Introduction to Quantum Groups

Lecture 14
Drinfeld Double

qft.itp.ac.ru/mbersht/quantum_groups.html

Remainder Classical double

- $(\mathfrak{g}, \delta)$ – Lie Bialg
- $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ – Manin triple. $\mathfrak{g}_+ = \mathfrak{g}$, $\mathfrak{g}_- = \mathfrak{g}^*$, $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$ as vect space.
- Th $\mathfrak{g}$ has str of cobound Lie Bialg given by $r \in \Lambda^2 \mathfrak{g}$, $r \leftrightarrow Id \in \mathfrak{g} \otimes \mathfrak{g}^*$. $\mathfrak{g} = D(\mathfrak{g})$ – classical double

If $\mathfrak{g} = \langle a_i \rangle$, $\mathfrak{g}^* = \langle a^i \rangle$ – dual bases, $r = \sum_i a_i \wedge a^i$

- Manin triple $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}, \mathfrak{g}_+ \oplus \mathfrak{g}_-)$
- Rk $\mathfrak{g}, \mathfrak{g}^*$ – sub Bialg. in $D(\mathfrak{g})$

Quantum (Drinfeld) Double

- **Def** Given Bialgebras A^-, A^+ a Bialgebra pairing
$$(\cdot,\cdot) : A^- \otimes A^+ \to \mathbb{C} \quad \text{s.t}$$

$$(a \cdot a', b) = (a \otimes a', \Delta(b)) = (a, b_{(1)})(a', b_{(2)})$$
$$(a, b \cdot b') = (\Delta^{op}(a), b \otimes b') = (a_{(2)}, b)(a_{(1)}, b')$$

- **Def** $A = A^- \otimes A^+$, s.t. $A^- \otimes 1$, $1 \otimes A^+$ sub bialgebras with the product $*$ s.t
$$\langle a_{(1)}, b_{(1)} \rangle a_{(2)} * b_{(2)} = b_{(1)} * a_{(1)} \langle a_{(2)}, b_{(2)} \rangle$$ — c.f $RTT = TTR$

is called Drinfeld double of A.

- **Problem** This is equiv to $a \in A^-$, $b \in A^+$

$$b * a = \langle a_{(1)}, b_{(1)} \rangle a_{(2)} * b_{(2)} \langle a_{(3)}, S^{-1}(b_{(3)}) \rangle$$ — c.f $TT = R^{-1} TTR$

Drinfeld center

- $(\mathcal{C}, \otimes)$ — tensor category

- <u>Def</u> $Z(\mathcal{C}) = \{(X, \varphi) \mid X \in \mathcal{C},$

$\varphi : X \otimes \_ \longrightarrow \_ \otimes X$ natural isomorphism of functors

s.t. $\varphi_{Y \otimes Z} = (\varphi_Y \otimes Id) \circ (Id \otimes \varphi_Z)$

$\forall \alpha : W \to W'$

$$\begin{tikzcd}
X \otimes W \arrow[r, "\varphi_X(W)"] \arrow[d, "id \otimes \alpha"'] & W \otimes X \arrow[d, "\alpha \otimes id"] \\
X \otimes W' \arrow[r, "\varphi_X(W')"] & W' \otimes X
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \otimes Z \otimes W \arrow[rr, "id \otimes \varphi_Z(W)"] \arrow[d, "\varphi_{Y \otimes Z}(W)"'] & & Y \otimes W \otimes Z \arrow[dll, "\varphi_Y(W) \otimes id"] \\
W \otimes Y \otimes Z & &
\end{tikzcd}$$

<u>Th</u> $Z(A\text{-mod}) = \mathcal{D}(A)\text{-mod}$.

- <u>Th</u> $R = \sum (e_i \otimes 1) \otimes (1 \otimes e^i) \in \mathcal{D}(A) \otimes \mathcal{D}(A)$ — universal R-matrix e_i, e^i dual bases in A, A^*

- <u>Problem</u> $A = \mathbb{C}[G]$, G - finite group $\mathcal{D}(A)$-? Reps $\mathcal{D}(A)$-?

