Introduction to quantum groups

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Problems for the course Skoltech, fall 2020. There are mistakes here, if you find some please write to mbersht@gmail.com

Definitions and hints are in slides and references.

1 Poisson algebras and quantization

Problem 1.1. *Show that Moyal formula defines associative product.*

Problem 1.2 (*)**.** *Find an example of the Poisson algebra which cannot be quantized.*

Problem 1.3 (*)**.** *Show that* $HH^2(U(\mathfrak{g})) = 0$ *for semisimple Lie algeba* $\mathfrak{g}$.

Problem 1.4. *Show that distribution* T^{Π} *is integrable.*

2 Poisson-Lie groups and Lie bialgebras

Problem 2.1. *Let* G *is Poisspn-Lie group,* $H \subset G$ *ia Poisson-Lie subgroup. Show that* $C^{\infty}(G)^H$ *is Posson subalgebra.*

Problem 2.2. *Let* $G = GL(2)$*, Poisson-Lie structure defined by* r *matrix with* $r = \frac{1}{4}h \otimes h + e \otimes f$, $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Find brackets of* a, b, c, d. *Check skew-commutativity. Check Poisson-Lie property.*

Problem 2.3. *For any finite dimensional Lie algebra* $\mathfrak{g}$ *there exists bijection between bialgebra structures on* $\mathfrak{g}$ *and Manin triples with* $\mathfrak{q}_+ \simeq \mathfrak{g}$.

Problem 2.4. *Find Lie bialgebra structure (i.e.* δ*) for Examples 1 and 4 and* $\mathfrak{g} = \mathfrak{sl}_2$.

3 Dual Poisson-Lie groups, symplectic leaves

Problem 3.1. *Let* $\mathfrak{q} = \mathfrak{sl}_2(\mathbb{C})$ *as* real *Lie algebra with scalar product* $(x, y) = \operatorname{Im}\operatorname{Tr}(xy)$ *and subalgebras* $\mathfrak{q}_+ = \mathfrak{su}_2$, $\mathfrak{q}_- = \{\begin{pmatrix} a & b+ic \\ 0 & -a \end{pmatrix} | a, b, c \in \mathbb{R}\}$.

a) Show that $\mathfrak{q}, \mathfrak{q}_+, \mathfrak{q}_-$ *is Manin triple. Find bialgebra structure on* $\mathfrak{su}_2$.
b) Show that $D(G) = G \times G^*$
c) Find symplectic leaves on $SU(2)$.

4 Classical r-matrices

Let $\delta_r(a) = \operatorname{ad}_a r$.

Problem 4.1. *Show that* δ_r *maps to* $\Lambda^2\mathfrak{g}$ *if and only if* $r_{12} + r_{21} \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$.

Problem 4.2. *Let* $r = r^S + r^A$, *where* $r^S = \alpha\Omega$, $r^A \in \Lambda^2\mathfrak{g}$. *Show that a)* $\delta_r = \delta_{r^A}$. *b)* $[[r, r]] = [[r^A, r^A]] + \alpha^2 c$.

Problem 4.3. *For standard bialgebra structure for simple* $\mathfrak{g}$

a) Find $\delta(h_i)$, $\delta(e_\alpha)$, $\delta(e_{-\alpha})$, *where* α *is simple.*
b) Find Lie algebra $\mathfrak{g}^*$.
c) Show that $r = \sum h_i \otimes h^i + 2\sum_{\alpha\in\Delta_+} e_\alpha \otimes e_{-\alpha}$ *defines the same* δ *and satisfies CYBE.*

Problem 4.4 (*). *Let* $r \in \Lambda^2\mathfrak{g}$ *satisfy MCYBE. Show that* $\Pi = (\lambda_g)_* r - (\rho_g)_* r$ *is Poisson bractet.*

5 Quantum group and algebras. Example of $\mathfrak{sl}_2$

Problem 5.1 (*). *Let* $U_\hbar(\mathfrak{g})$ *is quantization of universal enveloping* $U(\mathfrak{g})$. *Let* $\delta(a) = \frac{\Delta(a) - \Delta^{op}(a)}{\hbar} \bmod \hbar$. *Show that* δ *satisfies cocycle and coJacobi conditions.*

Problem 5.2. *Let* $g(\hbar) = 1 + O(\hbar) \in U(\mathfrak{h})[[\hbar]]$ *is group like element (i.e.* $\Delta(g) = g \times g$*), then* $g(\hbar) = \exp(\alpha H \hbar)$, *for* $\alpha \in \mathbb{C}[[\hbar]]$.

Problem 5.3. *Show that relation* $[E, F] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}}$ *agrees with coproduct* Δ.

Problem 5.4. *Show that exists homomorphism* $U_\hbar(\mathfrak{sl}_2) \to U(\mathfrak{sl}_2)[[\hbar]]$ *such that* $E \mapsto e$, $H \mapsto h$, *and* $F \mapsto \Phi(c, h) f$, *where* $c \in U(\mathfrak{sl}_2)$ *is Casimir.*

6 Hopf algebras

Problem 6.1. *Show that* S *is antihomomorphism of algebra and coalgebra.*

Problem 6.2. *a) Find formulas for action of* E, H, F *in basis* v_m. *b) Define basis* $\tilde{v}_m$.

