Itroduction to cluster algebras and varieties

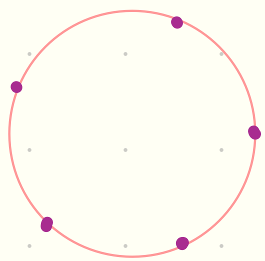
Lecture 12-14

Cluster structures on the moduli spaces of local systems

- We know ($\mathcal{X}$-cluster) coordinates on $\mathrm{Conf}_n(\mathbb{P}^1)$

Geometrically

Seeds $\longleftrightarrow$ triangulations of n-gon

$\longleftrightarrow$  triangulations of disk with n marked points on the boundary

$p_i \longrightarrow u_i \in \mathbb{P}^1$ up to PGL_2

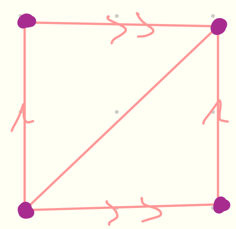
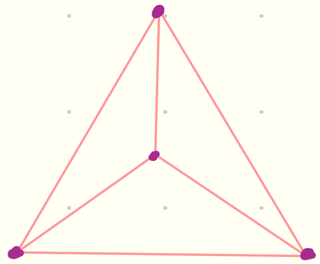
- <u>Question</u> Can we consider more general surfaces?

- Let S - surface with punctures (probably) with boundary and marked points on it.

<u>Assumption</u> $\forall$ bound. comp. has at least 1 marked point,
$\#$ marked points + $\#$ punctures ≥ 1
$\chi(S) < 0$

- Let $\hat{S}$ be universal cover of S

Triangulation with vertices in marked points and punctures

- <u>Example</u> (a)  (b)  S^2 with 4 punctures

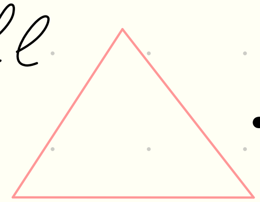
$\mathbb{T}$ with 1 puncture

- We can lift triangulation to the $\hat{S}$.

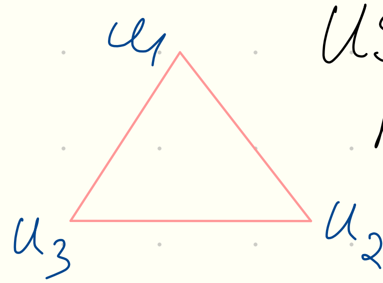
- <u>Assume</u> a map $X : \{\text{edges on } S\} \to \mathbb{C}^*$

  <u>Construct</u> a map $u : \{\text{vertices on } \hat{S}\} \to \mathbb{P}^1$ up to PGL_2

Take some triangle on $\hat{S}$

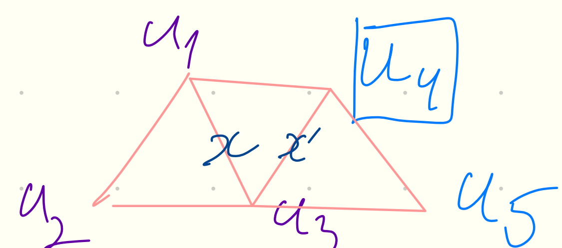


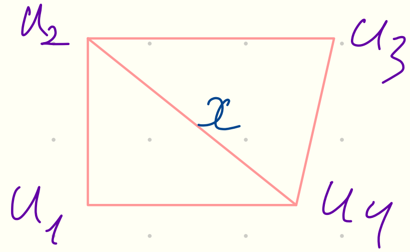
$\longrightarrow$



Assig some u-s to vertices using PGL_2 we can make $u_1 = \infty$, $u_2 = -1$, $u_3 = 0$.

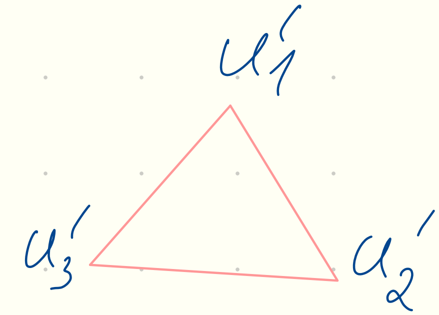
$\longrightarrow$ Using edge weigts (x-s) we can compute futher u-s



(Recall 

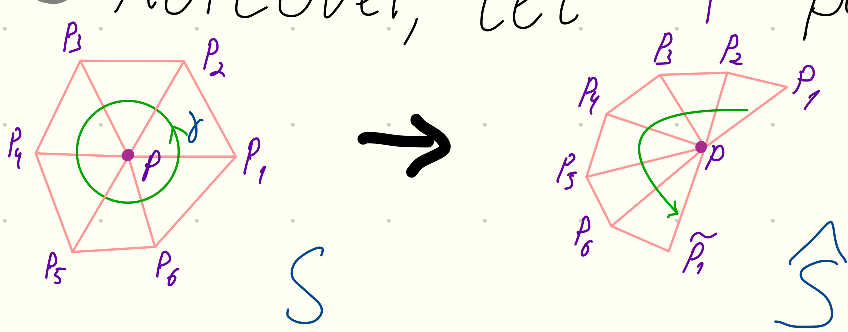
, hence $x=[u_1:u_2:u_3:u_4]$)

→ For $\forall \gamma \in \pi_1(S)$ path γ on $\hat{S}$ leads to another triangle

$\exists!\ \rho(\gamma)\in PGL_2$ s.t. $\gamma \longrightarrow$ 

$\rho(\gamma)u_1=u_1'$, $\rho(\gamma)u_2=u_2'$, $\rho(\gamma)u_3=u_3'$

- We obtained a representation of $\pi_1(\hat{S})$
  $\rho:\pi_1(S)\to PGL_2$ — "local system"
- Moreover, let p - puncture, γ - loop around p



Then $\rho(\gamma)u_p=u_p$

- Overall, we have

$$(\mathbb{C}^*)^{\#\,\text{edges}} \longleftrightarrow \mathrm{Hom}(\pi_1(S), PGL_2)/PGL_2$$

+ line in each marked point, invariant in each puncture

$\overset{\text{def}}{\longleftrightarrow}$

Framed PGL_2 local systems on S.

- 

"Principal PGL_2-bundle with flat connection"

/

monodromy of connection

$\mathrm{Hom}(\pi_1(S), PGL_2)/PGL_2$

- above u-line in assosiated vector bundle
$\longleftrightarrow$ point in assosiated $\mathbb{P}^1$ bundle

- Inverse map. Assume that we have

$\rho : \pi_1(S) \to PGL_2$, + line in each marked point, invariant in each puncture

framing

Lift to $\hat{S}$. The local system on $\hat{S}$ is trivial hence we can consider lines in fixed $\mathbb{C}^2$.

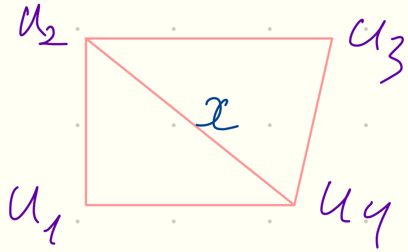
Hence we get $u : \{\text{vertices on } \hat{S}\} \to \mathbb{P}^1$

u is ρ equivariant, namely if $\gamma \cdot p = p'$ then $\rho(\gamma)\, u_p = u_{p'}$, where p, p' vertices on $\hat{S}$.

Computing cross ratios we have

$X : \{\text{edges on } \hat{S}\} \to \mathbb{C}^*$

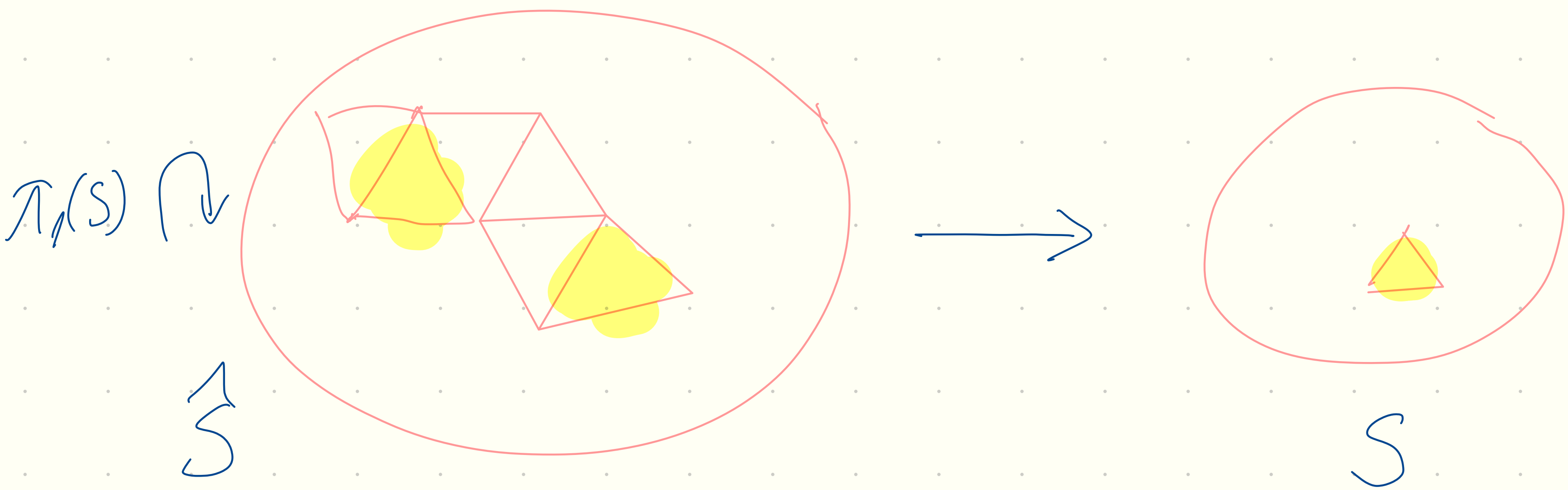
(Recall



, hence $x = [u_1 : u_2 : u_3 : u_4]$)