$U_q(\mathfrak{g})$ as Drinfeld double

- classically $\mathfrak{g} = D(\mathfrak{b}^+)/\mathfrak{h}$

- <u>th</u> (Drinfeld) $\exists!$ non degenerate pairing $U_\hbar(\mathfrak{b}^-)\otimes U_\hbar(\mathfrak{b}^+)\to\mathbb{C}$ s.t.

  - $(e^{\hbar H_i}, e^{\hbar H_j}) = e^{\hbar (H_i, H_j)}$
  - $(e^{\hbar H_i}, E_j) = (F_i, e^{\hbar H_j}) = 0$
  - $(F_i, E_j) = \delta_{i,j}\dfrac{1}{e^{\hbar}-e^{-\hbar}}$

- $D(U_\hbar(\mathfrak{b}^+)) = U_\hbar(\mathfrak{b}^-)\otimes U_\hbar(\mathfrak{b}^+)$ – product of subalg

$$\langle a_{(1)}, b_{(1)}\rangle a_{(2)} * b_{(2)} = b_{(1)} * a_{(1)} \langle a_{(2)}, b_{(2)}\rangle$$

$$\Delta E_i = E_i \otimes K_{i,+} + 1\otimes E_i$$
$$\Delta F_j = F_j\otimes 1 + K_{j,-}^{-1}\otimes F_j$$

$$\langle F_j, E_i\rangle K_{i,+} + \langle K_{j,-}, 1\rangle F_j E_i = E_i F_j \langle K_{i,+}, 1\rangle + K_{j,-}^{-1}\langle F_j, E_i\rangle$$

$$E_i F_j - F_j E_i = \delta_{i,j}\frac{K_{i,+} - K_{i,-}^{-1}}{q-q^{-1}}$$

$U_q(\mathfrak{sl})$ as Drinfeld double

- $(e^{\hbar H_i}, e^{\hbar H_j}) = e^{\hbar (H_i, H_j)}$ • $(e^{\hbar H_i}, E_j) = (F_i, e^{\hbar H_j}) = 0$ • $(F_i, E_j) = \delta_{i,j} \frac{1}{e^{\hbar} - e^{-\hbar}}$

- $D(U_\hbar(\mathfrak{b}^+)) = U_\hbar(\mathfrak{b}^-) \otimes U_\hbar(\mathfrak{b}^+)$ – product of subbialg

$$\langle a_{(1)}, b_{(1)} \rangle a_{(2)} * b_{(2)} = b_{(1)} * a_{(1)} \langle a_{(2)}, b_{(2)} \rangle$$

$$\Delta E_i = E_i \otimes K_{i,+} + 1 \otimes E_i \qquad \Delta F_j = F_j \otimes 1 + K_{j,-}^{-1} \otimes F_j \qquad \Delta K_{\ell,\pm} = K_{\ell,\pm} \otimes K_{\ell,\pm}$$

$$\langle K_{\ell,-}, 1 \rangle K_{\ell,-} E_i = K_{\ell,-} E_i \langle K_{\ell,-}, K_{i,+} \rangle \longrightarrow K_{\ell,-} E_i = q^{(H_\ell, H_j)} E_i K_{\ell,-}$$

$$\langle K_{j,-}^{-1}, K_{\ell,+} \rangle F_j K_{\ell,+} = K_{\ell,+} F_j \langle 1, K_{\ell,+} \rangle \longrightarrow K_{\ell,+} F_j = q^{-(H_\ell, H_j)} F_j K_{\ell,+}$$