Problem 6.3. *a) Show existence of natural morphisms* $V^* \otimes V \to \mathbb{C}$ *and* $\mathbb{C} \to V \otimes V^*$. *b) Show that* $(V \otimes W)^* = W^* \otimes V^*$.

Problem 6.4 (*). *a) Show directly that* $L_1 \otimes L_l \simeq L_{l+1} \oplus L_{l-1}$*, for* $l \geq 1$.
b) Show that $L_{l_1} \otimes L_{l_2} = \oplus L_l$*, where summation region* $|l_1 - l_2| \leq l \leq l_1 + l_2$ *and* $l + l_1 + l_2$ *is even.*

7 Quantum R-matrices

Problem 7.1. *For* $U_\hbar(\mathfrak{sl}_2)$ *show that* $\Delta^{op}(E)R = R\Delta(E)$.

Problem 7.2. *a) Show that* $C_\hbar = FE + \frac{e^{\hbar(H+1)} + e^{-\hbar(H+1)}}{(e^\hbar - e^{-\hbar})^2}$ *is central.*
b) Find action of $C_\hbar$ *and* $e^{-\hbar H}u$ *on* L_m.
c) Let $\Phi_\hbar^{-1} \colon U(\mathfrak{sl}_2)[[\hbar]] \to U_\hbar(\mathfrak{sl}_2)$ *isomorphism. Let* $c = fe + h(h+2)/4$. *Find* $\Phi_\hbar^{-1}(c)$. $\Phi_\hbar^{-1}(e^{\hbar c})$*, relate to central elements above.*

8 Drinfeld-Jimbo quantum groups

Problem 8.1. *Show that* $[F_i, q\text{-Serre } E_i] = 0$ *follows from quadratic relations.*

Problem 8.2. *Find* $H \in \mathfrak{h}$ *such that* $S^2(x) = e^{\hbar H} x e^{-\hbar H}$

Problem 8.3. *Using the Fact show nondegeneracy of the pairing* $U_\hbar(\mathfrak{b}^+) \otimes U_\hbar(\mathfrak{b}^-) \to \mathbb{C}$.

9 RTT realiztion

Problem 9.1. *a) Deduce quadratic relations on* E, F, H *from RTT relations.*
b) Deduce Serre relations from RTT relations.*

Problem 9.2. *Prove that* $U(R)$ *is generated by* $l^+_{ii}, l^+_{i,i+1}, l^-_{ii}, l^-_{i,i-1}$.

Problem 9.3. *Find formulas for* $L^+ = (\rho \otimes \mathrm{id})\mathfrak{R}$ *and* $L^- = (\mathrm{id} \otimes \rho)\mathfrak{R}^{-1}$ *for* $U_q(\mathfrak{sl}_2)$.

10 Functions on quantum group SL_2

Problem 10.1. *Show that qdet is central and group-like.*

Problem 10.2. *Show that* $L_l \otimes L_l^*$ *are linearly independend in* $U_q(\mathfrak{sl}_2)^\circ$ *for* $l \geq 0$.

Problem 10.3 (*). *Let* $U_\hbar(\mathfrak{g})$ *be quantum universal enveloping algebra. Let* $A = \{x \in U_\hbar(\mathfrak{g}) | (id - \epsilon)\Delta_n(x) \in U_\hbar(\mathfrak{g})^{\otimes n}, \forall n\}$. *Show that* A *is a Hopf algebra, cocommutative up to first order in* $\hbar$.

Problem 10.4 (*). *For* $\mathfrak{g} = \mathfrak{sl}_2$ *define* $U_\hbar(\mathfrak{g}^*)$ *for standard bialgebra structure.*

11 Functions on quantum group SL_n

Problem 11.1 (*). *For given $J = \{j_1 < \dots J_{r-1}\}$, $I = \{i_1, \dots, i_r\}$, $K = \{k_0, \dots k_r\}$ show relation*

$$\sum_{s=0}^{r} \mathrm{sgn}(J, k_s)(-q)^{-s} t^{i_1 \dots i_r}_{j_1 \dots k_s, \dots, j_{r-1}} t^{i_1 \dots i_r}_{k_0, \dots, \hat{k_s} \dots, k_r} = 0$$

Problem 11.2 (*). *Show that center of $\mathbb{C}[SL_n]_q$ is generated by qdet.*

Problem 11.3 (*). *a)* $t^{\Lambda}_{-w_0(\Lambda),\Lambda} t^{\Lambda'}_{-\mu,\lambda} = q^{(\Lambda,\lambda)-(w_0(\Lambda,\mu))} t^{\Lambda'}_{-\mu,\lambda} t^{\Lambda}_{-w_0(\Lambda),\Lambda}$.

b) $t^{\Lambda}_{-\Lambda,w_0(\Lambda)} t^{\Lambda'}_{-\mu,\lambda} = q^{(\Lambda,\mu)-(w_0(\Lambda),\lambda)} t^{\Lambda'}_{-\mu,\lambda} t^{\Lambda}_{-\Lambda,w_0(\Lambda)}$.

c) Elements $t^{\Lambda}_{-w_0(\Lambda),\Lambda}$, $t^{\Lambda'}_{-\Lambda',w_0(\Lambda')}$ *form commutative subalgebra.*

Problem 11.4 (*). *d) Subalgrebra A_+ is generated by $t^{1 \dots k}_{i_1 \dots i_k}$. Subalgebra A_- is generated by $t^{n-k+1 \dots n}_{i_1 \dots i_k}$.*

e) Commutative subalgebra from c) above is generated by $t^{1 \dots k}_{n-k+1 \dots n}$, $t^{n-k+1 \dots n}_{1 \dots k}$.

