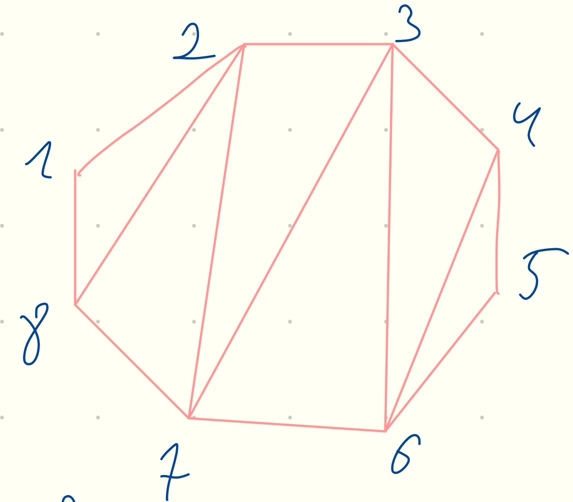
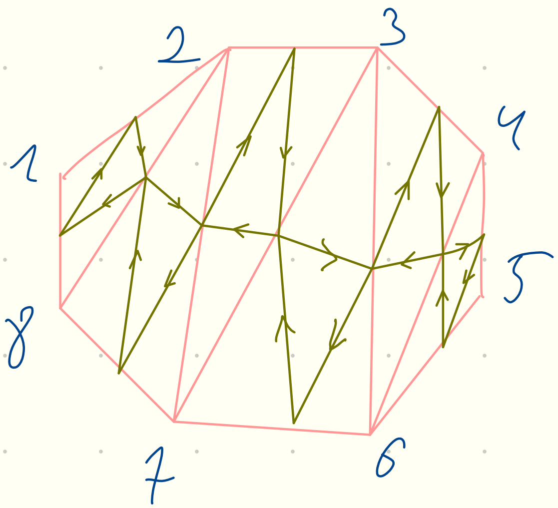
Itroduction to cluster algebras and varieties

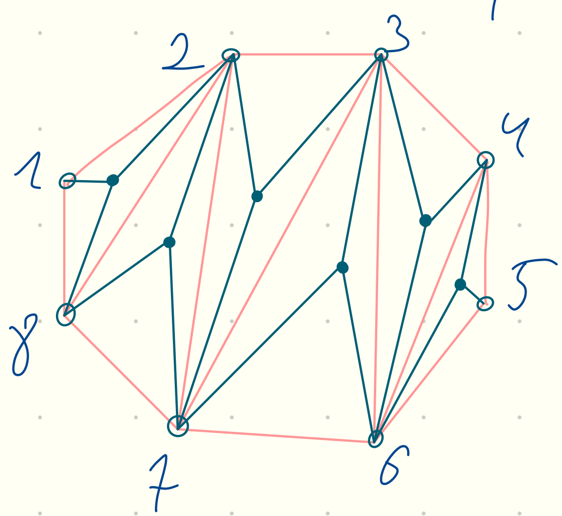
Lecture 9, 10

Plabic graphs. Grassmanians.

