Introduction to Quantum Groups

Lecture 1

Poisson algebras and quantization

qft.itp.ac.ru/mbersht/quantum_groups.html

Formal Deformation

- Quantum group = Deformation of "algebra"
- $\hbar$ - formal parameter. A_0 - comm. algebra

Def Formal deformation $A = A_0 \hat{\otimes} \mathbb{C}[[\hbar]]$

$$a * b = a \cdot b + \hbar \mu_1(a,b) + \hbar^2 \mu_2(a,b) + \dots$$

$a, b \in A_0$ $\quad \mu_i : A_0 \otimes A_0 \to A_0$

- $\mu_1, \mu_2, \dots$ satisfy quadratic relations

$$a \mu_1(b,c) - \mu_1(ab,c) + \mu_1(a,bc) - \mu_1(a,b)c = 0$$

$\dots$

- Gauge freedom

$v_i : A_0 \to A_0$

$$a \mapsto a + v_1(a)\hbar + v_2(a)\hbar^2 + \dots$$

Geometrical Setting

- $A_0 = C^\infty(M)$, or locally $C^\infty(\mathbb{R}^n)$

$\mu_1, \mu_2, \ldots \qquad \nu_1, \nu_2, \ldots$ — diff operators

- <u>Prop</u> If μ_1 satisfies

$$a\,\mu_1(b,c) - \mu_1(ab,c) + \mu_1(a,bc) - \mu_1(a,b)c = 0$$

then using ν we get

$$\mu_1(b,c) = \sum K_{ij}\left(\partial_i b\, \partial_j c - \partial_j b\, \partial_i c\right) \qquad K_{ij} \in A_0$$

Proof by an Example

- $a\mu_1(b,c) - \mu_1(ab,c) + \mu_1(a,bc) - \mu_1(a,b)c = 0$

$\mu_1(b,c) \mapsto \mu_1(b,c) + b\,\nu_1(c) + c\,\nu_1(b) - \nu_1(bc)$

Cases w.r.t. degree of μ_1.

- degree $0 \longrightarrow \mu_1(a,b) = ab \longrightarrow \nu_1(a) = a$
- $\mu_1(a,b) = k\,a\partial_1 b$

$kab\partial_1 c - kab\partial_1 c + ka\partial_1 bc + kab\partial_1 c - ka\partial_1 bc = 0 \Longrightarrow k=0$

- More generally, if $\mu_1(a,b) = k\,a\,\partial_B b$, then

term $kab\partial_B c$ doesn't cancel.

Proof by an Example

- $a\,\mu_1(b,c) - \mu_1(ab,c) + \mu_1(a,bc) - \mu_1(a,b)c = 0$

$\mu_1(b,c) \mapsto \mu_1(b,c) + b\,\nu_1(c) + c\,\nu_1(b) - \nu_1(bc)$

- $\mu_1(b,c) = k\,\partial_1\partial_2 b\,\partial_3 c + \dots$

Hence $\mu_1 = k(\partial_1\partial_2 b\,\partial_3 c + \partial_1 b\,\partial_2\partial_3 c + \partial_2 b\,\partial_1\partial_3 c +$
$+ \partial_1\partial_3 b\,\partial_2 c + \partial_2\partial_3 b\,\partial_1 c + \partial_3 b\,\partial_1\partial_2 c)$

$\nu_1(c) = k\,\partial_1\partial_2\partial_3 c$

Geometrical Setting

- $A_0 = C^\infty(M)$, or locally $C^\infty(\mathbb{R}^n)$

$$a * b = a \cdot b + \hbar \mu_1(a,b) + \hbar^2 \mu_2(a,b) + \dots$$

- $\mu_1(b,c) = \sum K_{ij} (\partial_i b \, \partial_j c - \partial_j b \, \partial_i c)$
- $O(\hbar^2)$: $[[a,b]_*, c]_* + [[b,c]_*, a]_* + [[c,a]_*, b]_* = 0$

$[a,b]_* = a*b - b*a = 2\hbar \mu_1(a,b) + O(\hbar)$

$\mu_1(\mu_1(a,b),c) + \dots = 0$

- <u>Corollary</u> μ_1 is Poisson Bracket.

