Introduction to Quantum Groups

Lecture 3

Dual P-L groups, symplectic leaves

qft.itp.ac.ru/mbersht/quantum_groups.html

Poisson – Lie groups

- G – Poisson-Lie group.
  Equivalent forms of condition on Π

ⓐ $\Pi(gh) = (\lambda_g)_* \Pi(h) + (\rho_h)_* \Pi(g)$

ⓑ $\Pi_e(gh) = \Pi_e(h) + (Ad_{h^{-1}})\Pi_e(g)$
where $\Pi_e(g) = (\lambda_{g^{-1}})_* \Pi(g) \in \Lambda^2 T_eG = \Lambda^2 \mathfrak{g}$

ⓒ (For connected G) $\Pi(e)=0$, $\delta = d\Pi_e : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$
cocycle $\delta([a,b]) = ad_a \delta(b) - ad_b \delta(a)$

- ⓑ $\Rightarrow$ ⓒ: $g=h=e \Rightarrow \Pi(e)=0$

$g = \exp(ta)$, $h = \exp tb$
$t^2 \delta([a,b]) = \Pi_e(gh) - \Pi_e(hg) = Ad_{h^{-1}} \Pi_e(g) - \Pi_e(g) = \ldots = -ad_b \delta(a) t + \ldots$

Dual group

- $(\mathfrak{g}, \delta)$ – Lie bialgebra

$[\cdot,\cdot] : \Lambda^2\mathfrak{g} \to \mathfrak{g}$ $\qquad \delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$

- $(\mathfrak{g}^*, \delta_{\mathfrak{g}}^*)$ – dual bialgebra

$\delta^* : \Lambda^2\mathfrak{g}^* \to \mathfrak{g}^*$ $\qquad \delta_{\mathfrak{g}^*} = [\cdot,\cdot]^* : \mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$

- G^* – corresponding connected, simply connected P-L group

Dressing Action

- $\ell : \mathfrak{g}^* \to \mathrm{Vect}(G)$

$$\alpha \mapsto \alpha_\ell \in \Omega^1(G) \xrightarrow{\Pi} V_\ell(\alpha) \in \mathrm{Vect}\, G$$
$\quad$ left. inv.

$$\alpha \mapsto \alpha_r \in \Omega^1(G) \xrightarrow{\Pi} -V_r(\alpha) \in \mathrm{Vect}\, G.$$

- Th $V_\ell : \mathfrak{g}^* \to \mathrm{Vect}\, G$ Lie alg anti homomorphism
  $V_r : \mathfrak{g}^* \to \mathrm{Vect}\, G$ Lie alg homomorphism
- V_ℓ, V_r integrates to actions of G^* on G.
- Orbits of this actions — symplectic leaves

Example

- $G \qquad \Pi = 0. \qquad \delta = 0$
  Dual Lie bialgebra: $\mathfrak{g}^*$ with zero bracket
  Dual P-L group $G^* = \mathfrak{g}^*$

- $\alpha \in \mathfrak{g}^* \to \alpha_e \in \Omega^1(G) \xrightarrow{\Pi} 0 \in Vect(G)$
  $G^* = \mathfrak{g}^*$ acts on G trivially.
  Orbits – points – symplectic leaves

- Dually
  $\xi \in \mathfrak{g} \rightsquigarrow \xi = \xi_e \in \Omega^1(\mathfrak{g}^*) = \Omega^1(\mathfrak{g}) \xrightarrow{\Pi} ad^*_\xi \in Vect(\mathfrak{g}^*)$
  G acts on $\mathfrak{g}^* = G^*$ by Ad.
  Coadj orbits – sympl leaves

Double

- $(\mathfrak{g}, d)$ – Lie bialgebra

$D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$ – Lie algebra (see Manin triples)

$[a+\alpha, b+\beta] = [a,b] + [\alpha,\beta] + ad^*_a \beta - ad^*_b \alpha - ad^*_\alpha b + ad^*_\beta a.$

$\mathfrak{g} \hookrightarrow D(\mathfrak{g}) \hookleftarrow \mathfrak{g}^*$ – Lie alg homomorphism

- $D(G)$ – Lie group corresp $\mathfrak{g} \oplus \mathfrak{g}^*$

$G \times G^* \longrightarrow D(G)$. local diffeomorphism.