- Recall for $Gr(2,n)$



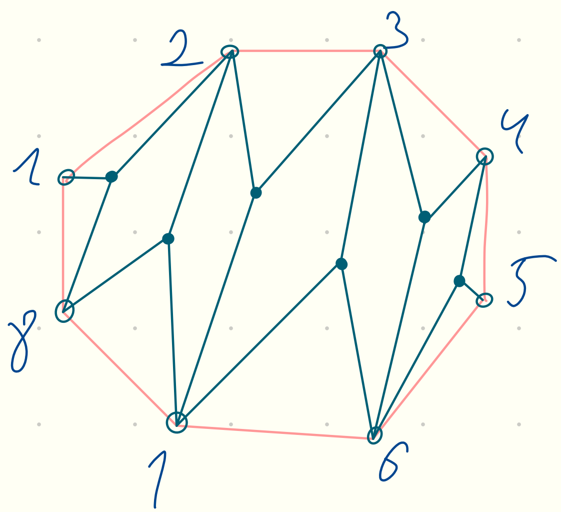


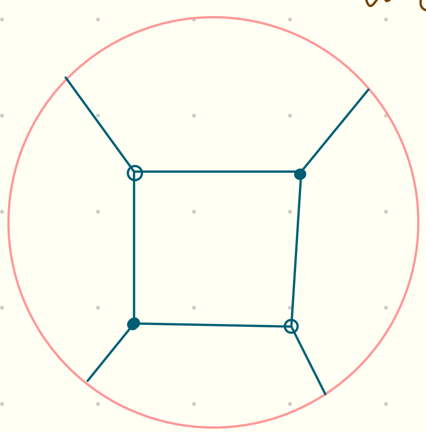


edges clockwise around black
counter clockwise around white

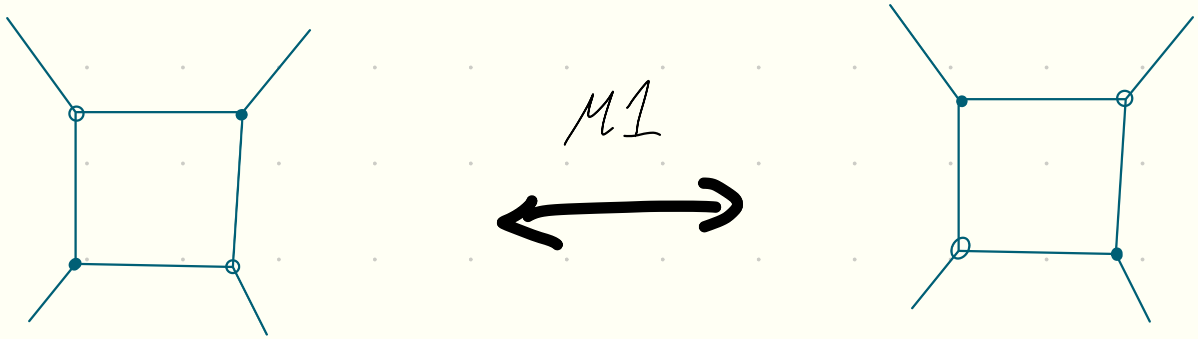
- **Def** Plabic graph – is a planar, bicolored graph
- Today: Plabic graphs embeded to disk
  - It has $n \neq 0$ boundary vertices ∂D labelled by $1, 2, .., n$
  - other vertices are internal.
  - (•) Boundary vertices have degree 1. (not always necessary, but we assume)