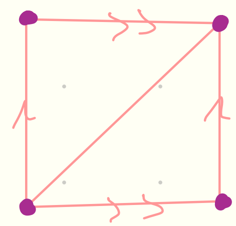
Due to ρ equivarience of $\mathcal{U}$ we have invariance of X i.e.

$$X : \{\text{edges on } \hat{S}\} \longrightarrow \mathbb{C}^*$$

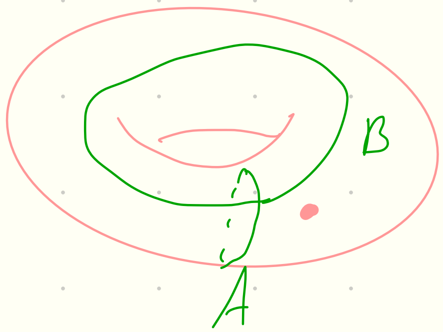


Example

(a)



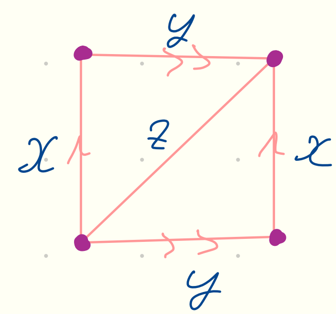
$\mathbb{T}$ with 1 puncture



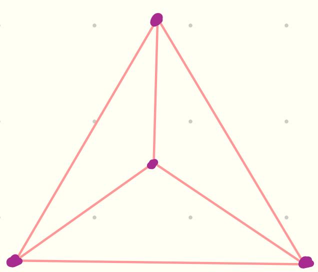
$\pi_1(S)$ — F_2 — free group generated by γ_A, γ_B

$$\dim\left(\mathrm{Hom}(\pi_1(S), PGL_2) / PGL_2 \right) = 3 = \#\text{ edges}$$

<u>Problem</u> Write representation corresp. to
Find corresp quiver.



(b)



S^2 with 4 punctures

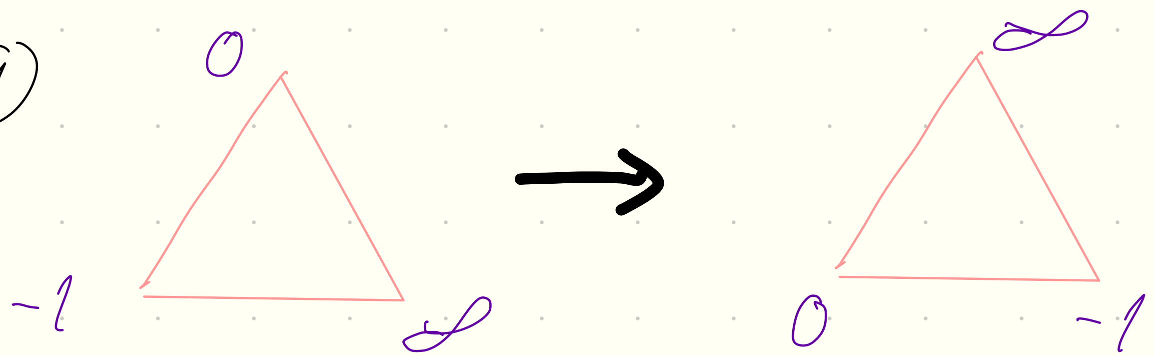
$$\pi_1(S) = \langle \mu_1, \mu_2, \mu_3\, \mu_4 \mid \mu_1 \mu_2 \mu_3 \mu_4 = e \rangle$$

$$\dim\left(\mathrm{Hom}(\pi_1(S), PGL_2) / PGL_2 \right) = 6 = \#\text{ edges}$$

- **Remark** The framed local systems form a stack. We work on open subset, where everything is in general position.

- How to construct monodromy more explicitly? Two transformations:

(1)



$$g = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

$$z \mapsto \langle e_1 + z e_2 \rangle$$

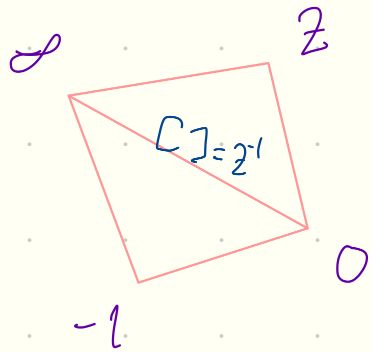
$$\langle e_1 \rangle \mapsto \langle e_2 \rangle$$

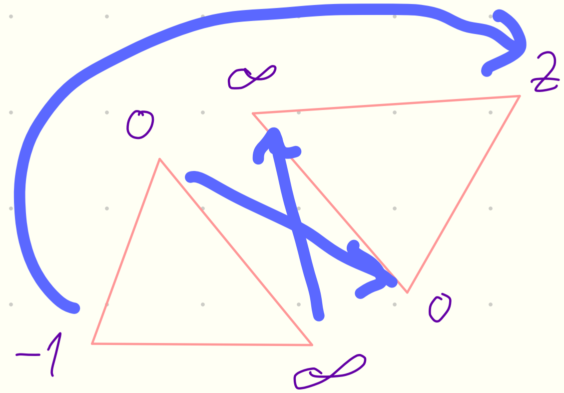
$$\langle e_2 \rangle \mapsto \langle e_1 - e_2 \rangle$$

$$\langle e_1 - e_2 \rangle \mapsto \langle e_1 \rangle$$

Note $g^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

②





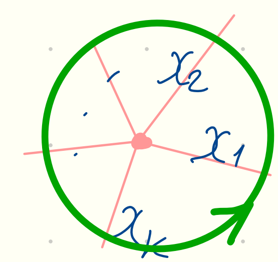
$$g = \begin{pmatrix} 0 & -z \\ 1 & 0 \end{pmatrix}$$

$\langle e_1 \rangle \mapsto \langle e_2 \rangle$

$\langle e_2 \rangle \mapsto \langle e_1 \rangle$

$\langle e_1 - e_2 \rangle \mapsto \langle e_1 + z e_2 \rangle$

- <u>Problem</u> Let P - puncture, γ - path around it, $x_1, \dots, x_k$ - weigts of edges adjacent to P.

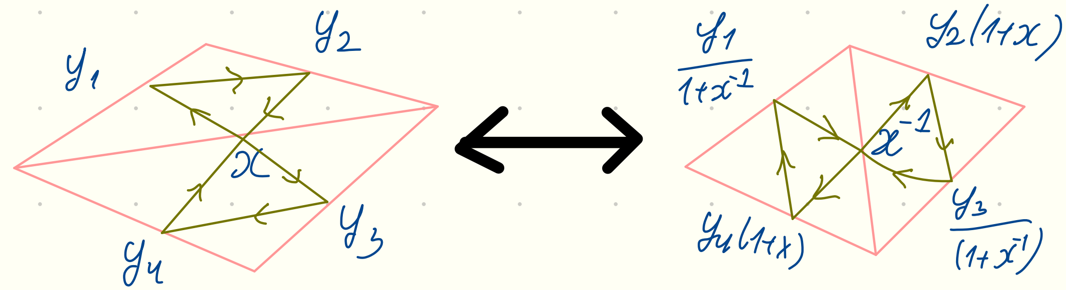


$$\rho(\gamma) \sim \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\lambda_1 / \lambda_2 = x_1 \cdots x_k$$

(in PGL_2 only ratio of eigenvalues has meaning)

- <u>Remark</u> Flip of triangulation — mutation of $\mathcal{X}$ variables



<u>Theorem</u> The space of framed PGL_2 local systems has structure of $\mathcal{X}$ cluster variety

seeds $\longleftrightarrow$ triangulations

(including triangulations with self-folded triangles)

- <u>Question</u> Higher rank generalization?

PGL_m - framed local systems?

Def [Fock Goncharov]

$G = PGL_m$

$$\mathcal{X}_{G,S} = \left\{ \begin{array}{l} G\text{-local systems } \mathcal{L} \\ + \text{ choice of flag} \\ \text{for each marked point} \\ \text{and invariant flag at} \\ \text{puncture} \end{array} \right\} \Big/ \sim$$

- (Complete) Flag $\quad 0 = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \dots \subset F_m = \mathbb{C}^m$

  $\dim F_k = k$.

Group PGL_m acts on $\{\text{flags}\}$

Stabilizer of standard flag $0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots$

is $B = \begin{pmatrix} * & \cdots & \cdots \\ 0 & * & \vdots \\ \vdots & 0 & \ddots \\ 0 & 0 & * \end{pmatrix}$ Hence $\{\text{flags}\} = PGL_m / B$

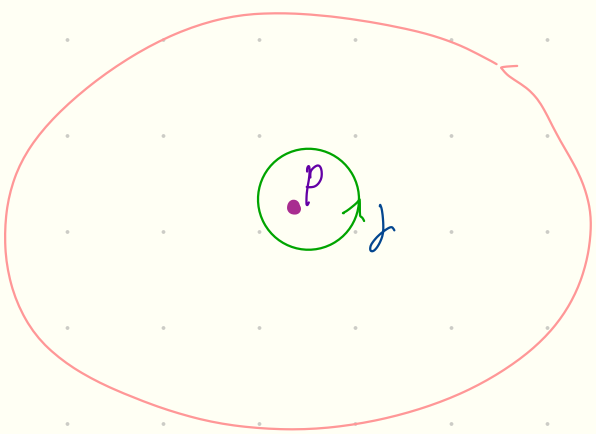
- Remark For $m=2$ we have $\{\text{flags}\} = \mathbb{P}^1$

- For any marked point or puncture p
  framing $p \longrightarrow G$ invariant section of $\mathcal{L} \times_G G/B$

in other words

local systems $\Longleftrightarrow$ monodromy of vector bundle with flat connection.