- Example $G, \Pi = 0$, $D(G) = T^*G = G \ltimes \mathfrak{g}^*$

as manyfold

as group

Refactorization

- $G \times G^* \to D(G)$ $\qquad G = G_+ \qquad G^* = G_-$

<u>Def</u> For $\forall\, g_+ \in G_+$, $g_- \in G_-$, represent $g_- g_+$ as an element of $G_+ \times G_-$

$$g_- g_+ =: (g_+)^{g_-} (g_-)^{g_+}, \qquad (g_+)^{g_-} \in G_+, \quad (g_-)^{g_+} \in G_-$$

- <u>Rem.</u> Works if $G_+ \times G_- \to D(G)$ is global diffeom. Otherwise — requires care.

- <u>Prop</u> $\left(g_-^{\,g_+}\right)^{h_+} = g_-^{\,g_+ h_+}, \qquad \left(g_+^{\,g_-}\right)^{h_-} = g_+^{\,h_- g_-}$

- <u>Ex</u>. G, $\Pi = 0$ $\qquad G_+ = G$, $G_- = \mathfrak{g}^*$

$$(g_+)^{g_-} = g_+ \qquad (g_-)^{g_+} = Ad^*_{g_+} g_-$$

Relation to dressing

- **Th** Turning left action of G_- on G_+ to right action we get action $V_r : G_- \curvearrowright G_+$

- **Cor** Symplectic leaf through $g \in G$ is an image $G^* \cdot g \subset D(G)$ under $\underset{G \times G^*}{pr} \; D(G) \to G$.

- **Ex** G, $\pi = 0$. . . . .

Nontrivial Example

- $(\mathfrak{d}, \mathfrak{d}_+, \mathfrak{d}_-)$, $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$.

$\mathfrak{d}_+ = \mathfrak{g} = \{(a,a) \in \mathfrak{g} \oplus \mathfrak{g}\}$ $\quad \mathfrak{d}_- = \left\langle (a, b) \,\middle|\, \begin{array}{l} a \in \mathfrak{b}_+,\ b \in \mathfrak{b}_-, \\ pr_+ a = -pr_- b \end{array} \right\rangle$

$((a_1, b_1), (a_2, b_2)) = (a_1, a_2) - (b_1, b_2)$ $\quad pr_+ : \mathfrak{b}_+ \to \mathfrak{h}$ $\quad pr_- : \mathfrak{b}_- \to \mathfrak{h}$

- $D(\mathfrak{g}) = G \times G$, $G_+ = G$, $G_- = G_+ \times_H G_-$ double Bruhat cell,
- $\forall w_+, w_- \in W$, let $C_{w_+, w_-} = B_+ w_+ B_+ \cap B_- w_- B_-$

If $g \in C_{w_+, w_-}$, then $G_- g = \{(b_+ g, b_- g) \in G \times G \mid b_+ \in B_+, b_- \in B_-\}$

Refactorization. $(b_+ g, b_- g) = (\tilde{g}\tilde{b}_+, g\tilde{b}_-)$, $\tilde{g} \in C_{w_+, w_-}$

- <u>Cor</u> C_{w_+, w_-} consist of G^* orbits on G.

<u>Cor</u> C_{w_+, w_-} consist of symplectic leaves

Problem

Let $\mathfrak{g} = sl_2(\mathbb{C})$ as real Lie algebra.

$\mathfrak{g}_+ = su_2$, $\mathfrak{g}_- = \left\{ \begin{pmatrix} a & b+ic \\ 0 & -a \end{pmatrix} \middle| \; a, b, c \in \mathbb{R} \right\}$

$(\cdot, \cdot) : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ $\quad x \otimes y \mapsto \operatorname{Im} \operatorname{Tr} xy$

(a) Show that $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is Manin triple.

Find bialgebra structure on $su(2)$.

(b) Show $\mathcal{D}(G) = G \times G^*$ as manifold.

(c) Find symplectic leaves on SU_2.

References

- Chari, Pressley *A guide to quantum groups* Sec. 1.5,
- Lu, Weinstein *Poisson-Lie groups, Dressing transformations and Bruhat decompositions*