Example

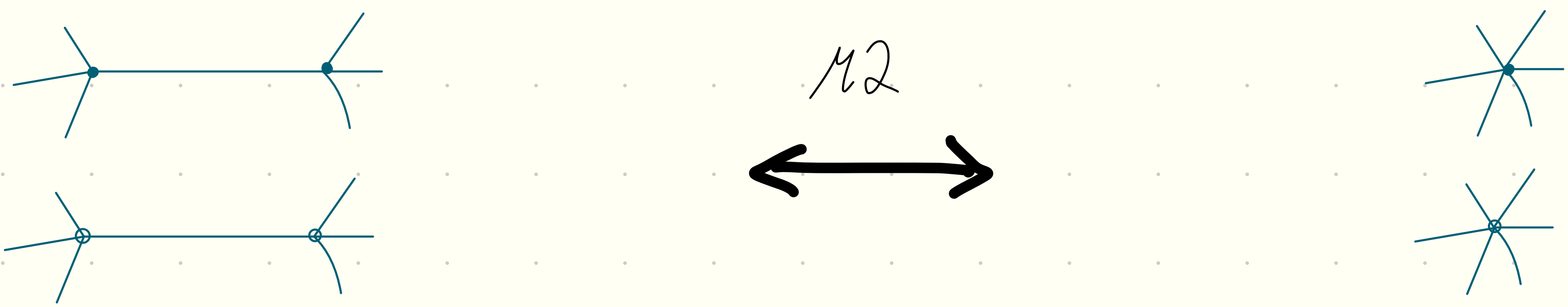


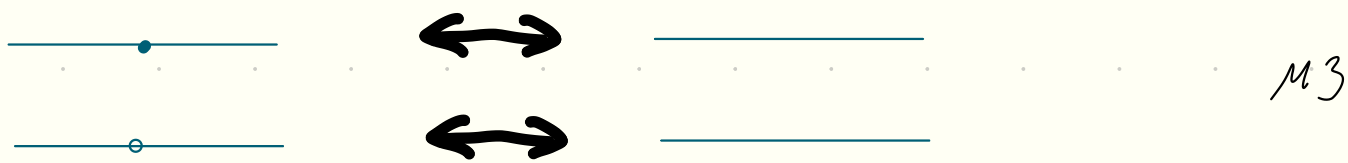


Local moves

- 

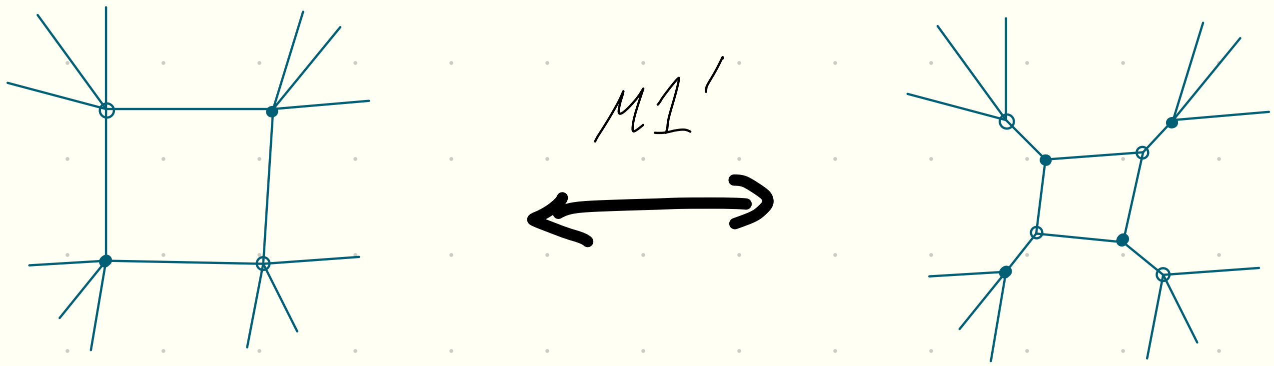
  for 4-gon face with vertices degree 3

- 

- 

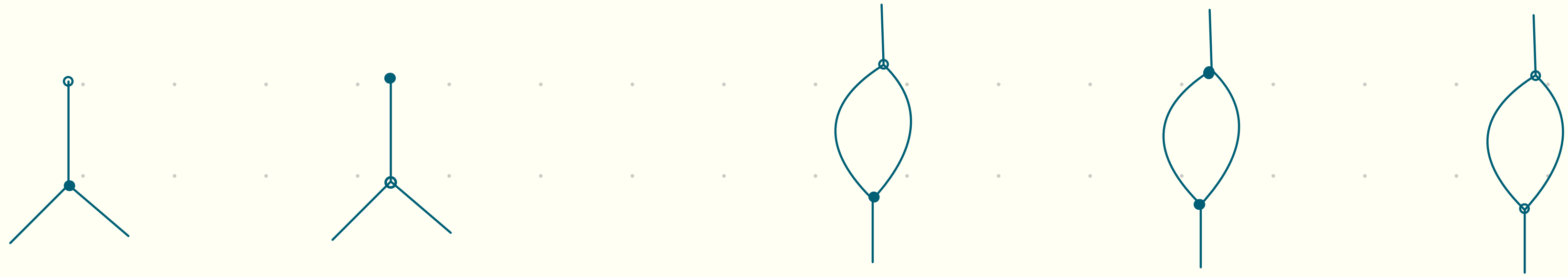
- Remark Plabic graph is not necessary bipartate. But it can be made bipartate using M2 & M3.

- Remark For bipartate graphs M1 can be restated as follows (using M2 and M3)



- The equivalence class $[G]$ graphs that can be obtained from G applying M_1, M_2 and M_3.

