \sum_{n\geq 0} (-1)^n q^{-\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E_\beta^n \otimes F_\beta^n$$

- <u>Thm</u> $\bar{R}_i \, \Delta(x) \, \bar{R}_i^{-1} = (T_i^{-1} \otimes T_i^{-1}) \Delta(T_i(x)) = \Delta^{s_i}$

  here $\bar{R}_i = \bar{R}_{\alpha_i}$

<u>Pf</u> Same as in f.d case, actually follows from rank 1, 2 $\square$

- For any $w \in W^{ae} \rightsquigarrow$ coproduct Δ^w conjugated by T_w

- Using thm we can compute $\Delta(E_\beta)$, $\Delta(F_{-\beta})$

Order on Φ^+

- PBW order

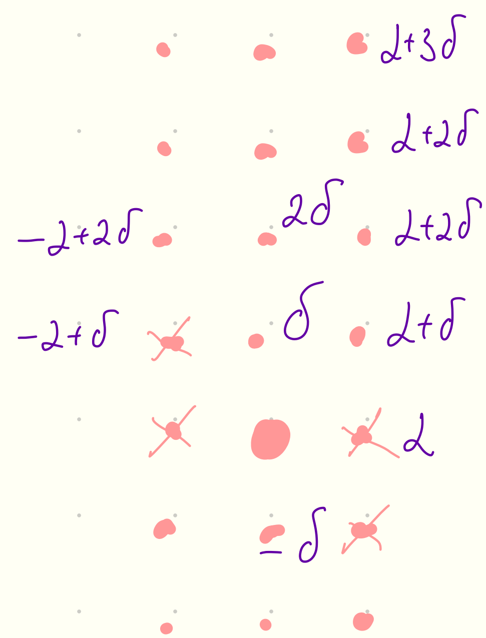
$$-\alpha+\delta < -\alpha+2\delta < \dots < \delta, 2\delta, 3\delta < \dots < \alpha+2\delta < \alpha+\delta < \alpha$$

$0 \quad -1 \quad -2 \qquad\qquad 3 \quad 2 \quad 1$

- <u>Th</u> (Levendorskii-Soibelman formula) For $\beta_1 < \beta_2$

$$E_{\beta_1}E_{\beta_2} - q^{(\alpha,\beta)} E_{\beta_2}E_{\beta_1} \in \mathbb{C}\langle E_\gamma \mid \beta_1 < \gamma < \beta_2 \rangle$$

<u>Pf</u> Case by case



- $\beta_1, \beta_2 \in \{\alpha + n\delta\}$
- $\beta_1, \beta_2 \in \{-\alpha + n\delta\}$
- $\beta_1 \in \{-\alpha+n\delta\}, \quad \beta_2 \in \{\alpha+n\delta\}$
- $\beta_1 \in \{n\delta\}$ or $\beta_2 \in \{n\delta\}$ $\square$

- Introduce subsets

$$\Phi^{+\infty} = \{\alpha + n\delta \mid n \geq 0\}, \quad \Phi^{-\infty} = \{-\alpha + n\delta \mid n > 0\}, \quad \Phi^{im} = \{n\delta \mid n > 0\}$$

$$\Phi^{+k} = \{\alpha + n\delta \mid 0 \leq n < k-1\} \quad \Phi^{-k+1} = \{-\alpha + n\delta \mid 0 < n < k\} \quad \underline{k \geq 1}$$

$$\Phi^{+\overline{\infty}} = \Phi^{+\infty} \sqcup \Phi^{im} \qquad \Phi^{-\overline{\infty}} = \Phi^{-\infty} \sqcup \Phi^{im}$$

In particular $\Phi^{+1} = \emptyset$, $\Phi^{+2} = \{\alpha\}$, $\Phi^{+3} = \{\alpha, \alpha+\delta\} \ldots$

$\Phi^{0} = \{\emptyset\}$, $\Phi^{-1} = \{-\alpha+\delta\}$, $\Phi^{-2} = \{-\alpha+\delta, -\alpha+2\delta\} \ldots$