Poisson algebras

- **Def** A_0 - <u>Poisson algebra</u> if $\{\cdot,\cdot\} : A_0 \otimes A_0 \to A_0$
  - $\{a,b\} = -\{b,a\}$ anti commutativity
  - $\{a, bc\} = \{a,b\}c + b\{a,c\}$. Leibniz rule
  - $\{\{a,b\},c\} + \{\{b,c\},a\} + \{\{c,a\},b\} = 0$ Jacobi identity
- **Def** <u>Formal deformation of the Poisson algebra</u> A_0 is $A = A_0 \otimes \mathbb{C}[[\hbar]]$, s.t.

$$a * b - b * a = 2\hbar\{a,b\} + O(\hbar^2)$$

Poisson Manifolds

- <u>Def</u> M – is Poisson monifold if $C^\infty(M)$ – Poisson algebra.

- In cordinates

$$\Pi = \sum \pi_{ij}\, \partial_i \wedge \partial_j \;:\; \{f,g\} = \sum \pi_{ij}\left(\partial_i f\, \partial_j g - \partial_j f\, \partial_i g\right)$$

- If (π_{ij}) is invertable, then $\omega = \Pi^{-1}$ is symplectic form

Example Constant Bracket

- $A_0 = \mathbb{C}(x_1, \dots x_n)$ $\quad \{x_i, x_j\} = \varepsilon_{ij}, \quad \varepsilon_{ij} \in \mathbb{C}$
  $\varepsilon_{ij} = -\varepsilon_{ji} \quad \{a, \{b, c\}\} = 0$

- Example $\{x, p\} = 1$ —— $[\hat{x}, \hat{p}] = \hbar$
  Moyal product $f * g = m\left(e^{\frac{1}{2}\hbar(\partial_x \otimes \partial_p - \partial_p \otimes \partial_x)} f \otimes g\right)$
  m – multiplication

$$x * p = m\left(x \otimes p + \frac{1}{2}\hbar(\partial_x \otimes \partial_p - \partial_p \otimes \partial_x)\, x \otimes p + \dots\right) = xp + \frac{1}{2}\hbar$$

$$p * x = xp - \frac{1}{2}\hbar$$

<u>Problem</u> Show that $*$ is assos..

- For any $\varepsilon_{ij} \rightsquigarrow \begin{pmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & 0 & 1 & & \\ & & & & -1 & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix} \to$ product of prev. cases

Example Linear Bracket

- $A_0 = \mathbb{C}[x_1, \dots, x_n]$, $\{x_i, x_j\} = \sum_k \varepsilon_{ij}^k x_k$ $\varepsilon_{ij}^k \in \mathbb{C}$

$\varepsilon_{ij}^k = -\varepsilon_{ji}^k$

ε_{ij}^k - structure constants of Lie algebra.

Jacobi relation $\longrightarrow$

$\mathfrak{g} = \langle x_1, \dots, x_n \rangle$

- Geometrically $\mathfrak{g}^*$, A_0 = Functions on $\mathfrak{g}^*$

$\mathfrak{g}^*$ is Poisson manifold.

$$A_0 = \mathbb{C}[x_1 \dots x_n] = S^{\cdot}\mathfrak{g}$$

Example Linear Bracket

- $A_0 = \mathbb{C}[x_1 \dots x_n] = S^{\bullet}\mathfrak{g} = T^{\bullet}\mathfrak{g} / x\otimes y - y\otimes x = 0$
- Quantization $- \mathcal{U}\mathfrak{g} = T^{\bullet}\mathfrak{g} / x\otimes y - y\otimes x - \hbar \sum \varepsilon_{ij}^{k} x_k = 0$
- $\mathcal{U}(\mathfrak{g})$ is filtred.

