Introduction to Quantum Groups

Lecture 10

Functions on quantum group SL_2

qft.itp.ac.ru/mbersht/quantum_groups.html

$$\mathbb{C}[GL_n]$$

- $\mathbb{C}[Mat_n] = \mathbb{C}[t_{ij}]$ $\quad 1 \le i,j \le n$.

$\mathbb{C}[Mat_n] \to \mathbb{C}[Mat_n] \otimes \mathbb{C}[Mat_n]$ – (dual to $M_1 M_2$)

$\Delta t_{ij} = \sum t_{ik} \otimes t_{kj}$

$\Delta T = T \otimes T$

$$T = \begin{pmatrix} t_{11} & \cdots & t_{1n} \\ & & \\ t_{n1} & & t_{nn} \end{pmatrix} = \sum t_{ij} E_{ij}$$

$\varepsilon(t_{ij}) = \delta_{ij}$ (evaluation at E)

- $S: \mathbb{C}[GL_n] \to \mathbb{C}[GL_n]$ $\quad \mathbb{C}[GL_n] = \mathbb{C}[t_{ij}, \det^{-1}]$

$T \mapsto T^{-1}$

- $\mathbb{C}[SL_n]$ $\quad \mathbb{C}[SL_n] = \mathbb{C}[t_{ij}]/\det = 1$

Comodules

- $GL(2) \times V \to V \qquad V = \mathbb{C}^2$

  $\Delta: \mathbb{C}[V] \to \mathbb{C}[GL_2] \otimes \mathbb{C}[V]$ homomorphism of algebras

  $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ corr $(g_1 g_2)v = g_1(g_2 v)$

- Explicitly $\mathbb{C}[V] = \mathbb{C}[x_1, x_2]$

  $$\Delta: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- $V^* \times GL_2 \to V^*$

  $$(x_1, x_2) \mapsto (x_1, x_2) \otimes \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$$

q-deformation of comodule

- $\mathbb{C}_q[V] = \mathbb{C}\langle x_1, x_2\rangle / (x_1x_2 - q^{-1}x_2x_1)$

- $\Delta : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \dot{\otimes} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$(\Delta \otimes id)\,\Delta X = (\Delta \otimes id)(T \otimes X) = T \otimes T \otimes X = (id \otimes \Delta)(T \otimes X) = (id \otimes \Delta)\Delta X$

- Δ is homomorphism

$x_1 x_2 = q^{-1} x_2 x_1 \longmapsto$

$(t_{11} \otimes x_1 + t_{12} \otimes x_2)(t_{21} \otimes x_1 + t_{22} \otimes x_2) = q^{-1}(t_{21} \otimes x_1 + t_{22} \otimes x_2)(t_{11} \otimes x_1 + t_{12} \otimes x_2)$

- $t_{11}t_{21} = q^{-1}t_{21}t_{11}, \quad t_{12}t_{22} = q^{-1}t_{22}t_{12}$

$t_{11}t_{22} + q^{\prime}\, t_{12}t_{21} = q^{-1}t_{21}t_{12} + t_{22}t_{11}$

$\mathbb{C}[Mat_2]_q$

- $t_{11}t_{21} = q^{-1}t_{21}t_{11}, \quad t_{12}t_{22} = q^{-1}t_{22}t_{12}$
  $t_{11}t_{22} + q t_{12}t_{21} = q^{-1}t_{21}t_{12} + t_{22}t_{11}$

- $\mathbb{C}_q[V^*] \to \mathbb{C}_q[V^*] \otimes \mathbb{C}_q[SL_2] \quad \rightsquigarrow$

  $t_{11}t_{12} = q^{-1}t_{12}t_{11} \qquad t_{21}t_{22} = q^{-1}t_{22}t_{21}$
  $t_{11}t_{22} + q t_{21}t_{12} = q^{-1}t_{12}t_{21} + t_{22}t_{11}$

- $t_{12}t_{21} = t_{21}t_{12}$

- $\mathbb{C}[Mat_2]_q = \mathbb{C}\langle t_{11}, t_{12}, t_{21}, t_{22}\rangle / (\text{all relations})$

