Itroduction to cluster algebras and varieties

Lecture 11

Geometric approach to cluster varieties

- Lattice

$N = \mathbb{Z}^n, \quad M = \mathrm{Hom}(N, \mathbb{Z}) \simeq \mathbb{Z}^n$

$(\cdot,\cdot) : N \times M \to \mathbb{Z}^n$

- Torus $\quad T_N = N \otimes_{\mathbb{Z}} G_m = \mathrm{Spec}\, \mathbb{C}[M]$

G_m – multiplicative group of base field (i.e. $\mathbb{C}^*$)

$\mathbb{C}[M] = \langle z^m \mid m \in M \rangle \qquad z^m \cdot z^{m'} = z^{m+m'}$

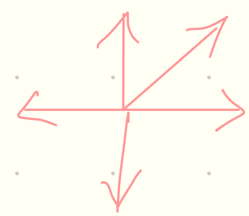
in coordinates $\quad T_N = G_m^n = (\mathbb{C}^*)^n$

$M = \langle f_1, \dots, f_n \rangle$ functions on T_N Laurent polynomials on $z^{f_1}, \dots, z^{f_n}$

- Toric varieties (up to codim 2)

Fan in M — set of rays $\mathbb{R}_{\geq 0} v_i$
Σ 0, $i=1,\dots,k$.

- Example $n=2$ figure like



- $TV_\Sigma \underset{\text{up to codim 2}}{=} T_N \sqcup \bigcup_i T_{\langle v\rangle^\perp}$

Here $\langle v\rangle^\perp \subset N$ sublattice of codim 1
$T_{\langle v\rangle^\perp}$ – torus of dimension $n-1$ glued to T_N
as a set $\{z^{v_i}=0\}$

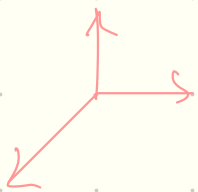
<u>Example</u> $n=1$

(a) $\longrightarrow$ $\mathbb{C}^* \cup \{z^v=0\} = \mathbb{C}$

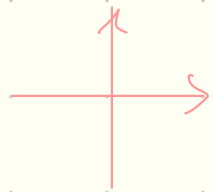
(b) $\longleftarrow\!\bullet\!\longrightarrow$ $\mathbb{C}^* \cup \{z^v=0\} \cup \{z^{-v}=0\} = \mathbb{P}^1$

Problem Find TV_Σ for the following Σ

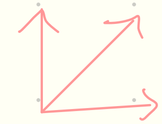
(a)



(b)



(c)



Date for cluster variety

Fixed date

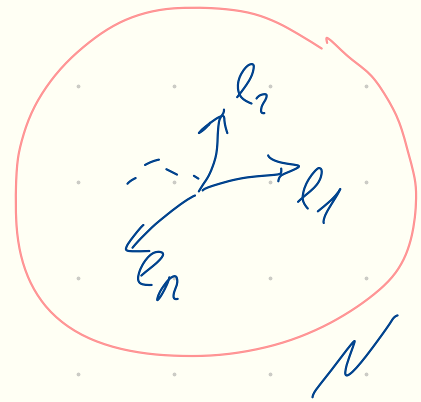
- Lattice N
- Skew-symmetric pairing $\langle \cdot , \cdot \rangle$ $N \otimes_{\mathbb{Z}} N \to \mathbb{Q}$
- $N_{uf} \subset N$ – unfrozen sublattice
  s.t $\langle N_{uf}, N \rangle \subset \mathbb{Z}$
- Index set I s.t. $|I| = \operatorname{rank} N$
  subset $I_{uf} \subset I$ s.t. $|I_{uf}| = \operatorname{rank} N_{uf}$ $I_{fr} = I \setminus I_{uf}$
- $M = \operatorname{Hom}(M, \mathbb{Z})$

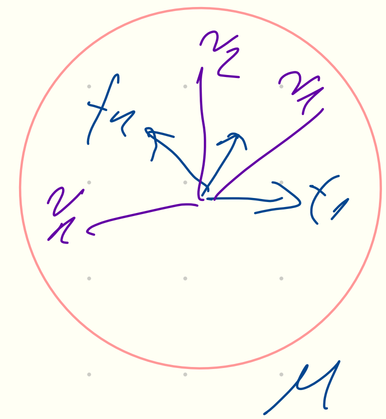
Seed date

$S = \{ e_i \in N \mid i \in I \}$ s.t.