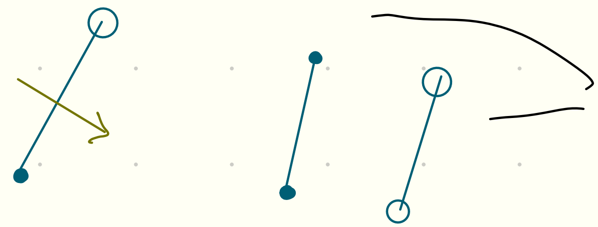
- Def If $\nexists$ $G' \in [G]$ which contains one of the following "badgones" then G is reduced



- Equivalently, any G' should not contain
  - hollow bigon
  - an internal leaf which is not lollipop (i.e or) and does not belong to collapsible tree.

- <u>Quiver</u> vertices — faces for G.

  arrows



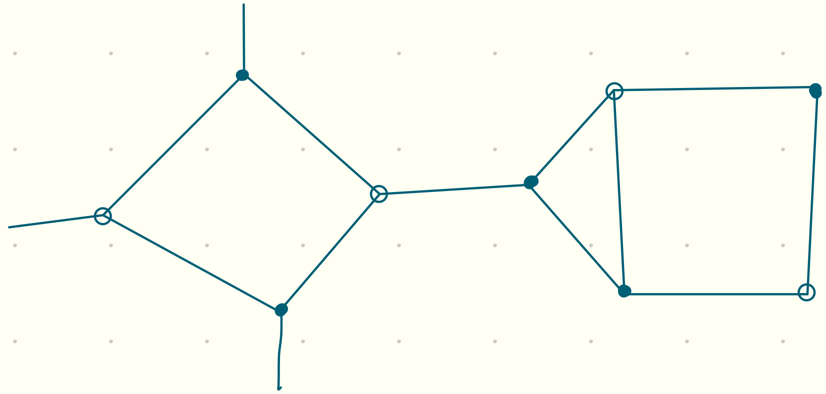
no arrow.

- <u>Prop</u> (a) M2 M3 preserve the quiver

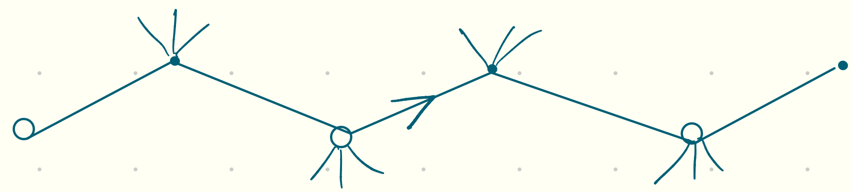
(b) M1 is a quiver mutation w.r.t corresp. vertex if

(*) Among 4 faces surrounding square the consequitive ones must be distinct

Example Restriction (*) fails for example for graph



- **Def** Zig-zag path (trip) is a path which moves right in black vertices and left in white



- **Remark** For any edge e there is a unique zig-zag path traversing e in each of two directions.

Remark Each zig-zag either begins on boundary or form a closed walk.

- **Problem** Show two zig-zags starting at different vertices terminate at different vertices.

- **Definition** G plabic graph with N boundary vertices. $\pi_G \in S_N$ - permutation defined by $\pi_G(i)=j$ if $\exists$ zig-zag originating on i and teminating on j.

- **Problem** Show that π_G is invariant under moves M1, M2, M3

- **Proposition** If G reduced, then
  (a) there are no closed zig-zags.
  (b) Each edge belongs to two different zig-zags.
  (c) Condition (*) is fulfilled.