$\mathbb{C}[Mat_2]_q$ vs RTT

- $t_{11} t_{21} = q^{-1} t_{21} t_{11}$, $t_{12} t_{22} = q^{-1} t_{22} t_{12}$

  $t_{11} t_{12} = q^{-1} t_{12} t_{11}$ $t_{21} t_{22} = q^{-1} t_{22} t_{21}$

  $t_{12} t_{21} = t_{21} t_{12}$

  $t_{11} t_{22} - t_{22} t_{11} + (q - q^{-1}) t_{12} t_{21} = 0$

- Matrix form $\sum_{k,k'} \tilde{R}^{kk'}_{ii'} t_{kj} t_{k'j'} = \sum t_{ik} t_{i'k'} \tilde{R}^{jj'}_{kk'}$

$$\tilde{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & (q-q^{-1}) & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \qquad \tilde{R} T_1 T_2 = T_1 T_2 \tilde{R}$$

$\mathbb{C}[GL_2]_q \qquad \mathbb{C}[SL_2]_q$

- $q\det(T) = t_{11}t_{22} - q^{-1}t_{12}t_{21}$

- <u>Prop</u> a) $\Delta q\det = q\det \otimes q\det$

  b) $q\det$ is central

- <u>Problem</u> Show Prop directly

- <u>Def</u> $\mathbb{C}[GL_2]_q = \mathbb{C}[Mat_2, q\det^{-1}]_q$

  $\mathbb{C}[SL_2]_q = \mathbb{C}[Mat_2]_q/(q\det - 1)$

dual Hopf algebra

- A – Hopf algebra $\to$ A^* Hopf algebra
- <u>Prop</u> Let A° – subalgebra in A^* generated by matrix elements of finite dim reps of A.
  Then A° – Hopf algebra
- <u>Rk</u> Peter-Weyl $\mathbb{C}[G] = \oplus L_\lambda \otimes L_\lambda^*$ for G – s/s
  Here by definition $U_q(\mathfrak{g})^\circ = \oplus L_{\lambda,q} \otimes L_{\lambda,q}^*$
- <u>Proof</u> V_1, V_2 – f.d. reps $t_{ij} \in V_1 \otimes V_1^*$, $t_{kl} \in V_2 \otimes V_2^*$

Coproduct $t_{ij} t_{kl}(x) = t_{ij} \otimes t_{kl}(\Delta(x))$ $t_{ij} t_{kl} \in (V_1 \otimes V_2) \otimes (V_1 \otimes V_2)^*$

Product $\Delta t_{ij}(x_1 \otimes x_2) = t_{ij}(x_1 \cdot x_2)$. $\Delta t_{ij} = \sum t_{ik} \otimes t_{kj}$

Isomorphism

- $\underline{\text{Th}}$ $\mathbb{C}[SL_2]_q = U_q(sl_2)^\circ$ $\qquad x \mapsto \begin{pmatrix} t_{11}(x) & t_{12}(x) \\ t_{21}(x) & t_{22}(x) \end{pmatrix}$

- $\underline{\text{Step 1}}$ $\quad \Psi: \mathbb{C}[SL_2]_q \to U_q(sl_2)^\circ$

  $t_{11}, t_{12}, t_{21}, t_{22} \longmapsto$ Matrix elements of $\mathbb{C}^2$

- $\underset{\Delta}{\mathbb{C}^2 \otimes \mathbb{C}^2} \xrightarrow{\tilde{R}} \underset{\Delta}{\mathbb{C}^2 \otimes \mathbb{C}^2}$ intertwining operator. Hence $\forall x \in U_q(sl_2)$

$$\tilde{R}^{\kappa\kappa'}_{ii'}(t_{\kappa j}\, t_{\kappa' j'})(x) = R^{\kappa\kappa'}_{ii'}(t_{\kappa j} \otimes t_{\kappa' j'})(\Delta(x)) = (t_{i\kappa} \otimes t_{i'\kappa'})\Delta(x) R^{jj'}_{\kappa\kappa'} = t_{i\kappa}\, t_{i'\kappa'}(x) R^{jj'}_{\kappa\kappa'}$$