12 Lusztig's braid group

Problem 12.1. *Chech that $[T_i(E_j), T_i(F_j)] = T_i([E_j, F_j])$ for $a_{ij} = -1$.*

Problem 12.2. *For $U_q(\mathfrak{g})^{coop}$ find $S(E_i)$. Show that $\mathrm{ad}_{\Delta^{op},E_i} = \mathrm{ad}_{q,E_i}$.*

Problem 12.3. *Fix reduced expression of $w_0 = s_{i_1} \cdots w_{i_N}$.*

a) If $a_{i_k,i_{k+1}} = 0$ then reversing $i_k, i_k + 1$ we get reduced expression $\vec{i'}$ with the same (but reordered) set of Cartan-Weyl elements.

b) If $i_k = i_{k+2}$, $a_{i_k,i_{k+1}} = a_{i_{k+1},i_{k+2}} = -1$ then $\beta_{k+1} = \beta_k + \beta_{k+2}$, $E_{\beta_{k+1}} = -[E_{\beta_k}, E_{\beta_{k+2}}]_{q^{-1}}$. Replacing $i_k, i_{k+1}, i_k \to i_{k+1} i_k i_{k+1}$ we get reducted expression $\vec{i'}$ and the set of Cartan-Weyl elements $\{E'_\beta\}$ differs from $\{E'_\beta\}$ only by $E'_{\beta_{k+1}}$ and $E_{\beta_{k+1}}$.

c) If $\beta_k = \alpha_i$ then $E_{\beta_k} = E_i$.

Problem 12.4. *Relate l^-_{ij} generators in RTT realization and Cartan-Weyl elements.*

13 Factorization of the universal R matrix

Problem 13.1 (*). *a) For $v \in L_l[m]$ show that $E^{(a)} F^{(b)} v = \sum_{t \geq 0} F^{(b-t)} E^{(a-t)} \begin{bmatrix} m-b+a \\ t \end{bmatrix}_q$.*

b) Let $v_l \in L_l$ be highest weight vector. Let $\tilde{v}_m = F^{(\frac{l-m}{2})} v_l \in L_l[m]$. Show that

$$t\tilde{v}_m = (-1)^{\frac{l-m}{2}} q^{-\frac{l-m}{2}\frac{l+m+2}{2}} \tilde{v}_{-m}.$$

c) Show that $tFv = -EKtv$, $tKv = K^{-1}tv$, $tEv = -k^{-1}Ftv$.

Problem 13.2 (*). *Let $\bar{\mathfrak{R}} = \sum_{n \geq 0} q^{\binom{n}{2}} \frac{(q-1/q)^n}{[n]_q!} E^n \otimes F^n$. Show that*
$\bar{\mathfrak{R}}^{-1} = \sum_{n \geq 0} (-1)^n q^{-\binom{n}{2}} \frac{(q-1/q)^n}{[n]_q!} E^n \otimes F^n$

Problem 13.3 (*). *Show that* $\Delta(t) = \bar{\mathfrak{R}}^{-1} t \otimes t$

Problem 13.4 (*). *Show that* $e^{-\hbar H^2/2} t^2$ *is central. Relate it to central elements from Lecture 7.*

Problem 13.5 (*). *For* $\mathfrak{g} = \mathfrak{sl}_n$ *compute* $\rho_{\mathbb{C}^n} \otimes \rho_{\mathbb{C}^n}(\mathfrak{R})$.

14 Drinfeld double

Problem 14.1 (*). *Show that relations* $\langle a_{(1)}, b_{(1)} \rangle a_{(2)} * b_{(2)} = b_{(1)} * a_{(1)} \langle a_{(2)}, b_{(2)} \rangle$ *is equivalent to* $b * a = \langle a_{(1)}, b_{(1)} \rangle a_{(2)} * b_{(2)} \langle a_{(3)}, S^{-1}(b_{(3)}) \rangle$

Problem 14.2 (*). *For* $A = \mathbb{C}[G]$, G *finite group, find* $D(A)$. *Describe representations of* $D(A)$.

Problem 14.3 (*). *Show that* $\langle R L_1^+ L_2^+ - L_2^+ L_1^+ R, - \rangle = 0$.

Problem 14.4 (*). *For* $\mathfrak{g} = \mathfrak{sl}_2$ *and* $V = \mathbb{C}^2$ *compute* C_V.