- **Proposition** Let G be reduced and $\pi_G(i)=i$. Then connected component of G containing i collapses to lollipop

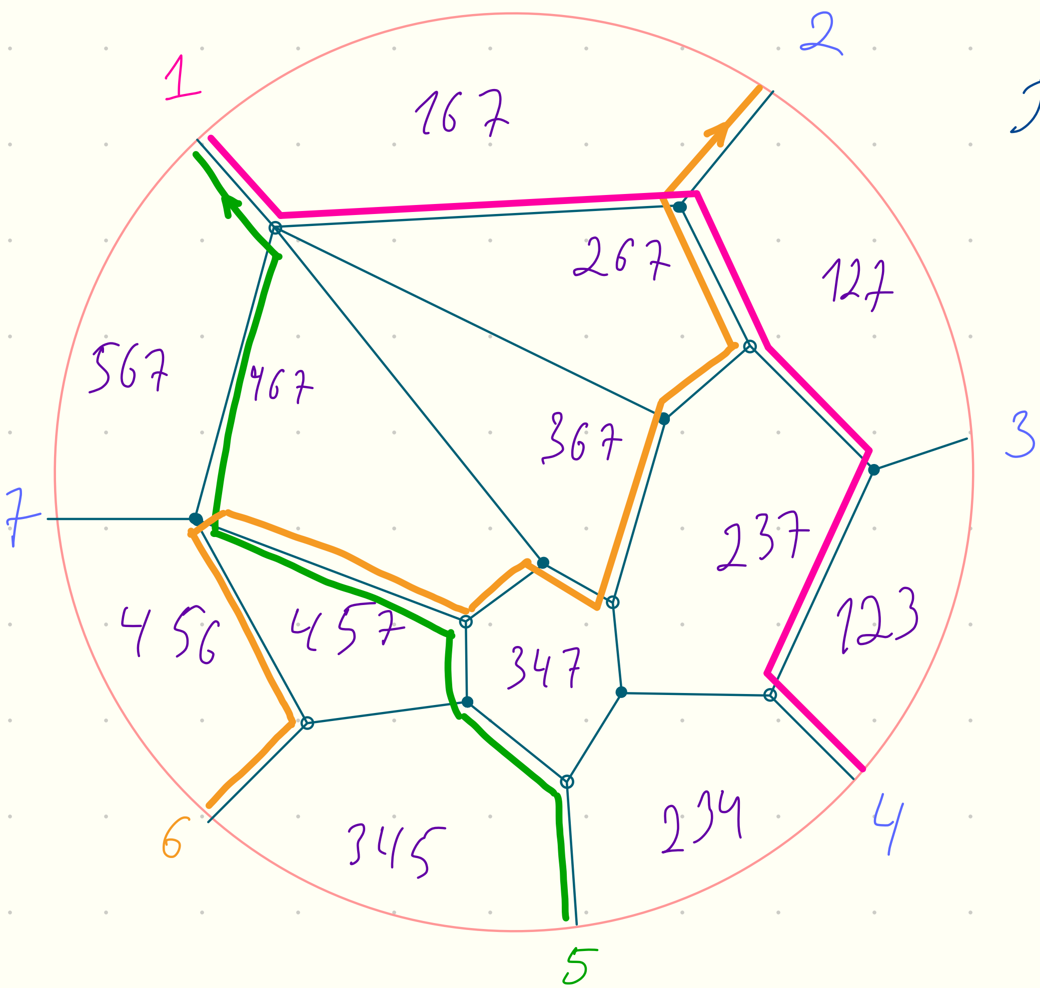
- **Definition**

$$\tilde{\pi}_G(i) = \begin{cases} \pi_G(i) & \text{if } \pi_G(i) \neq j \\ \bar{i} & \text{if conn. comp. of } i \text{ collapses to white lollipop} \\ \underline{i} & \text{if conn. comp. of } i \text{ collapses to black lollipop} \end{cases}$$

$\tilde{\pi}$ – decorated permutation.

- **Theorem** (Postnikov). Let G, G' are reduced. Then $\tilde{\pi}(G) = \tilde{\pi}(G') \iff G' \in [G]$
- For any decorated permutation $\tilde{\pi}$, $\exists G$ s.t. $\tilde{\pi} = \tilde{\pi}_G$.