$$\mathcal{U}(\mathfrak{g})_0 \subset \mathcal{U}(\mathfrak{g})_1 \subset \mathcal{U}(\mathfrak{g})_2 \subset \dots$$

- PBW $\mathcal{U}(\mathfrak{g})_n / \mathcal{U}(\mathfrak{g})_{n-1} = S^n(\mathfrak{g})$,

$$\bigoplus \mathcal{U}(\mathfrak{g})_n / \mathcal{U}(\mathfrak{g})_{n-1} = S^{\bullet}(\mathfrak{g})$$

Rees algebra

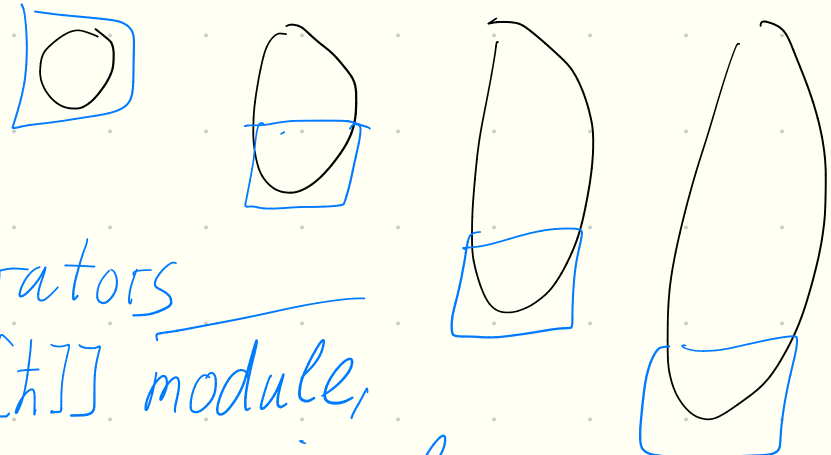
- $A = \left(\bigoplus \hbar^n \, U(\mathfrak{g})_n\right)[[\hbar]]$

elements: $u_0 + \hbar u_1 + \hbar^2 u_2 + \dots$;

$\deg u_i \leq i$

$\deg u_i < C$ for some C.

- $U_0 \quad U_1 \quad U_2 \quad U_3$



generators as $\mathbb{C}[[\hbar]]$ module, NOT canonical.

due to PBW

$A = S(\mathfrak{g}) \otimes \mathbb{C}[[\hbar]]$

- $\forall x, y \in \mathfrak{g} \subset U_1$, $(\hbar x)\cdot(\hbar y) - (\hbar y)\cdot(\hbar x) =$

$= \hbar\,(\hbar\,[x,y]) \rightsquigarrow$ quant.

Hochschild cohomology

- $C^n(A,A) =$ Linear maps $A^{\otimes n}$ to A

$d^n : C^n \to C^{n+1}$

$$d^n f(a_1, \dots, a_{n+1}) = a_1 f(a_2, \dots, a_{n+1}) - f(a_1 a_2, a_3, \dots, a_{n+1}) +$$
$$+ f(a_1, a_2 a_3, \dots, a_{n+1}) - \dots + (-1)^n f(a_1, \dots, a_n a_{n+1}) + (-1)^{n+1} f(a_1, \dots, a_n) a_{n+1}$$

$d^2 = 0$ $\quad \to C^{n-1} \to C^n \to C^{n+1} \to C^{n+2} \to \dots$

- <u>Def</u> $HH^n(A) = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}$

- <u>Rem</u> $HH^n(A) = \operatorname{Ext}^n_{A\text{-bimod}}(A,A)$

Hochschild cohomology

$$d^n f(a_1,\dots,a_{n+1}) = a_1 f(a_2,\dots,a_{n+1}) - f(a_1a_2, a_3,\dots,a_{n+1}) + f(a_1, a_2a_3,\dots,a_{n+1}) - \dots + (-1)^n f(a_1,\dots,a_n a_{n+1}) + (-1)^{n+1} f(a_1,\dots,a_n)a_{n+1}$$