- Let $U^+(k)$ – subalgebra U^+ generated by E_β, $\beta \in \Phi^k$

$U^-(k)$ — // — U^- —//— F_β —//—

$U^+(im)$, $U^-(im)$, $U^+(+\infty)$, $U^-(-\infty)$ —//—

Coproduct of real roots

- $E_{\alpha+n\delta}$, $n\geq 0$

<u>Example</u> $E_{\alpha+\delta}=(\tau T_1)^{-1}E_1=T_1^{-1}E_0$

$$\Delta(E_{\alpha+\delta})=\Delta(T_1^{-1}E_0)=\bar{R}_1^{-1}\,T_1^{-1}\otimes T_1^{-1}(E_0\otimes K_0+1\otimes E_0)\bar{R}_1$$

$$=\bar{R}_1^{-1}(E_{\alpha+\delta}\otimes K_{\alpha+\delta}+1\otimes E_{\alpha+\delta})\bar{R}_1=\sum_{n,m}\#_{n,m}\big(E_\alpha^n E_{\alpha+\delta}E_\alpha^m\otimes F_\alpha^n K_{\alpha+\delta}F_\alpha^m+$$

$$+E_\alpha^{n+m}\otimes F_\alpha^n E_{\alpha+\delta}F_\alpha^m\big)=\left|\begin{array}{l}\text{Using}\\ R_1^{-1}R_1=1\end{array}\right|=E_{\alpha+\delta}\otimes K_{\alpha+\delta}+1\otimes E_{\alpha+\delta}$$

$$+\sum_{n,m\geq 0}\#_{n,m}\big([E_\alpha^n\otimes F_\alpha^n,E_{\alpha+\delta}\otimes K_{\alpha+\delta}]E_\alpha^m\otimes F_\alpha^m+E_\alpha^{n+m}\otimes[F_\alpha^n,E_{\alpha+\delta}]F_\alpha^m=$$

$$=\left|\begin{array}{l}\text{Using } E_\alpha E_{\alpha+\delta}=q^{-2}E_{\alpha+\delta}E_\alpha,\ F_\alpha K_{\alpha+\delta}=q^2K_{\alpha+\delta}F_\alpha \Rightarrow [E_\alpha\otimes F_\alpha,E_{\alpha+\delta}\otimes K_{\alpha+\delta}]=0\\ \quad + \Delta E_{\alpha+\delta}\in\mathcal{U}^+\otimes\mathcal{U}^+\mathcal{U}^0, \text{ since } E_{\alpha+\delta}\in\mathcal{U}^+=\mathcal{U}(n^+)\end{array}\right|$$

$$=E_{\alpha+\delta}\otimes K_{\alpha+\delta}+1\otimes E_{\alpha+\delta}+\text{low terms}$$

low terms $\in\mathbb{C}[E_\alpha]\otimes\mathcal{U}^+\mathcal{U}^0=\mathcal{U}^+(\alpha)\otimes\mathcal{U}^+\mathcal{U}^0$

<u>Lemma</u> $\Delta(E_{\alpha+n\delta}) = E_{\alpha+n\delta} \otimes K_{\alpha+n\delta} + 1 \otimes E_{\alpha+n\delta}$ + "low terms"

"low terms" $\in \mathbb{C}\langle E_\alpha, E_{\alpha+\delta}, \dots E_{\alpha+(n-1)\delta} \rangle \otimes \mathcal{U}^+ \mathcal{U}^0 = \mathcal{U}^+(n+1) \otimes \mathcal{U}^+ \mathcal{U}^0$

<u>Pf</u> $E_{\alpha+n\delta} = (\tau T_1)^{-n} E_1 = \underbrace{T_1^{-1} T_0^{-1} \cdots T_i^{-1}}_{n} E_{1-i}$ where $i = n \bmod 2$

$$\Delta(E_{\alpha+n\delta}) = \Delta(T_1^{-1} T_0^{-1} \cdots T_i^{-1} E_{1-i}) = \bar{R}_1^{-1} T_1^{-1} \otimes T_1^{-1} (\bar{R}_0^{-1} T_0^{-1} \otimes T_0^{-1} (\cdots T_i^{-1} \otimes T_i^{-1} (\Delta(E_{1-i}) \cdots) \bar{R}_0) \bar{R}_1$$

$$= \bar{R}_\alpha^{-1} \bar{R}_{\alpha+\delta}^{-1} \cdots \bar{R}_{\alpha+(n-1)\delta}^{-1} (E_{\alpha+n\delta} \otimes K_{\alpha+n\delta} + 1 \otimes E_{\alpha+n\delta}) \bar{R}_{\alpha+(n-1)\delta} \cdots \bar{R}_{\alpha+\delta} \bar{R}_\alpha$$