- $K_{\ell,+} K_{\ell,-}^{-1}$ – central elements, generate $U_q(\mathfrak{h})$

$$U_q(\mathfrak{sl}) = D(U_q(\mathfrak{b}^+)) / U_q(\mathfrak{h})$$

RTT as double

- $\mathfrak{g} = \mathfrak{gl}_n \quad \ell^+_{ij},\ \ell^-_{ji},\quad 1 \le i \le j \le n.\quad \ell^-_{ii}\ell^+_{ii} = \ell^+_{ii}\ell^-_{ii} = 1$

$R L^+_1 L^+_2 = L^+_2 L^+_1 R, \quad R L^-_1 L^-_2 = L^-_2 L^-_1 R, \quad R L^+_1 L^-_2 = L^-_2 L^+_1 R$

$L^\pm_1 = L^\pm \otimes 1, \quad L^\pm_2 = 1 \otimes L$

- $\Delta(L^\pm) = L^\pm \otimes L^\pm, \quad S(L^\pm) = (L^\pm)^{-1}$

- $U_q(\mathfrak{b}^+)$ – algebra generated $\ell^-_{ji} \quad i \le j$

- $\langle L^+_1, L^-_2 \rangle = R_{12}$ pairing. $\langle \ell^+_{ij}, \ell_{i'j'} \rangle = R^{jj'}_{ii'}$

$\langle a_{(1)}, b_{(1)} \rangle a_{(2)} * b_{(2)} = b_{(1)} * a_{(1)} \langle a_{(2)}, b_{(2)} \rangle,$

$R_{12} L^+_1 L^-_2 = L^-_2 L^+_1 R_{12}$

RTT pairing vs relations

- Need to check that pairing is well defined

<u>Problem</u> $\langle R L_1^+ L_2^+ - L_2^+ L_1^+ R, - \rangle = 0$

- <u>Hint</u> a) Sufficcient to show $\langle R_{12} L_1^+ L_2^+, L_3^- L_4^- \cdots L_N^- \rangle$ $\forall N$

- consider $N=1$, other cases similar ($N=0$ requires definition)

$$\langle R_{12} L_1^+ L_2^+ - L_2^+ L_1^+ R_{12}, L_3^- \rangle \in \underset{1}{Mat_n} \otimes \underset{2}{Mat_n} \otimes \underset{3}{Mat_n}$$

$$\shortparallel$$

$$\langle R_{12} L_1^+ \otimes L_2^+ - L_2^+ \otimes L_1^+ R_{12}, L_3^- \otimes L_3^- \rangle$$

$$\shortparallel$$

$$R_{12} R_{13} R_{23} - R_{23} R_{13} R_{12} = 0$$

Pairing for sl_2

- $\langle F, E\rangle = \dfrac{1}{e^{\hbar}-e^{-\hbar}}$

<u>Lemma</u> $\langle F^n, E^m\rangle = \dfrac{[n]_q!}{(q-q^{-1})^n}\, q^{-\binom{n}{2}}\, \delta_{n,m}$

<u>Pf</u> $\langle F^n, E^m\rangle = \langle F\otimes\ldots\otimes F, \Delta_n(E)^m\rangle$

$\Delta(E) = \Delta_2(E) = E\otimes K + 1\otimes E, \quad \Delta_3(E) = E\otimes K\otimes K + 1\otimes E\otimes K + 1\otimes 1\otimes E$

$\Delta_n(E)^n = q^{\binom{n}{2}}[n]_q!\, E\otimes KE\otimes\ldots\otimes K^{n-1}E$ + [non important terms]

$(F, K^j E) = (\Delta^{op}F, K^j\otimes E) = (F\otimes K^{-1} + 1\otimes F, K^j\otimes E) = \dfrac{1}{q-q^{-1}}$

- $\langle e^{\hbar H_-}, e^{\hbar H_+}\rangle = e^{2\hbar}$

$\langle H_-, H_+\rangle = \frac{2}{\hbar} \longrightarrow \langle H_-^k, H_+^e\rangle = \delta_{k,e}\dfrac{2^k k!}{\hbar^k}$