$p \longleftrightarrow$ flag in the fiber of this bundle



μ_γ : fiber at p $\to$ fiber at p

flag $\mapsto$ flag.

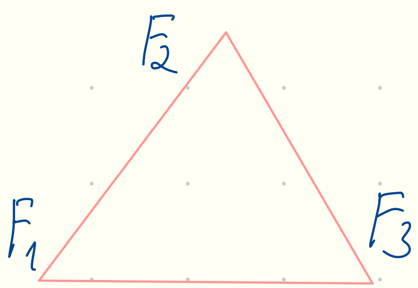
- <u>Remark</u> If μ_j is generic $\mu_j \sim \begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_m\end{pmatrix}$ then there exist $m!$ choices of invariant flags: $\sigma \in S_n$

$$F_\sigma = 0 \subset \langle e_{\sigma(1)}\rangle \subset \langle e_{\sigma(1)}, e_{\sigma(2)}\rangle \subset \dots \subset \mathbb{C}^m$$

Hence if S has no boundary:

$$\mathcal{X}_{G,S} \longrightarrow \{\text{Local } G \text{ systems on } S\} \qquad m!^{\#\text{punctures}} \to 1$$

- On boundary we have no conditions on flags



Hence $\dim \mathcal{X}_{G,S} \geq \{\text{local } G \text{ systems}\}$

E.g. for triangle $\qquad \dim\{\text{local system}\} = 0$

$$\dim \mathcal{X}_{G,S} = \underbrace{3(\dim G - \dim B)}_{3 \text{ flags}} - \underbrace{\dim G}_{\text{conjugation}} = 2\dim G - 3\dim B$$

- [Goncharov Shen]

$\mathcal{P}_{G,S} = \mathcal{X}_{G,S}$ + extra date called "pinnings"

- Pinning $B, B' \subset G$ pair of Borel subgroups

Assume $B \cap B' = H$ is abelian (i.e general position)

(e.g. $B = \{$upper triangular matrices$\}$
$B' = \{$lower triangular matrices$\}$)

For any simple positive coroot $\alpha_i^\vee$ we assign a homomorphism $\phi_i : SL_2 \to G$ s.t

$$\phi_i\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right) \in B \qquad \phi_i\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) \in H \qquad \phi_i\left(\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\right) \in B'$$

(a choice of Chevalley generators)

$H \to H \qquad E \to \lambda E \qquad F \to \lambda^{-1} F$

Example

$B = \{\text{upper triangular matrices}\}$
$B' = \{\text{lower triangular matrices}\}$

$$\phi_i : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2 \longmapsto \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & a & b & & \\ & & c & d & & \\ & & & & 1 & \\ & & & & & \ddots\ 1 \end{pmatrix}$$

Triple $(B, B', \{\phi_i\})$ is called a pinning over (B, B')

- More explicitly for PGL_m

$B \longleftrightarrow$ flag $(0 = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_m = \mathbb{C}^m)$
$B' \longleftrightarrow$ flag $(0 = F_0' \subsetneq F_1' \subsetneq F_2' \subsetneq \dots \subsetneq F_m' = \mathbb{C}^m)$

B, B' - in general position $\Longleftrightarrow F, F'$ - in general position

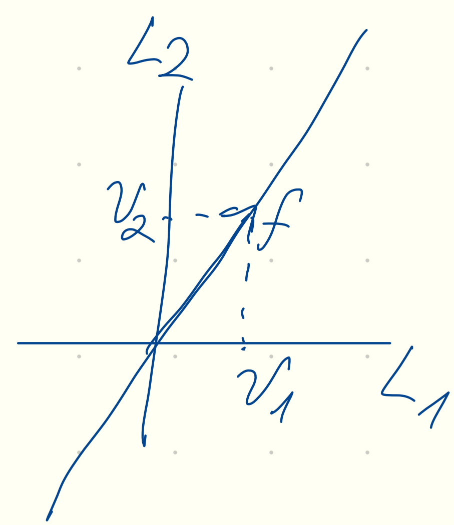
Hence $F_i \cap F'_{m+1-i} =: L_i$, $\dim L_i = 1$

$\mathbb{C}^n = L_1 \oplus L_2 \oplus \dots \oplus L_m$

(easy to see $F_i = L_1 \oplus \dots \oplus L_i$ $F'_i = L_m \oplus \dots \oplus L_{m+1-i}$)

<u>Problem</u> Pinning over (B, B') $\longleftrightarrow$ choice of $v_i \in L_i$ up to $\{v_i\} \to \{\lambda v_i\}$ $\lambda \in \mathbb{C}^*$ $\longleftrightarrow$ choice of line $\langle f \rangle \subset \mathbb{C}^m$

$\langle f \rangle$, $v_i := pr_{L_1 \oplus \dots \oplus \hat{L_i} \oplus \dots \oplus L_m} f$



<u>Problem</u> $\forall p, p'$ - pinnings $\exists!$ $g \in G$ s.t $g \cdot p = p'$

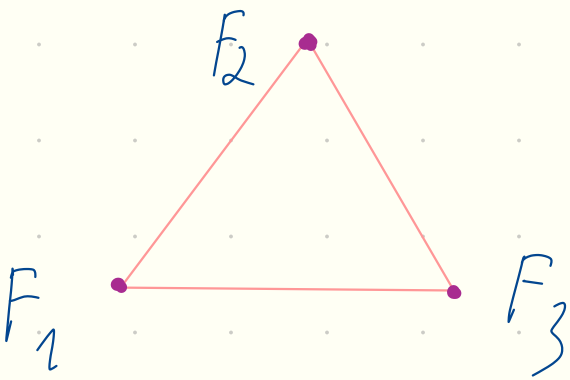
adjoint action on B, B', $\operatorname{im} \phi_i$

- **Def** $\mathcal{P}_{G,S} = \mathcal{X}_{G,S}$ + pinning assigned to any boundary segment

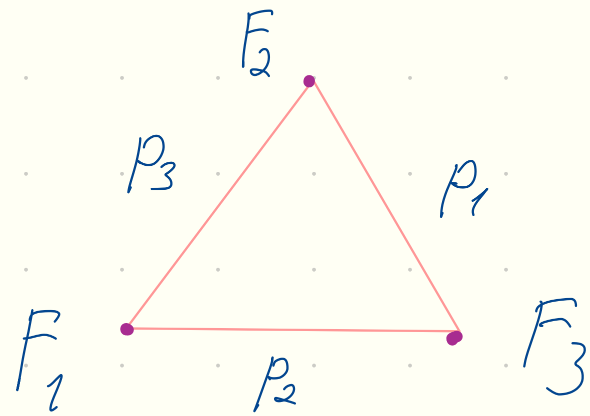
- **Remark** If S has no boundary $\Rightarrow$ $\mathcal{P}_{G,S} = \mathcal{X}_{G,S}$

more generally $\dim \mathcal{P}_{G,S} = \dim \mathcal{X}_{G,S} + \mathrm{rk}\, G \cdot \#\{\text{bound. marked pts}\}$

- **Example** disc with 3 marked points

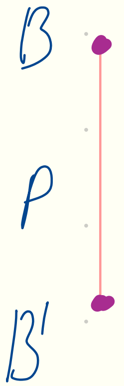


$\mathcal{X}_{G,S}$



$\mathcal{P}_{G,S}$

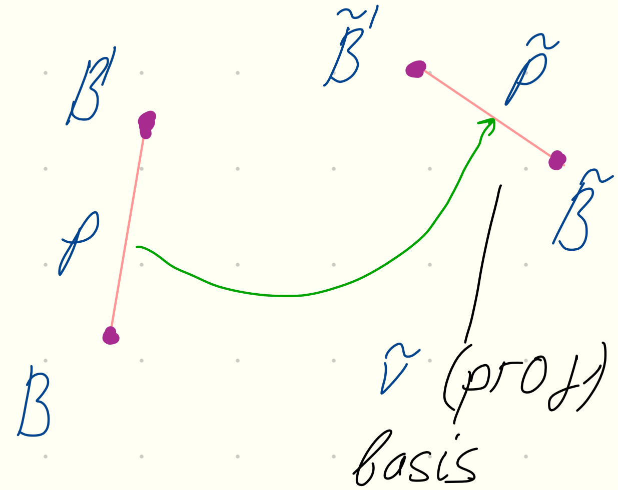
● Advantages. ⓐ We can define a projective basis on any boundary segment.