- e_i $i \in I$ form a basis in N
- e_i $i \in I_{un}$ form a basis in N_{un}

- <u>Remark</u> $b_{ij} = \langle e_i , e_j \rangle$ – antisymm





Notations

- $f_i = e_i^* \in M$ s.t. $(e_i, f_j) = \delta_{ij}$
  dual basis
- $v_i = \langle \cdot, e_i \rangle \in M$
- To each seed S we assosiate
  $\mathcal{X}_S = T_M$ $\qquad$ $\mathcal{A}_S = T_N$
- Coordinates $\qquad x_i = z^{e_i} \qquad A_i = z^{f_i}$

- **Mutation** Let $k \in I_{uf}$

notation

$$[r]_+ = \begin{cases} r & r \geq 0 \\ 0 & r < 0 \end{cases} = \max(r, 0)$$

- $$e_i' = \begin{cases} -e_k & i = k \\ e_i + [b_{ki}]_+ e_k & i \neq k \end{cases}$$

$\Downarrow$ Hence on dual basis

$$f_i' = \begin{cases} -f_k + \sum_j [b_{kj}]_+ f_j & \\ f_i & i \neq k \end{cases}$$

Remark $b_{ij} \longmapsto b_{ij}'$

- Algebraic (Geometric date)

$\mu_k : \mathcal{X}_S \longrightarrow \mathcal{X}_{S'}$	$\mu_k : \mathcal{A}_S \longrightarrow \mathcal{A}_{S'}$
$\mu_k^*(z^n) = z^n \cdot (1 + z^{e_k})^{\langle n, e_k \rangle}$	$\mu_k^* z^m = z^m (1 + z^{v_k})^{(e_k, m)}$

Problem This is equivalent to the standard formulas

- $$\mu_k^* A_k' = \frac{\prod_{j,\, b_{jk}>0} A_j^{b_{jk}} + \prod_{j,\, b_{jk}<0} A_j^{b_{kj}}}{A_k}$$ $\mu_k^* A_i' = A_i \quad i \neq k$

- $$\mu_k^* x_i' = \begin{cases} x_k^{-1} & j = k \\ x_j (1 + x_k^{\operatorname{sgn} b_{jk}})^{b_{jk}} & \end{cases}$$

$A_i = z^{f_i} \quad A_i' = z^{f_i'}$

$x_i = z^{e_i} \quad x_i = z^{e_i}$

- Remark μ_k^2

$$e_i' = \begin{cases} -e_k & i = k \\ e_i + [b_{ki}]_+ e_k & i \neq k \end{cases}$$

$$e_k \longmapsto -e_k \longmapsto e_k$$

$$e_i \longmapsto e_i + [b_{ki}]_+ e_k \longmapsto e_i + [b_{ki}]_+ e_k + [-b_{ki}]_+ (-e_k)$$

$$= e_i + b_{ki} e_k = e_i + \langle e_k, e_i \rangle e_k$$

- linear transformation

Hence

$\mu_k^2 = id$ on $\{e_i\}$

$\mu_k^2 = id$ on coordinates x_i, A_i and preserves b_{ij}

- $\mathcal{X}$ (or $\mathcal{A}$) cluster varieties (schemes) are gluing of all $\mathcal{X}_{S'}$, $\mathcal{A}_{S'}$ obtained from S by mutations

- <u>Remark</u> Often we do not want to invert A_i corresp. to $i \in I_{fr} = I \setminus I_{uf}$.

In order to do so geometrically

$$\mathcal{A}_{S'} \rightsquigarrow (\mathbb{C}^*)^{I_{uf}} \times (\mathbb{C})^{I_{fr}}$$ — toric variety corresp $\{\mathbb{R}_{\geq} e_i \mid i \in I_{fr}\}$.

All gluing definitions remain the same.