$=$ | Using $[E_{\alpha+k\delta} \otimes F_{\alpha+k\delta}, E_{\alpha+n\delta} \otimes K_{\alpha+n\delta}] \in \mathbb{C}(E_{\alpha+\ell\delta} \mid k<\ell<n) \otimes F_{\alpha+k\delta} K_{\alpha+n\delta}$ + $\bar{R}_\beta^{-1} \bar{R}_\beta = 1$ |

Levendorskii-Soibelman formula

$= E_{\alpha+n\delta} \otimes K_{\alpha+n\delta} + 1 \otimes E_{\alpha+n\delta}$ + "low terms" $\square$

- Similarly for other root vectors

<u>Lemma</u> ⓐ $\Delta(E_{-\alpha+n\delta}) = E_{-\alpha+n\delta} \otimes K_{-\alpha+n\delta} + 1 \otimes E_{-\alpha+n\delta} +$ "low terms" $\quad n \geq 1$

"low terms" $\in \mathcal{U}^+ \otimes \mathbb{C}\langle E_{-\alpha+\delta}, \dots, E_{-\alpha+(n-1)\delta} \rangle \mathcal{U}^0 = \mathcal{U}^+ \otimes \mathcal{U}^+(-n+1)\, \mathcal{U}^0$

ⓑ $\Delta(F_{\alpha-n\delta}) = F_{\alpha-n\delta} \otimes 1 + K_{\alpha-2\delta} \otimes F_{\alpha-n\delta} +$ "low terms" $\quad n \geq 1$

"low terms" $\in \mathbb{C}\langle F_{\alpha-\delta}, \dots, F_{\alpha-(n-1)\delta} \rangle \mathcal{U}^0 \otimes \mathcal{U}^- = \mathcal{U}^-(-n+1)\, \mathcal{U}^0 \otimes \mathcal{U}^-$

ⓒ $\Delta(F_{-\alpha-n\delta}) = F_{-\alpha-n\delta} \otimes 1 + K_{-\alpha-n\delta} \otimes F_{-\alpha-n\delta} +$ "low terms" $\quad n \geq 0$

"low terms" $\in \mathcal{U}^- \mathcal{U}^0 \otimes \mathbb{C}\langle F_{-\alpha}, \dots, F_{-\alpha-(n-1)\delta} \rangle = \mathcal{U}^- \mathcal{U}^0 \otimes \mathcal{U}^-(n+1)$

<u>Problem</u> Prove any one of the formulas ⓐ, ⓑ, ⓒ

<u>Hint</u> 1 method — as above

2 method — Induction + commutators with $E_\delta, F_{-\delta}$

coproduct of imaginary roots

- Example

$$\Delta(E_\delta) = \Delta(E_0E_1 - q^{-2}E_1E_0)$$

$$= (E_0\otimes K_0 + 1\otimes E_0)(E_1\otimes K_1 + 1\otimes E_1) - q^{-2}(E_1\otimes K_1 + 1\otimes E_1)(E_0\otimes K_0 + 1\otimes E_0)$$

$$= E_\delta\otimes K + (E_0\otimes K_0E_1 - q^{-2}E_0\otimes E_1K_0) + (E_1\otimes E_0K_1 - q^{-2}E_1\otimes K_1E_0) - 1\otimes E_\delta$$