Hence we can compute Wilson line: parallel transport from one boundary segment to another one

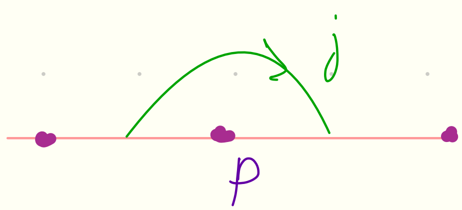
M_γ – parallel transport along γ

v – (proj) basis



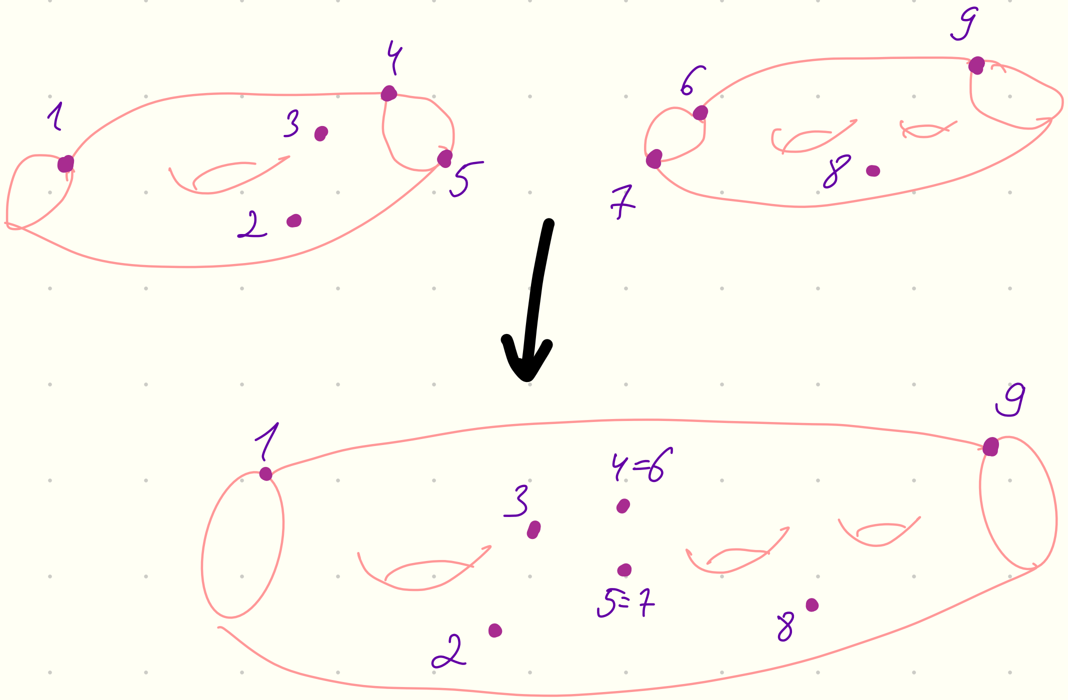
$\exists!\, g_\gamma$ s.t. $g_\gamma \cdot M_\gamma v = \tilde{v}$

In particular, now we have condition at marked points on boundary



$g_j F_p = F_p$

(b) Gluing map is avaible



we glue bases on corresp segments

$45 \longleftrightarrow 67$

v_{45} v_{67} $\exists!\, g_{45}$ s.t.

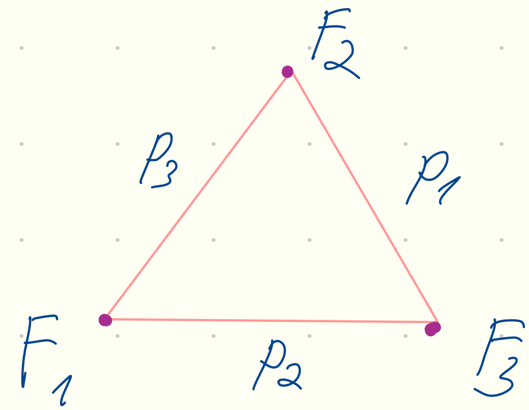
$$g_{45} v_{45} = v_{67}$$

$54 \longleftrightarrow 76$ $\exists!\, g_{54}$ s.t.

$$g_{54} v_{54} = v_{76}$$

- In particular, if we have triangulation of S we can glue (amalgamate) cluster structure on $\mathcal{P}_{G,S}$ from cluster structures on $\mathcal{P}_{G,\Delta}$.

• Case of the triangle

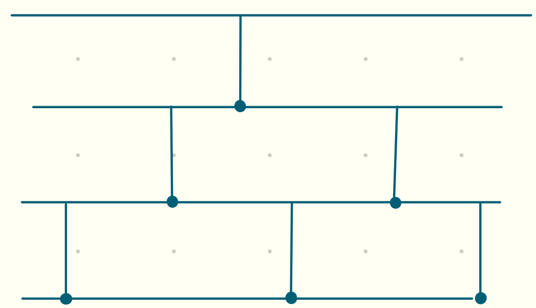


Local system is trivilar.
Monodromy $v_{F_2F_1} \to v_{F_2F_3}$ preserves flag F_2 hence it belongs to some Borel subgroup

recall that $G^{e,w_0} = B_+ \cap B_- w_0 B_- = B_+$

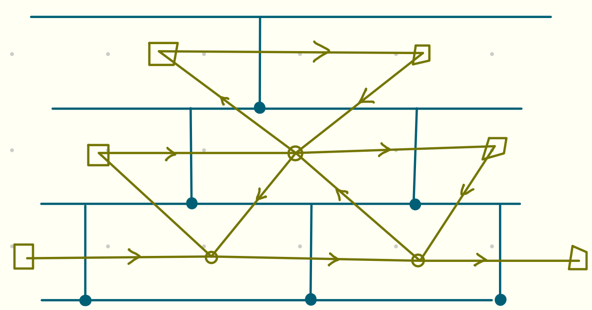
PGL_4

$w_0 = s_3\, s_2\, s_1\, s_3\, s_2\, s_3$



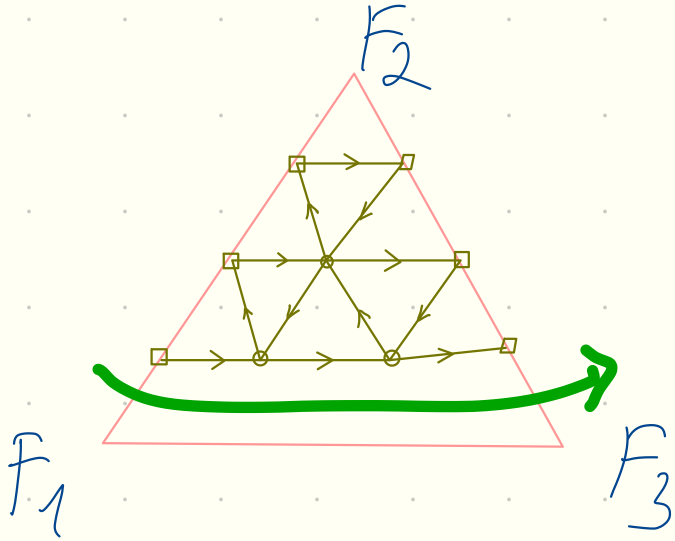
$s_3\ s_2\ s_1\ s_3\ s_2\ s_3$

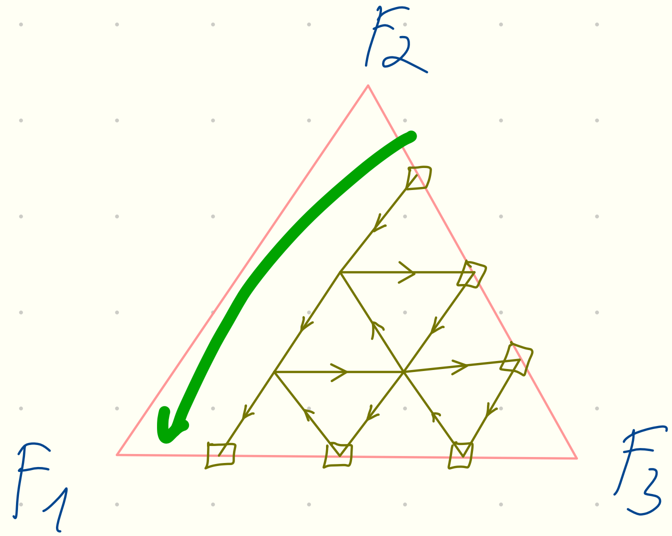
network
bipartita graph



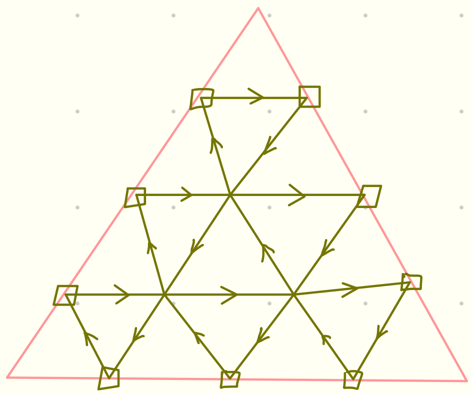
quiver

•

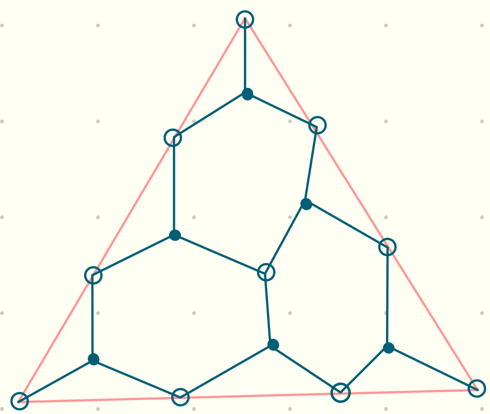




we define quiver for triangle as



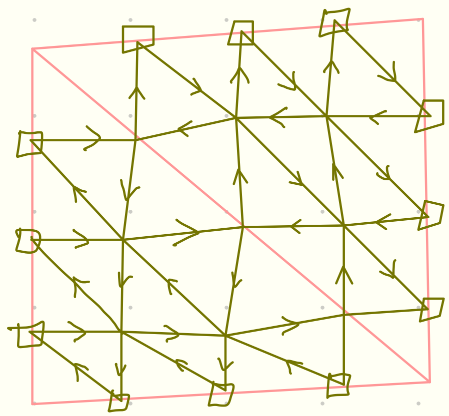
corresp. bipartite graph

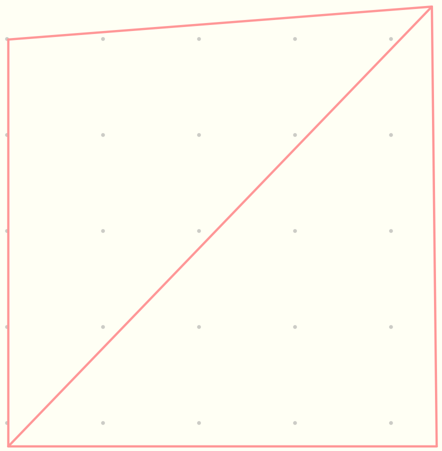


• Quiver for △ has 3 rk G frozen variables

- Gluing $\longleftrightarrow$ amalgamation of $\mathcal{X}$ cluster varieties (unfrozing vertices which become internal)