def. of product in $U_q(sl_2)^\circ$ $\qquad$ intertwining property

Hence $\tilde{R}\, T_1 T_2 = T_1 T_2 \tilde{R}$

- $e_1 \otimes e_2 - q\, e_2 \otimes e_1$ generates trivial submodule

$\forall x \in U_q(sl_2)$ $\quad \Delta(x)(e_1 \otimes e_2 - q\, e_2 \otimes e_1) = \varepsilon(x)$ $\quad \Longrightarrow t_{11} t_{22} - q t_{12} t_{21} = \varepsilon \in U_q(sl_2)^\circ$

$\quad \| $

$(t_{11} t_{12} - q\, t_{12} t_{11})(x)\, e_1 \otimes e_1 + (t_{11} t_{22} - q\, t_{12} t_{21})(x)\, e_1 \otimes e_2 + \ldots$

Proof

- Step 2 ψ is surjective since $\forall$ f.d. rep of $U_q(sl_2)$ is direct summand of $V^{\otimes n}$ for some n.

- Step 3 Introduce filtrations

$B = \mathbb{C}[SL_2]_q$, $B_0 \subset B_1 \subset B_2 \subset \ldots \subset B$, B_k - genereted $t_{i_1 j_1} \ldots t_{i_\ell j_\ell}$ $\ell \leq k$

$A = U_q(sl_2)^\circ$, $A_0 \subset A_1 \subset A_2 \subset \ldots \subset A$ A_k - generated $L_\ell \otimes L_\ell^*$ $\ell \leq k$

Problem Show that $L_\ell \otimes L_\ell^*$ linearly independent for different $\ell \in \mathbb{Z}_{\geq 0}$.

(Hint Define left action of $U_q(sl_2)$ on $U_q(sl_2)^\circ$. Use casimir C_q)

Hence $\dim A_k = 1 + \ldots + (k+1)^2$

B is generated by $t_{11}^{k_1} t_{21}^{k_2}$, $t_{11}^{k_0} t_{12}^{k_1} t_{21}^{k_2}$, $t_{22}^{k_0} t_{12}^{k_1} t_{21}^{k_2}$

$k_0 > 0$, $k_1, k_2 \geq 0$. Hence $\dim B_k / B_{k-1} \leq (k+1) + \frac{k(k+1)}{2} + \frac{k(k+1)}{2} = (k+1)^2$

$\psi(B_k) \subset A_k$, ψ surj $\Rightarrow$ ψ isomorphism.

Duality

- For Lie Bialgebra we have $U_\hbar(\mathfrak{g})$ and two dual objects $\rightsquigarrow U_\hbar(\mathfrak{g}^*)$ and $\mathbb{C}[G]_q$
They are closely related

- <u>Problem</u>* $U_\hbar(\mathfrak{g})$ – QUE, $\Delta_n = (\Delta \otimes id \otimes \ldots \otimes id) \ldots (\Delta \otimes id)\Delta$
Let $A = \{x \in U_\hbar(\mathfrak{g}) \mid (id - \varepsilon)^{\otimes n} \Delta_n(x) \in \hbar^n U_\hbar(\mathfrak{g})^{\otimes n}, \forall n\}$ Show that A is Hopf subalgebra cocomm up to first order in $\hbar$.
I.e. we found $\mathbb{C}[G^*]$ in $U_\hbar(\mathfrak{g})$

- <u>Problem</u>* For $\mathfrak{g} = sl_2$ define $U_\hbar(\mathfrak{g}^*)$ for standard Bialgebra structure.

References

- Chari, Pressley, A guide to quantum groups
Sec. 7.1

- Korogodski Soibelman Algebras of Functions on quantum group Sec 3.1, 3.3

- Etingof, Schiffmann, Lectures on quantum groups Ch 10