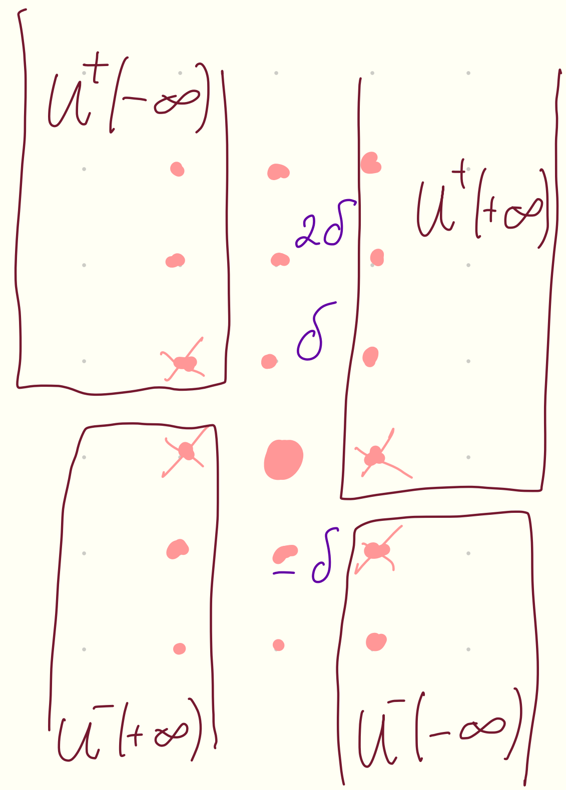
$$= E_\delta\otimes K + (q^2 - q^{-2})E_1\otimes K_1E_0 + 1\otimes E_\delta \in \mathcal{U}^{+}(+\infty)\otimes\mathcal{U}^{+}(-\infty)\,\mathcal{U}^0$$

<u>Lemma</u> ⓐ $\Delta(h_r) = h_r\otimes K_{r\delta} + 1\otimes h_r +$ "low terms"

"low terms" $\in \mathcal{U}^{+}(+\infty)\otimes\mathcal{U}^{+}(-\infty)\,\mathcal{U}^0$

ⓑ $\Delta(h_{-r}) = h_r\otimes 1 + K_{r\delta}^{-1}\otimes h_r +$ "low terms"

"low terms" $\in \mathcal{U}^{-}(-\infty)\,\mathcal{U}^0\otimes\mathcal{U}^{-}(+\infty)$



Duality

- Recall order

$$-\alpha+\delta < -\alpha+2\delta < \dots < \delta, 2\delta, 3\delta < \dots < \alpha+2\delta < \alpha+\delta < \alpha$$

$0 \quad -1 \quad -2 \qquad\qquad 3, \quad 2 \quad 1$

Let $\bar{E}_\gamma = E_\gamma$, $\bar{F}_\gamma = F_\gamma$ for $\gamma \in \Phi^{re}$, $\bar{E}_{r\delta} = h_r$, $\bar{F}_{r\delta} = h_{-r}$

- <u>Lemma</u> For $\beta_1 \le \beta_2 \le \dots \le \beta_k$, $\langle \bar{F}_{-\gamma}, \bar{E}_{\beta_1} \bar{E}_{\beta_2} \cdots \bar{E}_{\beta_k} \rangle = 0$
unless $k=1$, $\beta_1 = \gamma$

<u>Pf</u> In order to have nonzero $\sum \beta_i = \gamma$

- Let $\gamma = -\alpha + n\delta$ or $n\delta$. Then $\Delta(F_\gamma) \in U^-(-\infty) U^0 \otimes U^-$

If $k>1$ then $\beta_k = \alpha + p\delta$ or $\beta_k = p\delta$ Then

$$\langle F_{-\gamma}, E_{\beta_1} E_{\beta_2} \cdots E_{\beta_k} \rangle = \langle \Delta(F_{-\gamma}), E_{\beta_k} \otimes E_{\beta_1} \cdots E_{\beta_{k-1}} \rangle = 0$$

- Let $\gamma = \alpha + n\delta$. Then $\Delta(F_\gamma) \in U^- U^0 \otimes U^-(\infty)$

If $k>1$ then, $\beta_1 = -\alpha + p\delta$ or $\beta_1 = p\delta \Rightarrow \langle \cdots \rangle = 0$ $\square$