<u>Example</u>

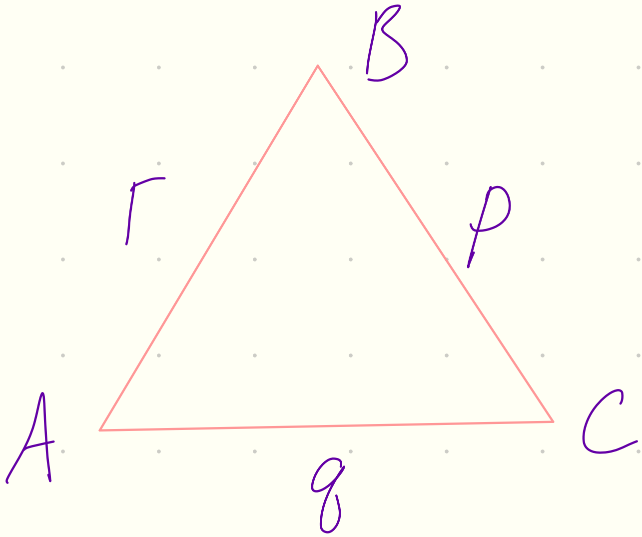




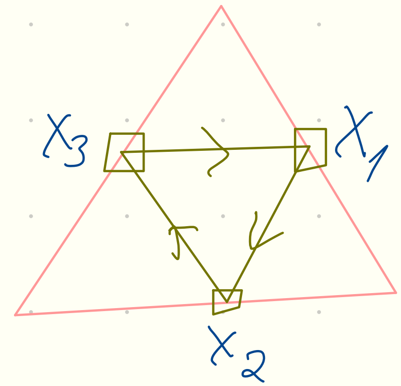
<u>Problem</u> Quivers and bipartite graphs for triangulations related by flip are mutation equivalent (Check for PGL_3, PGL_4)

- Cluster variables, coordinates on $\mathcal{J}_{a,s}$
Inverse map: monodromy from cluster coor.

- PGL_2 revisited A, B, C – flags, p, q, r – pinnings



want



$\mathcal{T}_{a,s}$

$X_{a,s} = pt$

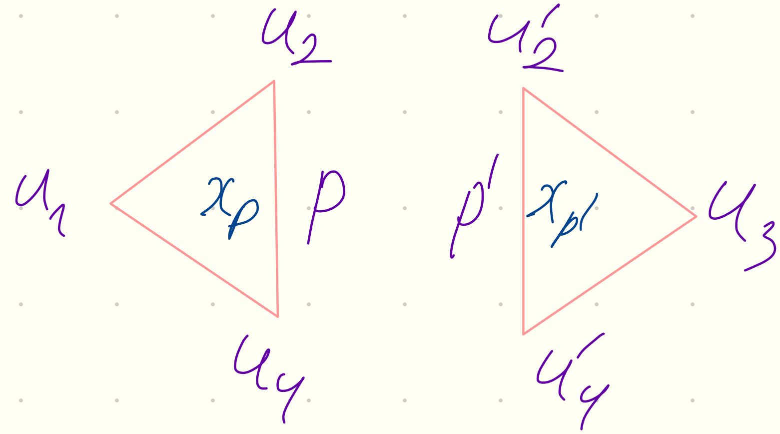
Let $A = \{0 \subset \langle a \rangle \subset \mathbb{C}^2\}$, $B = \{(0) \subset \langle b \rangle \subset \mathbb{C}^2\}$, $C = \{0 \subset \langle c \rangle \subset \mathbb{C}^2\}$

<u>Define</u> $x_1 = [a, b, p, c]$, $x_2 = [b, c, q, a]$, $x_3 = [c, a, r, b]$.

(Recall $[a_1, a_2, a_3, a_4] = -\dfrac{(a_1 - a_2)}{(a_2 - a_3)} \dfrac{(a_3 - a_4)}{(a_4 - a_1)}$)

- <u>Remark</u> Definition requires choice of trivialization of local system on $\triangle$.

• Gluing

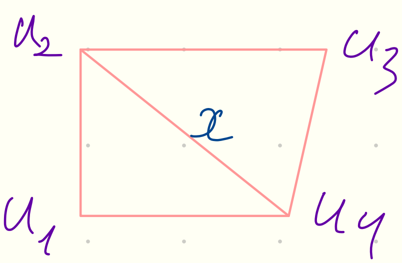


Fix trivialization of local system on square

Gluing conditions $u_2'=u_2$, $u_4'=u_4$, $[u_2, p, u_4, p'] = 1$

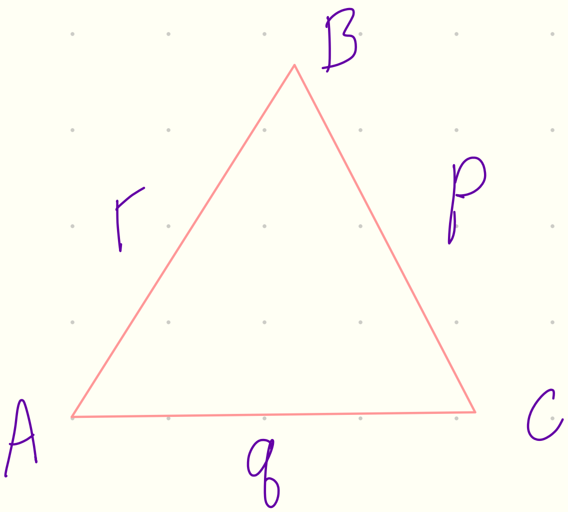
$$x_p \cdot x_{p'} = [u_1\, u_2\, p\, u_4][u_3\, u_4\, p'\, u_2] = \frac{u_1-u_2}{u_2-p}\,\frac{p-u_4}{u_4-u_1}\,\frac{u_3-u_4}{u_4-p'}\,\frac{p'-u_2}{u_2-u_3} = -\frac{u_1-u_2}{u_4-u_1}\,\frac{u_3-u_4}{u_2-u_3} =$$

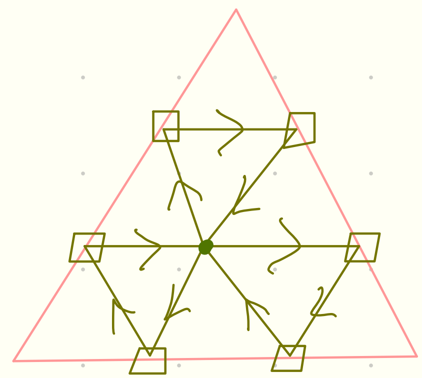
$$= [u_1, u_2, u_3, u_4]$$



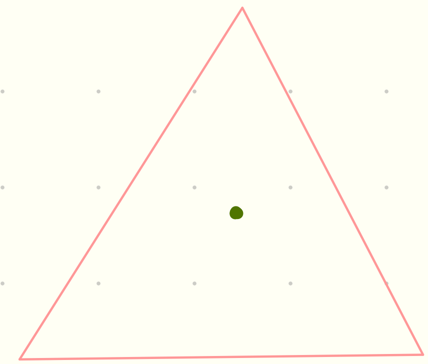
, hence $x = [u_1 : u_2 : u_3 : u_4]$ agrees with "old" definition.

- PGL_3 triangle (disk with 3 marked points)





$\mathcal{J}_{GL_3,\Delta}$



$X_{GL_3,\Delta}$

$$\dim \mathcal{J}_{GL_3,\Delta} = \frac{\dim(A,B,C,p,q,r)}{PGL_3} = 3(3+2)-8 = 7$$

$$\dim X_{GL_3,\Delta} = \frac{\dim(A,B,C)}{PGL_3} = 3\cdot 3 - 8 = 1.$$

Def Let $A = 0 \subset \langle a_1 \rangle \subset \langle a_1, a_2 \rangle \subset \mathbb{C}^3$

$B = 0 \subset \langle b_1 \rangle \subset \langle b_1, b_2 \rangle \subset \mathbb{C}^3$

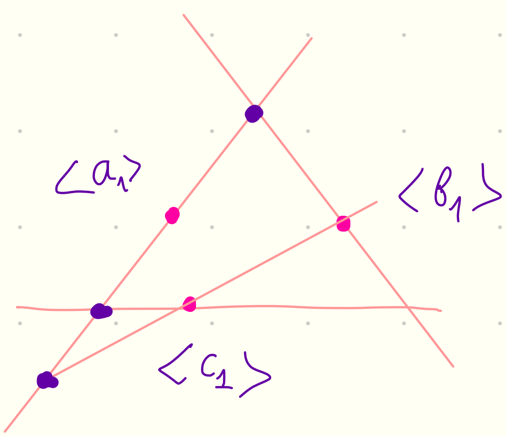
$C = 0 \subset \langle c_1 \rangle \subset \langle c_1, c_2 \rangle \subset \mathbb{C}^3$

$$[u, v, w] = \frac{\det(a_1, a_2, b_1)\,\det(b_1, b_2, c_1)\,\det(c_1, c_2, a_1)}{\det(a_1, a_2, c_1)\,\det(b_1, b_2, a_1)\,\det(\ _1,\ _2,\ _1)}$$

Lemma $[u, v, w]$ does not depend on choices of det or basic vectors $a_1, a_2, b_1, b_2, c_1, c_2$.