R matrix for sl_2

- $$\langle H_+^k E^n, H_-^\ell F^m \rangle = \delta_{k,\ell} \frac{2^k k!}{\hbar^k} \delta_{m,n} \frac{[n]_q!}{(q-q^{-1})^n} q^{-\binom{n}{2}}$$

For $D(U_q(b^+))$ $$R = \sum_k \frac{\hbar^k}{k!\, 2^k} H_+ \otimes H_- \sum_n \frac{(q-q^{-1})^n}{[n]_q!} q^{\binom{n}{2}} E^n \otimes F^n$$
$$= e^{\frac{\hbar}{2} H_+ \otimes H_-} \overline{R}$$

- For $U_q(sl) = D(U_q(b^+)) / (H_+ - H_-)$ $\qquad R = e^{\frac{\hbar}{2} H \otimes H} \overline{R}$

Pairing of PBW bases

- $w_0 = s_{i_1} \cdots s_{i_N}$ reduced decomp

  $E_{\gamma_k} = T_{i_N}^{-1} T_{i_{N-1}}^{-1} \cdots T_{i_{k+1}}^{-1}(E_{i_k})$, $F_{\gamma_k} = T_{i_N}^{-1} T_{i_{N-1}}^{-1} \cdots T_{i_{k+1}}^{-1}(F_{i_k})$ – Cartan-Weyl elements

- <u>Lemma</u> $\langle F_{\gamma_1}^{n_1} \cdots F_{\gamma_N}^{n_N}, E_{\gamma_1}^{m_1} \cdots E_{\gamma_N}^{m_N} \rangle = \prod_{k=1}^{N} \left(\delta_{n_k, m_k} \frac{[n_k]_q!}{(q-q^{-1})^{n_k}} q^{-\binom{n_k}{2}} \right)$

<u>Pf</u> $\Delta E_{\gamma_k} = \prod_{e>k} \bar{R}_{\gamma_k}^{-1} (E_{\gamma_k} \otimes K_{\gamma_k} + 1 \otimes E_{\gamma_k}) \prod_{e>k} \bar{R}_{\gamma_k}$

Hence $\langle X_\lambda, \Delta E_{\gamma_k} \rangle = 0$ for $X_\lambda \in \mathcal{U}(n^-)_\lambda$ if $\lambda \notin \{ \sum_{e>k} i_e \gamma_e \mid i_e \in \mathbb{Z}_{\geq 0} \}$

$$\langle F_{\gamma_1}^{n_1} \cdots F_{\gamma_N}^{n_N}, E_{\gamma_1}^{m_1} \cdots E_{\gamma_N}^{m_N} \rangle = \langle F_{\gamma_1} \otimes F_{\gamma_1}^{n_1-1} \cdots F_{\gamma_N}^{n_N}, \Delta(E_{\gamma_1})^{m_1} \cdots \Delta(E_{\gamma_N})^{m_N} \rangle =$$

$$= [m_1] q^{1-m_1} \langle F_{\gamma_1}, E_{\gamma_1} \rangle \langle F_{\gamma}^{n_1-1} \cdots F_{\gamma_N}^{n_N-1}, K_{\gamma_1} E_{\gamma_1}^{m_1-1} \cdots E_{\gamma_N}^{m_N} \rangle = \ldots = \prod_{k=1}^{N} \delta_{n_k, m_k} \frac{[n_k]_q!}{(q-q^{-1})^{n_k}} q^{-\binom{n_k}{2}}$$

- <u>Corollary</u> For $\mathcal{U}_q(\mathfrak{g})$

$$R = R_H \sum \prod_k \left(\frac{(q-q^{-1})^{n_k}}{[n_k]_q!} q^{\binom{n_k}{2}} \right) E_{\gamma_1}^{n_1} \cdots E_{\gamma_N}^{n_N} \otimes F_{\gamma_1}^{n_1} \cdots F_{\gamma_N}^{n_N} = R_H R_{\gamma_1} \cdots R_{\gamma_N}$$