Theorem $\langle F_{\alpha-\delta}^{a_1} F_{\alpha-2\delta}^{a_2} \cdots F_{\delta}^{b_1} F_{2\delta}^{b_2} \cdots F_{-\alpha-\delta}^{c_2} F_{-\alpha}^{c_1}, E_{-\alpha+\delta}^{a_1'} E_{-\alpha+2\delta}^{a_2'} \cdots E_{\delta}^{b_1'} E_{2\delta}^{b_2'} \cdots E_{\alpha+\delta}^{c_2'} E_{\alpha}^{c_1'} \rangle =$

$$= \prod \delta_{a_n,a_n'} \frac{[a_n]!}{q^{\binom{a_n}{2}}} \langle F_{\alpha-n\delta}, E_{-\alpha+n\delta} \rangle^{a_n} \delta_{b_n,b_n'} b_n! \langle F_{-n\delta}, E_{n\delta} \rangle^{b_n} \delta_{c_n,c_n'} \frac{[c_n]!}{q^{\binom{c_n}{2}}} \langle F_{-\alpha-n\delta}, E_{\alpha+n\delta} \rangle^{c_n}$$

Pf $\langle \cdots, \cdots \rangle = \langle F_{\alpha-\delta} \otimes F_{\alpha-\delta}^{a_1-1} \cdots, \prod \Delta(E_\beta)^{\#} \rangle$

From Lemma have to find $E_{-\alpha+\delta}$ on first factor

$\Delta E_{\alpha+n\delta}, E_{n\delta} \in U^+(\overline{\infty}) \otimes U^+$ $\Rightarrow$ no $E_{-\alpha+\delta}$ on first factor

$\Delta E_{-\alpha+n\delta} - E_{-\alpha+n\delta} \otimes K_{-\alpha+n\delta} + 1 \otimes E_{-\alpha+n\delta} + U^+U^0 \otimes U^+(-\infty)$
no weight space $\alpha-\delta$ on first factor

Hence only important terms are

$$\Delta(E_{-\alpha+\delta}^{a_1'}) = (E_{-\alpha+\delta} \otimes K_{-\alpha+\delta} + 1 \otimes E_{-\alpha+\delta})^{a_1'} = \dots + q^{-a_1'+1}[a_1] E_{-\alpha+\delta} \otimes K_{-\alpha+\delta} E_{-\alpha+\delta}^{a_1'-1} + \dots$$

Then by induction $\square$

Pairing of root vectors

- Example $\langle F_{-\delta}, E_\delta\rangle = \langle F_1 F_0 - q^2 F_0 F_1, E_\delta\rangle =$

$$= \langle F_1 \otimes F_0,\ E_\delta \otimes K + (q^2-q^{-2}) E_1 \otimes K_1 E_0 + 1\otimes E_\delta\rangle = \frac{q^2-q^{-2}}{(q-q^{-1})^2}$$

- Example $\langle F_{-\alpha-\delta}, E_{\alpha+\delta}\rangle = \frac{1}{[2]_q}\langle [F_{-\delta}, F_{-\alpha}], E_{\alpha+\delta}\rangle =$

$$= \frac{-1}{[2]_q}\langle [F_{-\delta}, F_{-\alpha}],\ 1\otimes E_{\alpha+\delta} + E_{\alpha+\delta}\otimes K_{\alpha+\delta} + (q-q^{-1}) E_\alpha \otimes K_1 E_\delta + \# E_1^2 \otimes K_1 E_0\rangle$$

$$= \frac{(q-q^{-1})}{[2]_q}\langle F_{-\alpha} F_{-\delta},\ E_\alpha \otimes K_1 E_\delta\rangle = \frac{1}{(q-q^{-1})}$$

- Problem $\Delta(E_{\alpha+n\delta}) = 1\otimes E_{\alpha+n\delta} + E_{\alpha+n\delta}\otimes K_{\alpha+n\delta} + (q-q^{-1})\sum_{p=1}^{n-1} E_{\alpha+p\delta}\otimes K_{\alpha+p\delta} E_{(n-p)\delta} +$ very low terms

where very low terms of the form $a\otimes b$ with a contains at least two $E_{\alpha+p\delta}$

Hint Induction. For step $\Delta E_{\alpha+(n+1)\delta} = \frac{1}{[2]_q}[\Delta(E_\delta), \Delta(E_{\alpha+n\delta})]$

- **Corol** $\langle F_{-\alpha-n\delta}, E_{\alpha+n\delta}\rangle = \dfrac{1}{q-q^{-1}}$