Elementary transformations

- N lattice $\qquad e \in N$ primitive

$\pi : N \to N/\mathbb{Z}e \qquad \pi : T_N \to T_{N/\mathbb{Z}e}$

in coordinates if $e = e_1, e_2, \dots, e_n$, $\quad f_1, \dots, f_n$ ~ dual basis $\quad x_i = z^{f_i}$

$\pi : (x_1, x_2, \dots, x_n) \mapsto (x_2, \dots, x_n)$

- Let $F \in \mathbb{C}[T_{N/\mathbb{Z}e}]$ $\qquad$ in coord. F – Laurent polynomial on $x_2, \dots, x_n$

- we can consider F as a map $T_N \to T_N$ given by composition $\quad T_N \to T_{N/\mathbb{Z}e} \xrightarrow{F} \mathbb{C} \dashrightarrow T_{\mathbb{Z}e} \hookrightarrow T_N$

- Introduce $\mu_F : T_N \dashrightarrow T_N$
$$t \mapsto F(\pi(t))^{-1} \cdot t$$

In coordinates as above $(x_1, x_2, \dots, x_n) \mapsto (F^{-1} x_1, x_2, \dots, x_n)$

- Let X_f be a gluing of two T_N via μ_F
- $\Sigma_{e,+} = \{\mathbb{R}_{\geq 0} e, 0\}$ fan in N

$TV_{\Sigma_{e,+}} = \mathbb{A}^1 \times T_{N/\mathbb{Z}e}$ in coord.

$TV_{\Sigma_{v,+}} = \{(x_1, \dots, x_n) \mid x_1 \in \mathbb{C}, x_2, \dots, x_n \in \mathbb{C}^*\}$

$\mathcal{D}_+ = \{x_1 = 0\} \subset TV_{\Sigma_{e,+}}$ - divisor

$Z_+ = \pi^{-1}(V(F)) \cap \mathcal{D}_+$, here $V(F)$ subscheme corresp. to (F).

$\widetilde{TV}_{\Sigma_{v,+}}$ — blowup of Z_+

$\mathcal{D}_+$ — proper transform $\mathcal{D}_+$

$$\begin{array}{ccc} \mathcal{D}_+^{+} & \subset & \widetilde{TV}_{\Sigma_{e,+}} \\ & & \downarrow \\ \mathcal{D}_+ & \subset & TV_{\Sigma_{e,+}} \end{array} \qquad \begin{array}{l} \mathcal{U}_{e,+} = \widetilde{TV}_{\Sigma_{e,+}} \setminus \widetilde{\mathcal{D}}_+ \\ \\ T_N = TV_{\Sigma_{e,+}} \setminus \mathcal{D}_+ \end{array}$$

- Example

T_N $\qquad$ $TV(\Sigma_{e,+})$ $\qquad$ $\widetilde{TV}(\Sigma_{e,+})$ $\qquad$ $\mathcal{U}_{e,+}$

- <u>Lemma 1</u> There is an open immersion $X_F \hookrightarrow \mathcal{U}_{e,+}$ s.t $\mathcal{U}_{e,+} \setminus X_F$ is codim ≥ 2.

- <u>Example</u> Typically mutation

$$(x_1, x_2, x_3, \dots, x_n) \longleftrightarrow (x_1(1+x_2^{-1})^{-1}, x_2^{-1}, x_3, \dots, x_n)$$

in terms of first chart in gluing we add points $x_1=0$, $x_2=-1$, $x_1/_{1+x_2^{-1}}$ - const $\longleftrightarrow$ blowup

in terms of second chart in gluing we add points $x_1'=\infty$ $x_2'=-1$, $x_1(1+x_1')$ - const

- Introduce fan $\Sigma_e = \{\mathbb{R}_{\geq 0} e, \mathbb{R}_{\leq 0} e, 0\}$ it would correspond to two charts.