Remark $\langle a_1 \rangle, \langle b_1 \rangle, \langle c_1 \rangle$ — points on $\mathbb{P}^2$

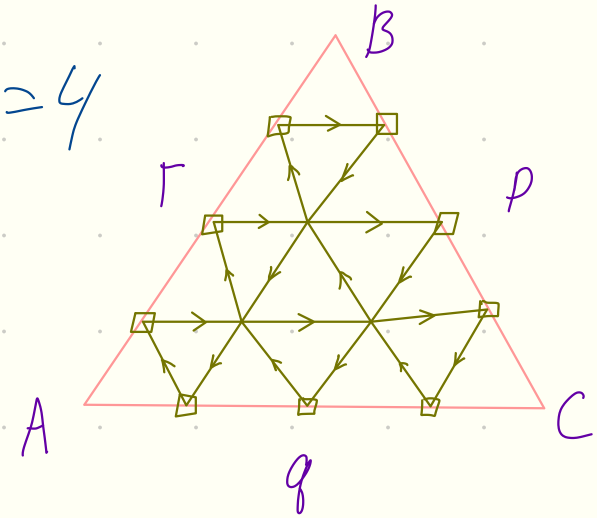
$\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle, \langle c_1, c_2 \rangle$ — lines on $\mathbb{P}^2$



cross-ratio of these 4 points $\longleftrightarrow [A, B, C]$

- PGL_m case

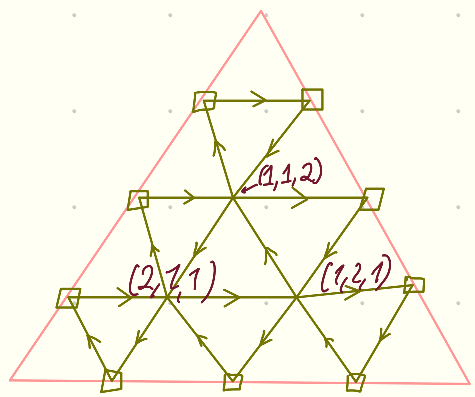
Figures $m=4$



- Triangle

@ Internal $v \to (a,b,c)$ – distances to the sides

<u>Properties</u> $a,b,c>0, \quad a+b+c=m$



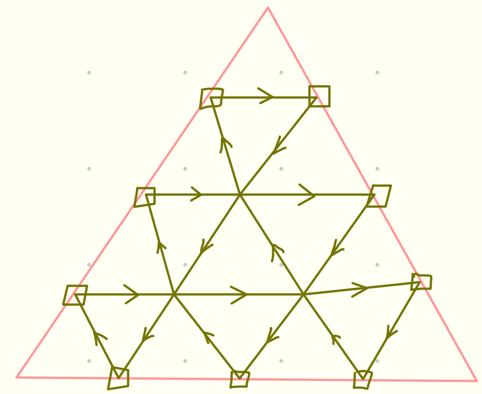
$\mathbb{C}^m / A_{a-1} \oplus B_{b-1} \oplus C_{c-1}$ – 3 dimensional subspace

$\pi_{a,b,c} : \mathbb{C}^m \to \mathbb{C}^m / A_{a-1} \oplus B_{b-1} \oplus C_{c-1}$ regires trivializ.

Define $x_{a,b,c} = [\pi_{a,b,c}(A), \pi_{a,b,c}(B), \pi_{a,b,c}(C)]$

Image of $A_a \subset A_{a+1}$ $B_b \subset B_{b+1}$ $C_c \subset C_{c+1}$

(B) Boundary vertices



$v \in (1,3) \longrightarrow (a,0,c)$

distances to the sides

Properties $\quad a,c>0 \qquad a+c=m$

$\mathbb{C}^m / A_{a-1} \oplus C_{c-1}$ — 2 dimensional subspace

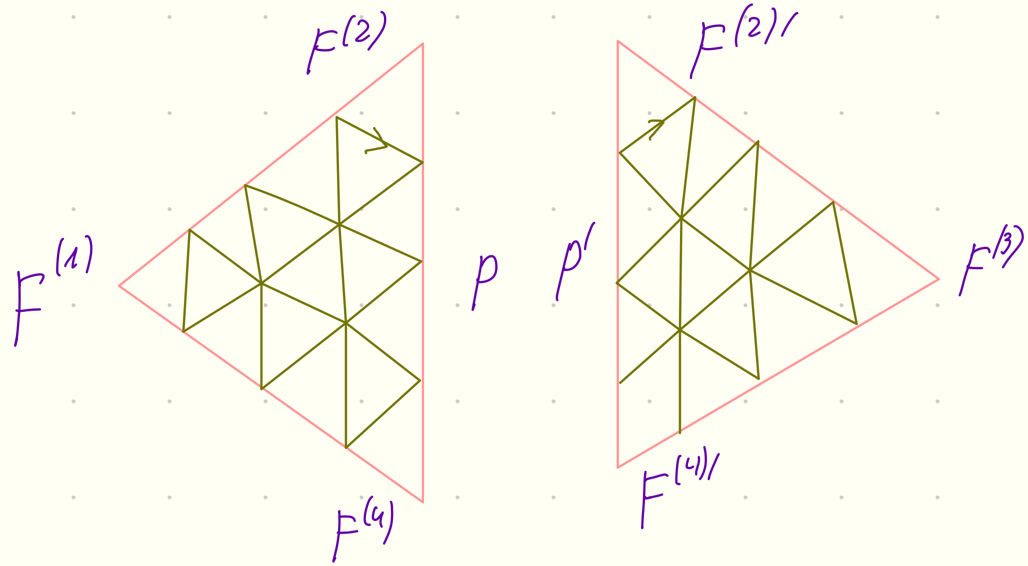
$\pi_{a,0,c}: \mathbb{C}^m \longrightarrow \mathbb{C}^m / F^{(1)}_{a-1} \oplus F^{(3)}_{c-1}$

Define $\quad x_{a,0,c} = [\pi_{a,0,c}(B), \pi_{a,0,c}(C), \pi_{a,0,c}(P_2), \pi_{a,0,c}(A)]$

Image of $\quad B_1 \qquad A_a \qquad P_2 \qquad C_c$

• Gluing

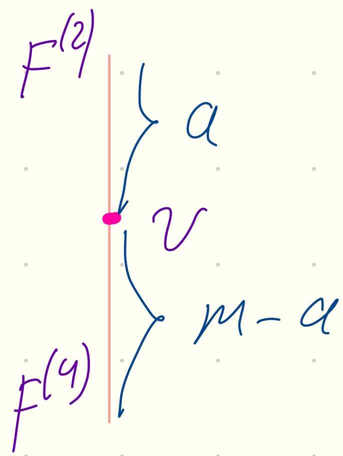
Conditions



$$F^{(2)} = F^{(2)\prime}, \quad F^{(4)} = F^{(4)\prime}$$

ⓐ Internal vertices — same as above

ⓑ Boundary vertices



$$\pi_a : \mathbb{C}^m \longrightarrow \mathbb{C}^m / F^{(2)}_{m-a-1} \oplus F^{(4)}_{a-1}$$

$$x_v = \left[\pi_a(F_1^{(1)}), \pi_a(F^{(2)}_{m-a}), \pi_a(F_1^{(1)}), \pi(F_a^{(4)})\right] = x_{v,\triangleleft}\, x_{v,\triangleright}$$

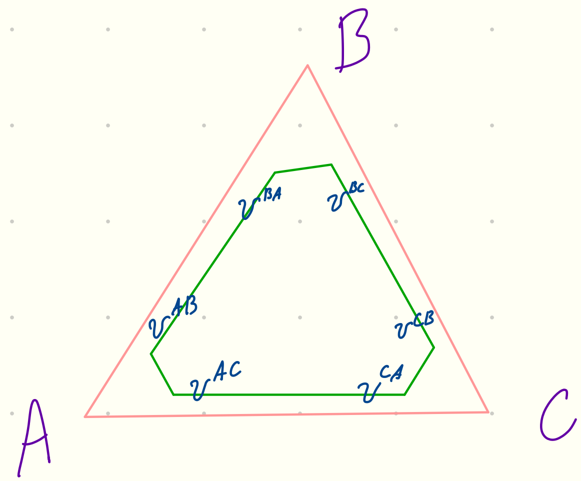
Requires $\left[\pi_a(F^{(2)}_{m-a}), \pi_a(p), \pi(F_a^{(4)}), \pi_a(p')\right] = 1 \quad \forall a$ gluing conditions

- <u>Problem</u> Let $F^{(2)} = \{0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \ldots \subset \langle e_1, \ldots, e_{m-1} \rangle \subset \mathbb{C}^m\}$
$F^{(4)} = \{0 \subset \langle e_m \rangle \subset \langle e_m, e_{m-1} \rangle \subset \ldots \subset \langle e_m, e_{m-1}, \ldots, e_2 \rangle \subset \mathbb{C}^m\}$,

$p = \langle \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_m e_m \rangle$. Find p' s.t. gluing conditions are satisfied

Inverse map

- **Def** Projective basis in $\mathbb{C}^m$ - basis $v_1, \dots, v_m$ up to common rescaling.