Pf Induction $\langle F_{-\alpha-(n+1)\delta}, E_{\alpha+(n+1)\delta}\rangle = \dfrac{1}{[2]_q}\langle [F_{-\alpha-n\delta}, F_{-\delta}], \Delta E_{\alpha+(n+1)\delta}\rangle$

$$= \frac{(q-q^{-1})}{[2]_q}\langle F_{-\alpha-n\delta}, E_{\alpha+n\delta}\rangle\langle F_{-\delta}, E_{\delta}\rangle = \frac{1}{q-q^{-1}} \qquad \square$$

- Similarly

Lemma (a) $\Delta(F_{\alpha-n\delta}) = K_{\alpha-n\delta}\otimes F_{\alpha-n\delta} + F_{\alpha-n\delta}\otimes 1 +$

$$+ (q^{-1}-q)\sum_p K_{-(n-p)\delta}F_{\alpha-p\delta}\otimes F_{-(n-p)\delta} + \text{very low terms}$$

(b) $\langle F_{\alpha-n\delta}, E_{-\alpha+n\delta}\rangle = \dfrac{1}{q-q^{-1}}$

- Pairing of imaginary roots

Lemma (a) $\langle F_{-n\delta}, E_{n\delta}\rangle = q^{-2n+2}\dfrac{[2]}{(q-q^{-1})}$

(b) $\langle h_{-n}, h_n\rangle = (q^{2n}-q^{-2n})/n(q-q^{-1})^2 = \dfrac{[2n]}{n(q-q^{-1})}$

Product formula

- <u>Thm</u> $R = \overline{R}_H \sum \# E_{-j} \otimes F_j$ ← dual bases

$$= \overline{R}_H \prod_{r>0} \left(\sum_{n\geq 0} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E^n_{-\alpha+r\delta} \otimes F^n_{\alpha-r\delta} \right)$$
$$\left(\prod_{r>0} \sum_{n\geq 0} \frac{1}{n!} \left(\frac{r(q-q^{-1})}{[2r]} \right)^n h_r^n \otimes h_{-r}^n \right) \prod_{r\geq 0} \left(\sum_{n\geq 0} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E^n_{\alpha+r\delta} \otimes F^n_{-\alpha-r\delta} \right)$$

Here $\overline{R}_H = e^{\hbar\left(\frac{1}{2} H_1 \otimes H_1 + k\otimes d + d \otimes k\right)}$ with $K_1 = e^{\hbar H_1}$, $K = e^{\hbar k}$, $q = e^{\hbar}$

- We have $\overline{R}_H E_\alpha \otimes F_\alpha = (K_\alpha^{-1} E_\alpha \otimes F_\alpha K_\alpha) \overline{R}_H$. Hence

$R = R^- R^0 R^+$ where

$$R^- = \prod_{r>0} \left(\sum_{n\geq 0} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} (K_{\alpha-r\delta} E_{-\alpha+r\delta})^n \otimes (F_{\alpha-r\delta} K_{-\alpha+r\delta})^n \right)$$

$$R^0 = R_H \left(\prod_{r>0} \sum_{n\geq 0} \frac{1}{n!} \left(\frac{r(q-q^{-1})}{[2r]} \right)^n h_r^n \otimes h_{-r}^n \right) \qquad R^+ = \prod_{r\geq 0} \left(\sum_{n\geq 0} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E^n_{\alpha+r\delta} \otimes F^n_{-\alpha-r\delta} \right)$$

For f.d. reps. $V_1 \otimes V_2$, we have $k=0$,

$R = R^- R^0 R^+$ — Gauss decomposition

- **Problem** ⓐ For $R: \mathbb{C}^2(u_1) \otimes_\Delta \mathbb{C}^2(u_2) \to \mathbb{C}^2(u_1) \otimes_{\Delta^{op}} \mathbb{C}^2(u_2)$ find Gauss decomposition $R = R^- R^0 R^+$

low unitriangular — diagonal — upper unitriangular

ⓑ Show that $R^-_{\mathbb{C}^2(u_1)\otimes\mathbb{C}^2(u_2)} = R^-$, $R^+_{\mathbb{C}^2(u_1)\otimes\mathbb{C}^2(u_2)} = R^+$