$TV_{\Sigma_e} = \mathbb{P}^1 \times (\mathbb{C}^*)^{n-1}$

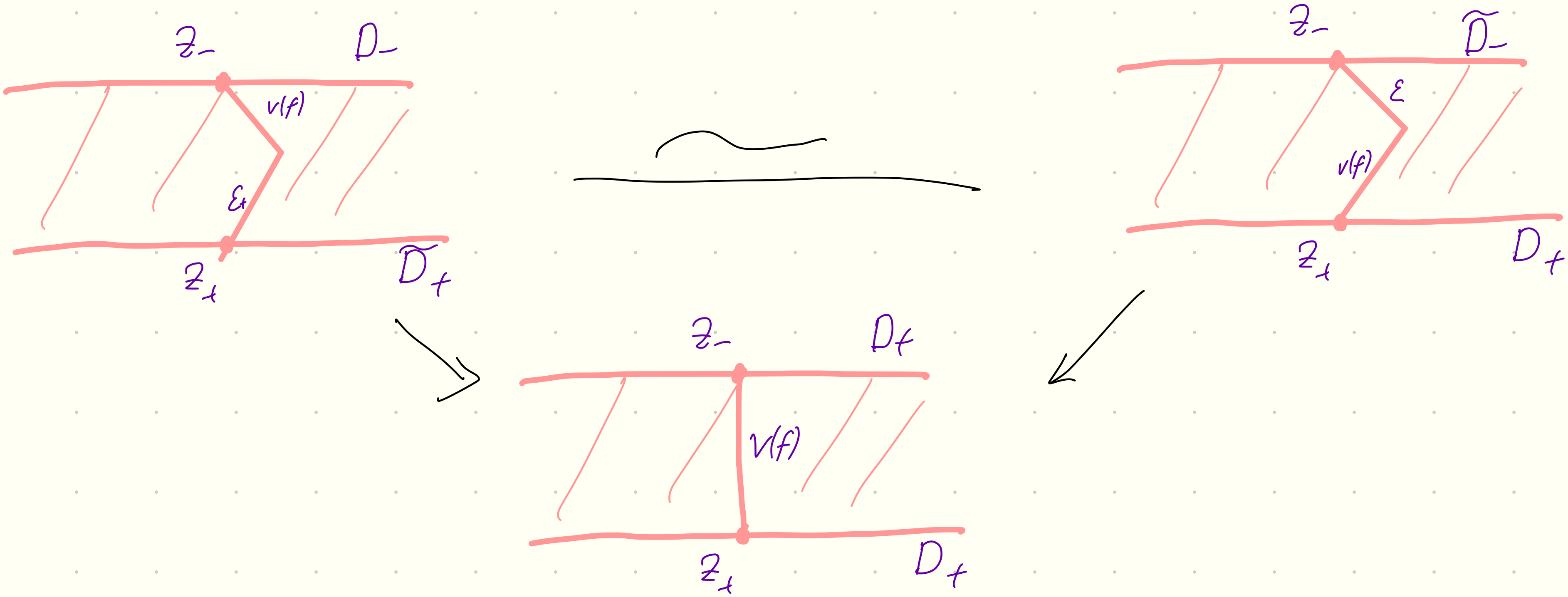
Divisors $\quad D_+ = \{x_1 = 0\} \quad D_- = \{x_1^{-1} = 0\}$

Codim 2 subsets $\quad Z_+ = D_+ \cap V(F \circ \pi) \quad Z_- = D_- \cap V(F \circ \pi)$

Blowups

$$\begin{tikzcd}
\widetilde{\mathbb{P}}_- \arrow[dr, "\text{blowup } z_-"'] & & \widetilde{\mathbb{P}}_+ \arrow[dl, "\text{blowup } z_+"] \\
& \mathbb{P} &
\end{tikzcd}$$

- <u>Lemma 2</u> The birational map $\mu_F : X_N \dashrightarrow X_N$ extends to isomorphism $\mu_f : \widetilde{\mathbb{P}}_+ \to \widetilde{\mathbb{P}}_-$



- <u>Corollaly</u> Denote by $\widetilde{\mathcal{D}}_+, \widetilde{\mathcal{D}}_-$ proper transforms of $\mathcal{D}_+, \mathcal{D}_-$. Then X_F is up to codim 2 isomorphic to both $\widehat{\mathbb{P}}_+ \setminus (\widetilde{\mathcal{D}}_+ \cup \widehat{\mathcal{D}}_-)$ and $\widehat{\mathbb{P}}_- \setminus (\widetilde{\mathcal{D}}_- \cup \widehat{\mathcal{D}}_+)$.

- <u>Problem</u> Prove Lemma 1.