- To any triangle we assign 6 projective bases in $(\mathbb{C}^m)^*$:

$v^{AB}, v^{AC}, v^{BA}, v^{BC}, v^{CA}, v^{CB}$

Construction is cyclically symmetric. Hence sufficient to define v^{AC}, v^{CA}

Sometimes in order to stress triangle and orientation we will write $v^{A^B_C}, v^{^A_C B}, v^{B^C A}, v^{B_A C}, v^{B^C A}$

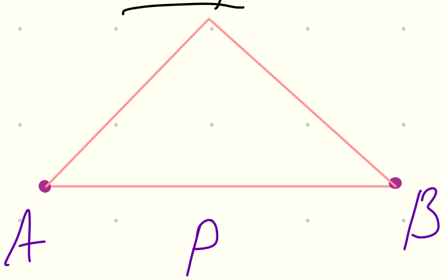
- **Def** $v^{A^B_C} = v^{AC} = (v_1^{AC}, \dots, v_m^{AC})$ s.t $v_i^{AC} \in A_{m-i}^{\perp} \cap C_{i-1}^{\perp}$
  $v_i^{AC} + v_{i+1}^{AC} \in B_1^{\perp}$

Here $A_{m-i}^{\perp} = \{ v \in (\mathbb{C}^m)^* \mid (v, w) = 0, \forall u \in A_{m-i} \}$ and similar for other

Note that $A_{m-i}^{\perp} \cap C_{i-1}^{\perp}$ - is 1 dimensional.

- **Def** $v^{C^B A} = v^{CA} = (v_1^{CA}, \dots, v_m^{CA})$ s.t $v_i^{CA} \in C_{m-i}^{\perp} \cap A_{i-1}^{\perp}$
  $v_i^{AC} - v_{i+1}^{AC} \in B_1^{\perp}$

- **Def** On the boundary with pinning



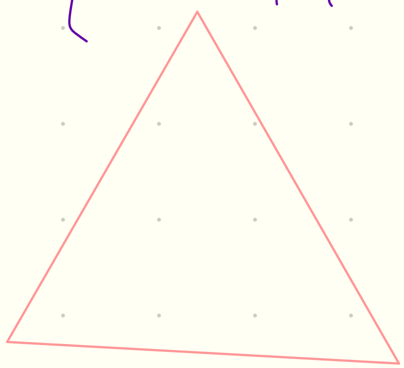
$v^{A_P B} = (v_1^{A_P B}, \dots, v_m^{A_P B})$ s.t $v_i^{A_P B} \in A_{m-i}^{\perp} \cap B_{i-1}^{\perp}$
$v_i^{A_P B} - v_{i+1}^{A_P B} \in P^{\perp}$

$v^{B^P A} = (v_1^{B^P A}, \dots, v^{B^P A})$ s.t $v_i^{B^P A} \in B_{m-i}^{\perp} \cap A_{i-1}^{\perp}$
$v_i^{B^P A} + v_{i+1}^{B^P A} \in P^{\perp}$

- **Example** $m=3$

Let $f_1, f_2, f_3 \in (\mathbb{C}^m)^*$ dual to basis $e_1, e_2, e_3 \in \mathbb{C}^m$.

$B = \{0 \subset \langle \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \rangle \subset \ldots\}$



$A = \{0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \mathbb{C}^3\}$ $\quad$ $C = \{0 \subset \langle e_3 \rangle \subset \langle e_2, e_3 \rangle \subset \mathbb{C}^3\}$

Then

$$v^{AC} = (\alpha_3^{-1} f_3, -\alpha_2^{-1} f_2, \alpha_1^{-1} f_1)$$
$$v^{CA} = (\alpha_1^{-1} f_1, \alpha_2^{-1} f_2, \alpha_3^{-1} f_3)$$

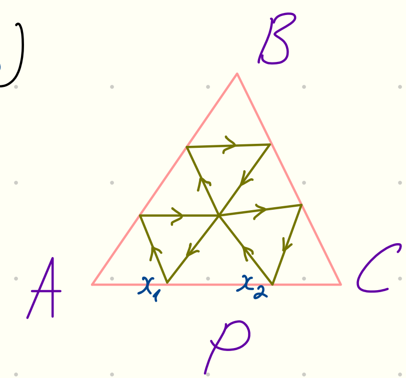
- **Problem** Show that $\left(v^{\overset{B}{AC}}\right) \cdot S = v^{C_B A}$ where

$$S = \begin{pmatrix} 0 & & & 1 \\ & & -1 & \\ & 1 & & \\ \cdot^{\cdot^{\cdot}} & & & 0 \end{pmatrix}$$

- Recall notation

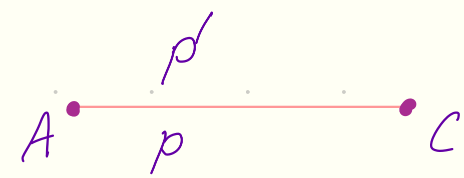
$$H_i(x) = \begin{array}{c} \\ \\ i \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} & & i & & \end{array} \\ \begin{pmatrix} x & & & & \\ & \ddots & & & \\ & & x & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \end{array} \in PGL_m$$

Problem Show some of the relations
(notations in figures for $m=3$)

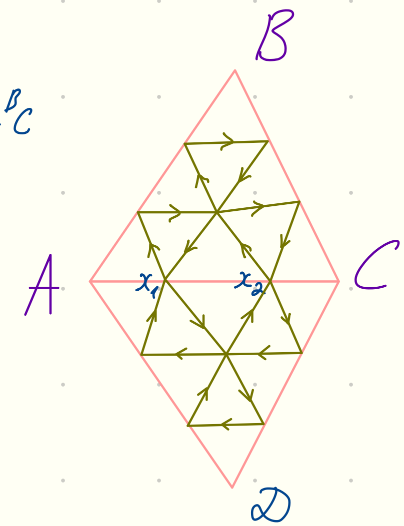


(a) $v^{A_pC} H_1(x_1)\cdots H_{m-1}(x_{m-1}) = v^{A^BC}$

(b) Gluing condition is satisfied iff



$v^{A_pC} = v^{A^{p'}C}$

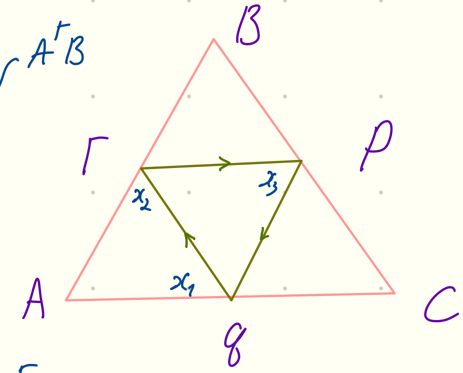


(c) For internal edge $v^{A_DC} H_1(x_1)\cdots H_m(x_m) = v^{A^BC}$

- Recall $E_i = \begin{array}{c c} & \begin{array}{ccccc} & & & i+1 & \end{array} \\ i & \begin{pmatrix} 1 & & & & \\ & \ddots & & \vdots & \\ \cdots & \cdots & 1 & 1 & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \end{array}$

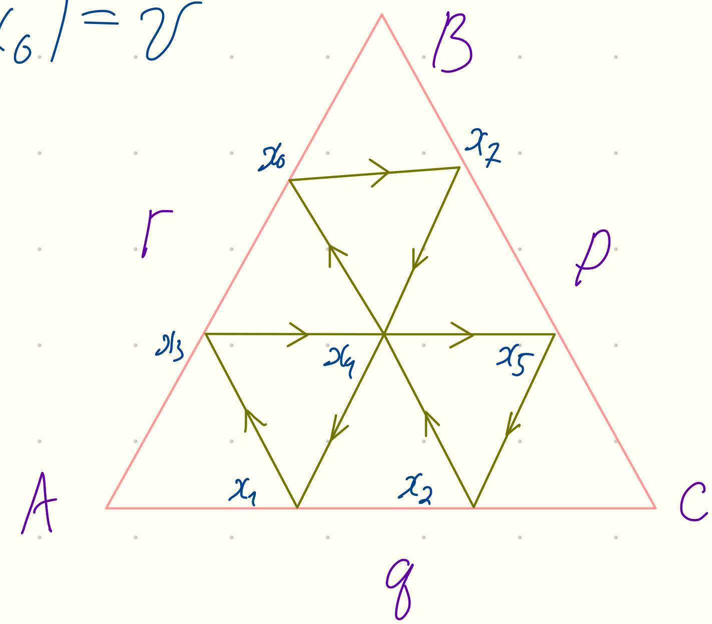
Problem*

ⓐ For $m=2$ $\quad v^{A_q C} H_1(x_1) E_1 H_1(x_2) = v^{A^r B}$



ⓑ For $m=3$

$$v^{A_q C} H_1(x_1) H_2(x_2) E_2 E_1 H_2(x_4) E_2 H_1(x_3) H_2(x_6) = v^{A^r B}$$