ⓒ* Show that $R^-_{\mathbb{C}^2(u_1)\otimes\mathbb{C}^2(u_2)} = f(u_1/u_2)\, R^-$ for some function f

Hint ⓑ • In this case $E_2 \otimes F_2 = K_2^{-1} E_2 \otimes F_2 K_2$

• For comparsion of calculations $\psi^+(z) = \psi^-(z) = \begin{pmatrix} \frac{v - q^2 z}{q(v-z)} & 0 \\ 0 & \frac{q^2 v - z}{q(v-z)} \end{pmatrix}$

for $v = -uq^{-1}$

New realization

- **Th** $\mathcal{U}_q(\widehat{sl}_2)$ has presentation with generators $X^+[n], X^-[n], n\in\mathbb{Z}$, h_r, h_{-r} $r\in\mathbb{Z}_{>0}$, K, K^{-1} and relations
  - $$[X^+(z), X^-(w)] = \frac{1}{q-q^{-1}}\left(\psi^+(z)\,\delta\left(\frac{Kw}{z}\right) - \psi^-(w)\,\delta\left(\frac{w}{Kz}\right)\right)$$
  - $$[h_r, h_s] = \frac{[2r]}{r}\frac{K^r - K^{-r}}{q-q^{-1}}\,\delta_{r+s}$$
  - $$[h_r, X^+(w)] = \frac{[2r]}{r} w^r X^+(w) \qquad [h_{-r}, X^+(w)] = \frac{[2r]}{r} K^{-r} w^{-r} X^+(w)$$
    $$[h_r, X^-(w)] = -K^r\frac{[2r]}{r} w^r X^-(w) \qquad [h_{-r}, X^-(w)] = -\frac{[2r]}{r} w^{-r} X^-(w)$$
  - $$X^+(z)X^+(w)(z-q^2w) + X^+(w)X^+(z)(w-q^2z) = 0$$
    $$X^-(z)X^-(w)(z-q^{-2}w) + X^-(w)X^-(z)(w-q^{-2}z) = 0$$

Drinfeld coproduct

- In terms of modes

$$\Delta^D X^+[n] = 1 \otimes X^+[n] + X^+[n] \otimes K_{\alpha+n\delta} + (q^{-1}-q) \sum_{p>0} X^+[n+p] \otimes E_{-p\delta} K_{-\alpha+n\delta}$$

$$\Delta^D h_r = h_r \otimes K^r + 1 \otimes h_r$$

$$\Delta^D X^-[n] = X^-[n] \otimes 1 + K^{-1}_{-\alpha+n\delta} \otimes X^-[n] + (q-q^{-1}) \sum_{p>0} K^{-1}_{\alpha-n\delta} E_{p\delta} \otimes X^-[n-p]$$

$$\Delta^D h_{-r} = h_{-r} \otimes 1 + K^{-r} \otimes h_{-r}$$

- <u>Currents</u>

$$\Delta^D X^+(z) = 1 \otimes X^+(z) + X^+(K_{(2)} z) \otimes \psi^-(K_{(2)} z) \qquad \Delta^D K = K \otimes K$$

$$\Delta^D \psi^+(z) = \psi^+(z K_{(2)}^{-1}) \otimes \psi^+(z)$$

$$\Delta^D X^-(z) = X^-(z) \otimes 1 + \psi^+(K_{(1)} z) \otimes X(K_{(1)} z)$$

$$\Delta^D \psi^-(z) = \psi^-(z) \otimes \psi(K_{(1)}^{-1} z)$$

- **Problem** Check that Δ^D preserves some (say a couple) of relations.

- **Remark** Coproduct Δ^D is topological since we have infinite sums on right side.

Clearly $\Delta^D(X)$ well defined on $V_1 \otimes V_2$ for $\forall X \in U_q(\widehat{sl}_2)$ and V_1, V_2 – h.w. reps.