- <u>Problem</u> Prove Lemma 2

More general construction

- Fan $\Sigma = \{\mathbb{R}_{\geq 0} v_i \mid 1 \leq i \leq \ell\} \cup \{0\} \subset N$, vectors v_i primitive
$w_1, \dots, w_\ell \in M \quad (w_i, v_i) = 0$

Let $a_1, \dots, a_\ell \in \mathbb{Z}_{>0}$, $\quad c_1, \dots, c_\ell \in \mathbb{C}^*$, $\quad F_i = (1 + c_i z^{w_i})^{a_i}$
and $\mu_i = \mu_{F_i} : T_N \dashrightarrow T_N$ defined by v_i, F_i as above

TV_Σ - toric variety,
D_i - divisor corresp $\mathbb{R}_{\geq 0} v_i$
$Z_i = D_i \cap V(F_i)$
$\pi : \widetilde{TV} \to TV(\Sigma)$ the blowup along $\bigcup_{i=1}^{\ell} Z_\ell$
$\widetilde{D}_j$ the proper transform of D_j

- Remark We allow some of the v_i-s coincide.

Two main examples

- $\Sigma_{S,\mathcal{A}} := \{0\} \cup \{\mathbb{R}_{\geq 0} e_i \mid i \in I_{uf}\}$ fan in N
  vectors in dual lattice — $v_i = \langle \cdot, e_i \rangle \in M$

$$Z_{\mathcal{A},i} = D_i \cap V(1+z^{v_i}) \subset TV_{S,\mathcal{A}}$$

- $\Sigma_{S,\mathcal{X}} := \{0\} \cup \{-\mathbb{R}_{\geq 0} v_i \mid i \in I_{uf}\}$ fan in M (N.B. roles of N and M are interchanged)
  vectors in dual lattice — $e_i \in M$.

$$Z_{\mathcal{X},i} = D_i \cap V(1+z^{e_i})^{\operatorname{ind} v_i} \subset TV_{S,\mathcal{X}}$$

here $\operatorname{ind} v$ is g.c.d of components of v.

- Let $T_0, T_1, \dots, T_\ell$ - copies of T_N
  Let X be gluing of $T_0, \dots, T_\ell$ via $\mu_i = \mu_{F_i} : T_0 \dashrightarrow T_i$

<u>Lemma 3</u> There is a natural morphism $\psi : X \to U_\Sigma = \widetilde{TV_\Sigma} \setminus \cup \tilde{\mathcal{D}}_i$ s.t. if $\dim Z_i \cap \dim Z_j < \dim Z_i \ \forall\, i \neq j$ then ψ is an isomorphism up to codim ≥ 2.

- <u>Remark</u> ⓐ In case of $\Sigma_{S, A}$ above, for $i \neq j$
  $\mathcal{D}_i \cap \mathcal{D}_j = \{ z^{e_i} = 0, z^{e_j} = 0, 1 + z^{v_i} = 0, 1 + z^{v_j} = 0 \}$ has codim ≥ 3
  (Actually since Σ is 1dim fan $\mathcal{D}_i \cap \mathcal{D}_j = \phi$ on TV_Σ)

  ⓑ In case of $\Sigma_{S, \chi}$ above, for $i \neq j$
  $\mathcal{D}_i \cap \mathcal{D}_j = \{ z^{-v_i} = 0, z^{-v_j} = 0, 1 + z^{e_i} = 0, 1 + z^{e_j} = 0 \}$ has codim ≥ 3
  (N.B we might have v_i proportional to v_j)

Mutation

For given $v \in N$, $w \in M$ with $(v, w) = 0$ we define

$$T_{v,w} : N_{\mathbb{R}} \to N_{\mathbb{R}} \qquad n \mapsto n + [(n, w)]_- v$$

Fix some k and let

$$\Sigma_+ = \Sigma \cup \{\mathbb{R}_{\leq 0} v_k\} \qquad \Sigma_- = T_{-v_k,\, a_k w_k}(\Sigma_+)$$

Divisors $\mathcal{D}_{k,+} \subseteq TV(\Sigma_+)$ corresp $\mathbb{R}_{\geq 0} v_k$ in Σ_+
$\mathcal{D}_{k,-} \subseteq TV(\Sigma_-)$ corresp $\mathbb{R}_{\leq 0} v_k$ in Σ_-

for $k \neq j$ $\mathcal{D}_{j,+} \subseteq TV(\Sigma_+)$ corresp $\mathbb{R}_{\geq 0} v_j$ in Σ_+
$\mathcal{D}_{j,-} \subseteq TV(\Sigma_-)$ corresp $\mathbb{R}_{\geq 0} T_{-v_k, a_k w_k} v_j$ in Σ_-