Some central elements

- Let $\lambda : U_q(\mathfrak{g}) \to \mathbb{C}$ s.t $\lambda(xy) = \lambda(y\, S^2(x))$

Example V - $U_q(\mathfrak{g})$ - module take $\lambda_V(x) = Tr|_V(x q^{2\rho})$

Pf $\lambda_V(xy) = Tr|_V(xyq^{2\rho}) = Tr_V(y\, q^{2\rho} x q^{-2\rho} q^{2\rho}) = \lambda_V(y\, S^2(x))$

Since $S^2(x) = q^{2\rho} x q^{-2\rho}$

- Prop $(id \otimes \lambda)(R_{21} R_{12})$ - central element $U_q(\mathfrak{g})$

- Prop $V \mapsto \lambda_V \mapsto C_V$: algebra hom. $K(U_q(\mathfrak{g})\text{-mod}) \to Z(U_q(\mathfrak{g}))$

(i.e ⓐ $0 \to V' \to V \to V'' \to 0 \rightsquigarrow C_V = C_{V'} + C_{V''}$, ⓑ $V = V' \otimes V''$ $C_V = C_{V'} C_{V''}$)

- Problem For $\mathfrak{g} = \mathfrak{sl}_2$, $V = \mathbb{C}^2$ compute C_V.

Proof of the prop

- **Prop** $(id\otimes\lambda)R_{21}R_{12}$ – central element $U_q(\mathfrak{sl}_2)$

Pf If $\Delta(x)=\sum y_i\otimes z_i$, then $\sum(1\otimes S^{-1}(z_i))\Delta(y_i)=x\otimes 1$

using prop of λ

$$x(id\otimes\lambda)[R_{21}R_{12}]=(id\otimes\lambda)[(x\otimes 1)R_{21}R_{12}]=(id\otimes\lambda)[\sum(1\otimes S^{-1}(z_i))\Delta(y_i)R_{21}R_{12}]=$$
$$=(id\otimes\lambda)[\sum\Delta(y_i)R_{21}R_{12}(1\otimes S(z_i))]=(id\otimes\lambda)[R_{21}R_{12}\sum\Delta(y_i)(1\otimes S(z_i))]=$$
$$=(id\otimes\lambda)[R_{21}R_{12}(x\otimes 1)]=(id\otimes\lambda)[R_{21}R_{12}]\cdot x$$

using prop of R.

- **Prop** $V\mapsto\lambda_V\mapsto C_V$: algebra hom. $K(U_q(\mathfrak{sl}_2)\text{-mod})\to Z(U_q(\mathfrak{sl}_2))$

(i.e. ⓐ $0\to V'\to V\to V''\to 0 \rightsquigarrow C_V=C_{V'}+C_{V''}$, ⓑ $V=V'\otimes V''$ $C_V=C_{V'}C_{V''}$)

Pf ⓐ – triv, ⓑ $\lambda_V=\lambda_{V'}\lambda_{V''}$

$$(id\otimes\lambda_V)[R_{21}R_{12}]=(id\otimes\lambda_{V'}\otimes\lambda_{V''})(id\otimes\Delta)[R_{21}R_{12}]=(id\otimes\lambda_{V'}\otimes\lambda_{V''})[R_{21}R_{31}R_{13}R_{12}]$$
$$=(id\otimes\lambda_{V'})[R_{21}(C_{V''}\otimes 1)R_{12}]=C_{V'}C_{V''}$$

since $C_{V''}$ central

References

- Chari, Pressley *A guide to quantum groups* Sec. 4.2
- Korogodski, Soibelman *Algebras of Functions on quantum group* Sec 2.2, 4.3, 4.4
- Drinfeld *Almost cocommutative Hopf algebras*