Another way: $\Delta^D(x)$ belong to completion, neighborhoods of zero are spanned by

$$\prod F_{-\beta_\kappa}^{n_\kappa} (\text{Cartan}) \prod E_{\beta_\kappa}^{m_\kappa} \otimes \prod F_{-\beta_\kappa}^{n'_\kappa} (\text{Cartan}) \prod E_{\beta_\kappa}^{m'_\kappa}$$

s.t

$$\sum (n_\kappa + n'_\kappa + m_\kappa + m'_\kappa) \, |\beta_\kappa| > N$$

where $|\beta|$ — height of root β

- **Thm** (Khoroshkin–Tolstoy) Coproducts

$$\Delta^{[n]}(x) = (\mathcal{T}T_1)^{-n} \otimes (\mathcal{T}T_1)^{-n} \Delta((\mathcal{T}T_1)^n x)$$

tend to Δ^D.

- **Corollary** $\Delta^D(x) = \left(\prod_{n\geq 0} \bar{R}_{\alpha+n\delta}\right) \Delta(x) \left(\prod \bar{R}^{-1}_{\alpha+n\delta}\right)$

Pf $\Delta^{[n]}(x) = (T_1^{-1} \otimes T_1^{-1}) \cdot (T_0^{-1} \otimes T_0^{-1}) \cdots T_i^{-1} \otimes T_i^{-1} \Delta(T_i \cdots T_0 T_1(x))$

$= (T_1^{-1} \otimes T_1^{-1}) \cdots (T_{1-i}^{-1} \otimes T_{1-i}^{-1}) \bar{R}_i \Delta(T_{i-1} \cdots T_1(x)) \bar{R}_i^{-1}$

$= (T_1^{-1} \otimes T_1^{-1}) \cdots \bar{R}_{T_{1-i}^{-1}(E_i)} \bar{R}_{1-i} \Delta(\cdots T_1(x)) \bar{R}_{1-i}^{-1} R_{T_{1-i}^{-1}(E_i)}$

$= \bar{R}_{\alpha+(n-1)\delta} \cdots \bar{R}_{\alpha+\delta} \bar{R}_\alpha \Delta(x) \bar{R}_\alpha^{-1} \bar{R}_{\alpha+\delta}^{-1} \cdots \bar{R}^{-1}_{\alpha+(n-1)\delta}$ $\square$

Using

$\bar{R}_i \Delta(x) \bar{R}_i^{-1} = (T_i^{-1} \otimes T_i^{-1}) \Delta(T_i x)$

- **Corollary** For $R^D = \prod_{r\geq 0} \bar{R}^{-1}_{\alpha+r\delta} \bar{R}_H \prod_{r>0} R_{-\alpha+r\delta} \prod_{r>0} R_{r\delta}$

we have $R^D \Delta^D = \Delta^{D,op} R^D$

R matrix for Drinfeld coproduct

Pf (Of Theorem) compute for $X^+[m]$

$(\mathcal{T}_1)^{n+m} X^+[m] = X^+[-n] = -F_{\alpha-n\delta} K_{-\alpha+n\delta}$ for $n+m>0$

Using formula for $\Delta(F_{\alpha-n\delta})$ above

$$\Delta(-F_{\alpha-n\delta} K_{-\alpha+n\delta}) = 1 \otimes (-F_{\alpha-n\delta} K_{-\alpha+n\delta}) + (-F_{\alpha-n\delta} K_{-\alpha+n\delta}) \otimes K_{-\alpha+n\delta}$$

$$+ \sum_p (-F_{\alpha-p\delta} K_{-\alpha+p\delta}) \otimes (q^{-1}-q) F_{-(n-p)\delta} K_{-\alpha+n\delta} + \text{"very low terms"}$$

$$= 1 \otimes X^+[-n] + \sum_{p=1}^{n} X^+[-p] \otimes \Psi^-[p-n] K^n + \text{"v.l.t."}$$

Hence

$$\Delta^{[n+m]}(X^+[m]) = 1 \otimes X^+[m] + \sum_{p=1}^{n} X^+[n+m-p] \otimes \Psi^-[p-n] K^{-m} + \text{"v.l.t"}$$

Here "v.l.t." contains $(\mathcal{T}_1)^{-n-m}(F_{\alpha-r_1\delta} F_{\alpha-r_2\delta})$ with $r_1+r_2<n$

i.e $E_{\alpha+(n+m-r_1)\delta} E_{\alpha+(n+m-r_2)\delta}$, the height $\geq 2m+n \to \infty$ $\square$

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