Blowup centers

$$Z_{j,+} = V(F_j) \cap \mathcal{D}_{j,+}$$

$$Z_{j,-} = \begin{cases} V(F_j) \cap \mathcal{D}_{j,-} & \text{if } (w_k, v_j) \geq 0 \\ V\left(\left(1 + c_j c_k^{a_k (w_j, v_k)} z^{w_j + a_k (w_j, v_k) w_k}\right)^{a_j}\right) \cap \mathcal{D}_{j,-} & \text{if } (w_k, v_j) \leq 0 \end{cases}$$

$\widetilde{TV}_{\Sigma_+}$, $\widetilde{TV}_{\Sigma_-}$ blowups of TV_{Σ_+}, TV_{Σ_-} on $\cup Z_{j,+}$, $\cup Z_{j,-}$ correspondingly

<u>Lemma 4</u> $\mu_k = \mu_{F_k} : T_N \dashrightarrow T_N$ defines a birational map $\mu_k : \widetilde{TV}_{\Sigma_+} \dashrightarrow \widetilde{TV}_{\Sigma_-}$. If $\dim V(F_k) \cap Z_{j,+} < \dim Z_{j,+}$ whenever $(w_k, v_j) = 0$, then this μ_k is isomorphism up to codim ≥ 2.

- <u>Remark</u> ⓐ In case of $\Sigma_{S,A}$ above, for $i \neq j$ $V(F_k) \cap Z_j = \{1 + z^{v_k} = 0, 1 + z^{v_j} = 0, z^{e_k} = 0\}$ has codim 3 if v_k and v_j are not proportional. Hence if seed is <u>coprime</u> conditions of the lemma holds

ⓑ In case of $\Sigma_{S,\chi}$ above, for $i \neq j$ $V(F_k) \cap Z_j = \{z^{-v_j} = 0, 1 + z^{e_j} = 0, 1 + z^{e_k} = 0\}$ has codim ≥ 3

- As a corallary we have description of cluster variety up to codim 2.

- For any S let $\mathcal{U}_{S,\mathcal{X}}$ be gluing tori T_N, $T_{N,i}$ $i \in I_{uf}$ via $\mu_i : T_N \to T_N$. Due to Lemma ③ above $\mathcal{U}_{S,\mathcal{X}} \simeq \widetilde{TV}_{\Sigma_{S,\mathcal{X}}} \setminus \cup \widetilde{D}_i$ isomorphism up to codim ≥ 2
  Due to Lemma 4 $S \overset{k}{—} S'$ map $\mu_k : \mathcal{U}_{S,\mathcal{X}} \dashrightarrow \mathcal{U}_{S',\mathcal{X}}$ isomorphism up to codim ≥ 2

  Theorem Up to codim ≥ 2 $\mathcal{X}$ cluster variety is isomorphic to $\mathcal{U}_{S,\mathcal{X}}$

- Similarly we define $\mathcal{U}_{S,A}$. If seed is totally coprime (e.g. matrix B is non-degenerate) we can similarly apply Lemmas 3, 4

Theorem Up to codim ≥ 2 $\mathcal{A}$ cluster variety is isomorphic to $\mathcal{U}_{S,\mathcal{A}}$ if seed is totally coprime.

- **Corollary** ⓐ If F is Laurent polynomial in $\mathcal{X}(S)$ as well in $\mathcal{X}(S^{(k)})$ for any mutation $\mu_k : S \xrightarrow{k} S^{(k)}$ then F is Laurent in $\mathcal{X}(S')$ for any seed S' connected to S by sequence of mutations

ⓑ Any cluster variable $A_i \in \mathcal{A}(S)$ is Laurent in $\mathcal{A}(S')$ for any seed S' connected to S by sequence of mutations

Pf ⓐ Follows from Theorem, F is regular on $\mathcal{U}_{S,\mathcal{X}}$ hence regular on whole $\mathcal{X}$ variety. ⓑ Add frozen variables s.t. B is non degenerate, then the same argument □

- This is Laurent phenomenon again.

References

- Gross, Hacking, Keel Birational Geometry of Cluster Algebras.