- $HH^0(A) = Z(A)$ - center

$\operatorname{Ker} d^1$ - derivations of A, $\operatorname{Im} d^0$ - inner derivations

- In formal deformation $\mu_1 \in \operatorname{Ker} d^2$,

μ_1 is defined up to $d^1 v_1 \longrightarrow [\mu_1] \in HH^2(A)$

- <u>Th.</u> (Hochshild-Kostant-Rozenberg)

If $A = C^\infty(M)$ then $HH^n(A) = \Lambda^n(TM)$ - polyvector fields

- <u>Th</u> $\longrightarrow$ Prop above

Hochschild cohomology

- **Problem*** Show that $HH^2(U(\mathfrak{g}))=0$, where $\mathfrak{g}$ – simple Lie algebra

Hint PBW, HKR $\to$ $H(\mathfrak{g}, S^\cdot(\mathfrak{g})) \to H(\mathfrak{g}, \mathbb{C}) \otimes Z(U(\mathfrak{g}))$, (actually spectral sequence degenerates at E^2)

- Th (Kontsevich) $A = C^\infty(M)$, $\forall \{\,;\,\}$ $\exists$ deform. quant.

Rm For M - symplectic – (Fedosov, ...

- Ex $M = T^*X$, $\mathrm{Diff}(X)$ is filtred by degree of diff operator. $\mathrm{Diff}_0 \subset \mathrm{Diff}_1 \subset \ldots$,

A = Rees algebra $\mathrm{Diff}\, X$ — deform quant

- **Problem*** $\exists$ Poisson alg. A s.t. no quant

Symplectic Leaves

- Π – Poisson structure on M.

$\forall x \in M \quad \Pi \in \Lambda^2 T_x M \rightsquigarrow \Pi : T_x^* M \to T_x M \rightsquigarrow T_x^\Pi = \mathrm{Im}\,\Pi \subset T_x M$

Equivalently $\quad T_x^\Pi = \langle V_H \mid H \in C^\infty(M) \rangle$, V_H – ham. vect. field.

Equivalently $\quad \Pi = \sum \Pi^i_{(1)} \otimes \Pi^i_{(2)}$, $\quad T_x^\Pi = \langle \Pi^i_{(1)} \rangle = \langle \Pi^i_{(2)} \rangle$

- <u>Problem</u> Show that distribution T^Π is integrable

<u>Hint</u> Use Frobenius theorem, it is sufficient to check integrability for vector fields V_H.

(Actually Frobenius theorem works only on open subset of fixed rank. In general case Stefan–Sussmann or Weinstain splitting theorem.)

- <u>Def</u> Symplectic leaves – submanifolds tangent to T^Π

Symplectic Leaves

- Another way: $x \sim y$ if $\exists$ curve piecewise smooth curve $\gamma: x - y$ s.t. γ tangent to T^Π

  Symplectic leaves: classes for $\sim$

- <u>Rem</u> Symplectic leaves generally are not submanifolds, irrational winding $\exists$.

- <u>Ex</u>. $M = \mathfrak{g}^*$, $\forall \xi \in \mathfrak{g}$, ξ - function on M,

  V_ξ - ham. vect. field $\Leftarrow$ $\forall \lambda \in \mathfrak{g}^*$, $\eta \in \mathfrak{g}$

  $$(V_\xi \eta)(\lambda) = \{\xi, \eta\}(\lambda) = (\lambda, [\xi, \eta]) = (ad^*_\xi \lambda, \eta)$$

  $\Rightarrow V_\xi = ad^*_\xi \Longrightarrow$ Symplectic leaves = coadjoint orbits

References

- Etingof, Schiffmann, Lectures on quantum groups Ch 1
- Chari, Pressley A guide to quantum groups Sec. 1.1, 1.6
- Kontsevich Deformation quantization of Poisson manifolds
- Calaque, Rossi Lectures on Duflo isomorphisms in Lie algebras and complex geometry Sec 1, 2, 3.