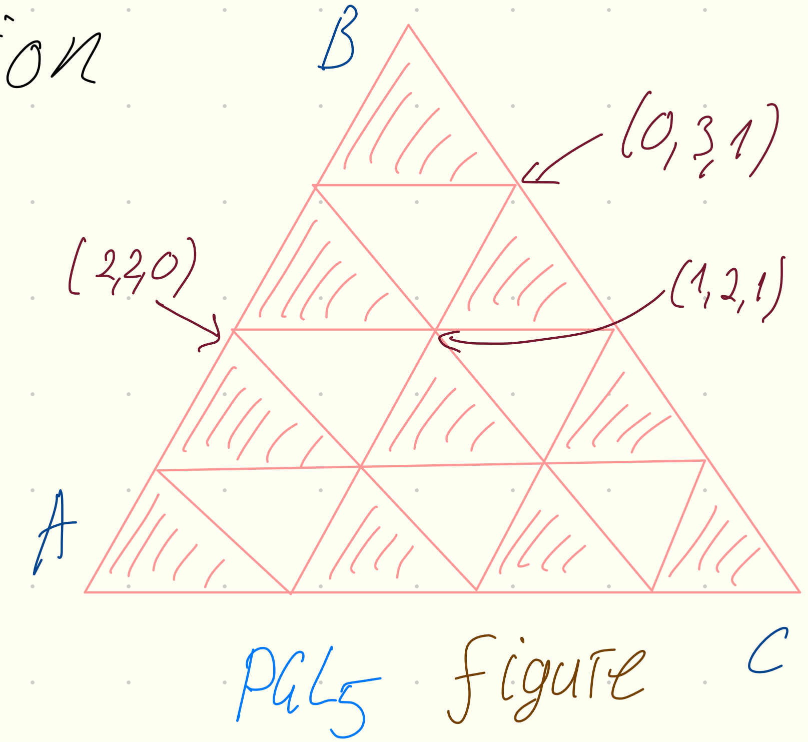
ⓒ For generic m

$$v^{A_q C} \mathbb{E}_{w_0}(\bar{x}) = v^{A^r B}$$

factorization scheme for G^{e,w_0}.

- Long hint

- Consider $m\text{-}1$ - triangulation of triangle



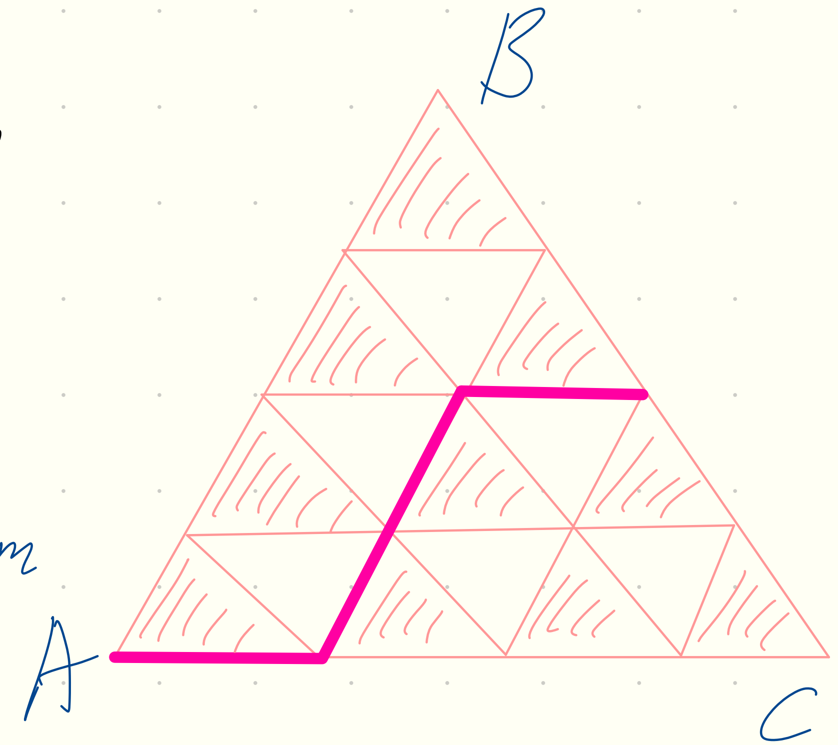
- Each vertex (a,b,c) distances

$a+b+c=m-1$

Assign $\mathcal{I}_{a,b,c}=A_a^{\perp}\cap B_b^{\perp}\cap C_c^{\perp}\subset(\mathbb{C}^m)^*$

$\mathcal{I}_{a,b,c}$ is a line

- Consider a snake path from A to opposite side with steps parallel to (AC) and (AB)



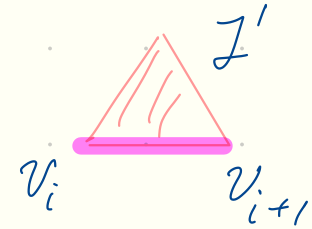
The length of the snake is m

we have sequence of lines

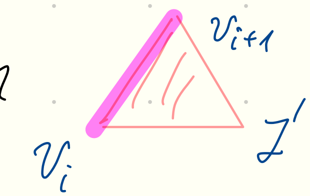
$\mathcal{I}_1^{snk},\ \mathcal{I}_2^{snk},\ \ldots,\ \mathcal{I}_m^{snk}$

- Assign to each snake a projective basis in $(\mathbb{C}^m)^*$

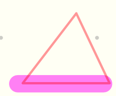
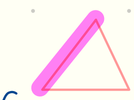
$v^{snk} = (v_1^{snk}, v_2^{snk}, \dots, v_m^{snk})$ s.t. $v_i^{snk} \in \mathcal{J}_i^{snk}$

for any step of the form 

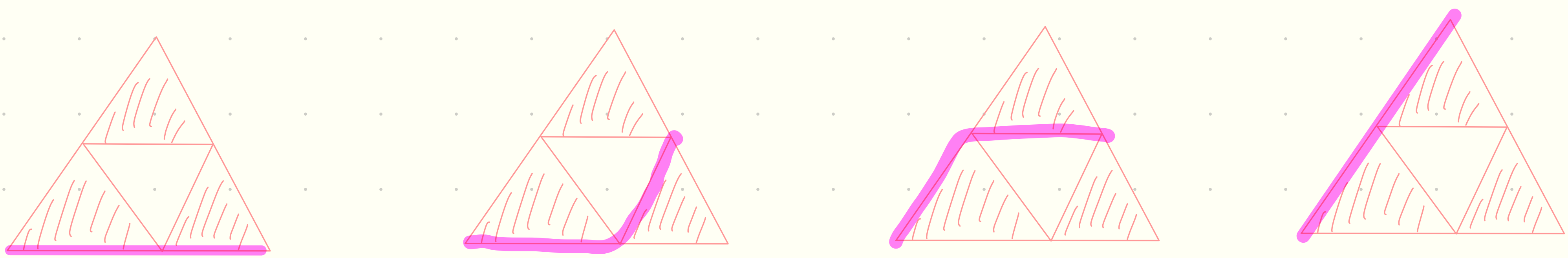
$v_i + v_{i+1} \in \mathcal{J}'$

for any step of the form 

$v_i - v_{i+1} \in \mathcal{J}'$

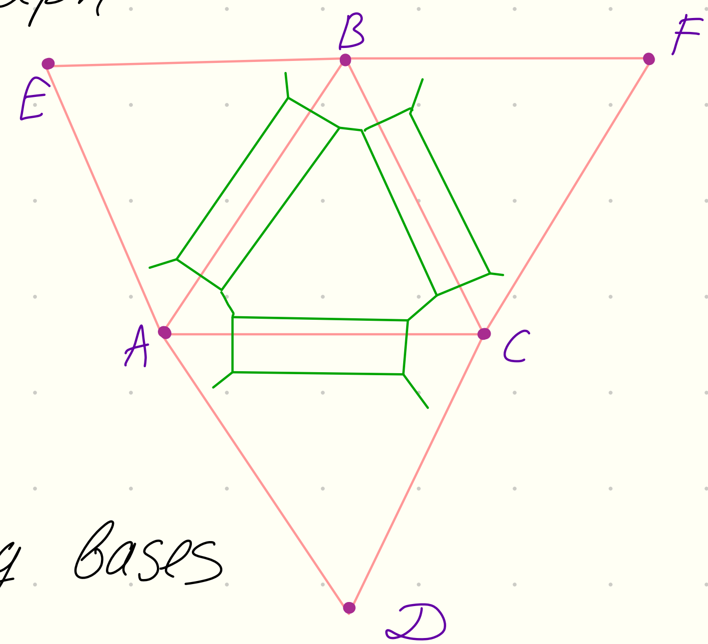
- Example v $= v^{A^{B}C}$, v $= v^{A_{C}B}$

- We can transform $v^{A^{B}C}$ to $v^{A_{C}B}$ like this:



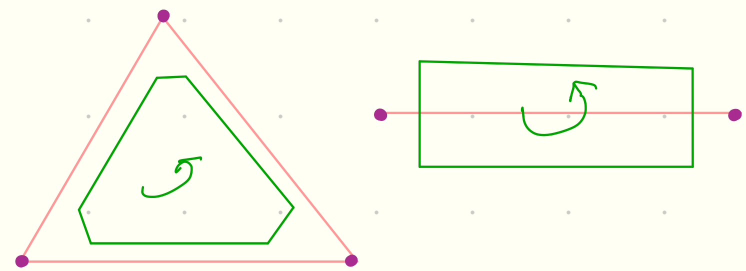
- Overall. For any triangulation of S we have auxiliary graph (in green)

To each vertex of auxiliary graph we assigned a projective basis (in the local trivialization, not global)



To each oriented edge of auxiliary graph we assigned an element of PGL_m transforming bases

Monodromy over contractable loop is trivial



We got a PGL_m local system

References

- Fock Goncharov Moduli spaces of local systems and higher Teichmüller theory Sec 9
- Goncharov Shen Quantum geometry of moduli spaces of local systems and representation theory Sec 3,