Example



$$\pi_G = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \end{pmatrix}$$

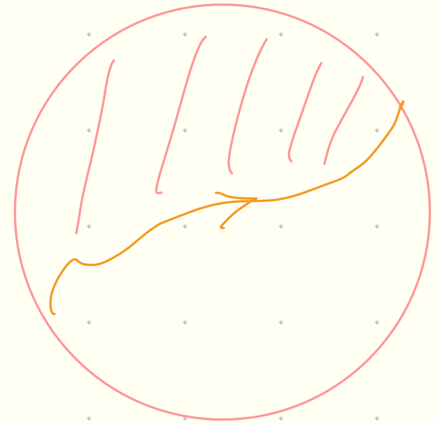
- Let $\pi_{k,N} \in S_N$ s.t. $\pi_{k,N}(i) = i + k \mod N$

$$\begin{matrix} 1 & 2 & \dots & N-1 & N \\ k+1 & k+2 & & k-1 & k \end{matrix}$$

Below $\left\{ \begin{matrix} \text{seeds} \\ \text{for} \end{matrix} \ \widehat{Gr}(k,N) \right\} \longleftrightarrow$ reduced plabic graphs G, s.t. $\pi_G = \pi_{k,N}$

- $\pi_{k,N}(i) \neq i$, so no decoration.

Face variables



- For any zig-zag z assign $\mathcal{D}_z$ - union of faces left to z.

  Face f $\to$ numbers (of starting points) of z s.t. $f \in \mathcal{D}_z$

  (c.f. figure above)

- <u>Properties</u> Let G - reduced, $\tilde{\pi}_G = \pi_{k,N}$

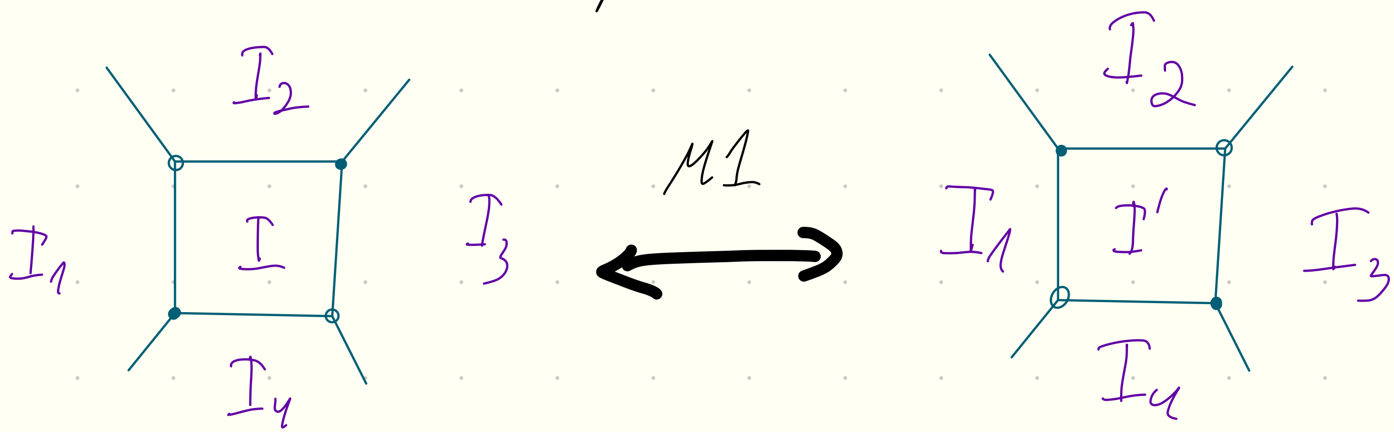
  ⓐ Labeling of boundary faces are $\{i, i+1, \dots, i+k-1\}$, for $1 \le i \le N$.

  ⓑ Any face is labelled by k distinct numbers $1 \le i_1 \le i_2 \le \dots \le i_k \le N$

$$f \mapsto i_1, \dots, i_k \rightsquigarrow \Delta_{i_1 \dots i_k} \in \mathbb{C}[\widehat{Gr}(k,n)]$$

- <u>Problem</u> (a) Moves M2, M3 preserve these Δ_I

  (b) Move M1 corresponds to mutation of Δ_I

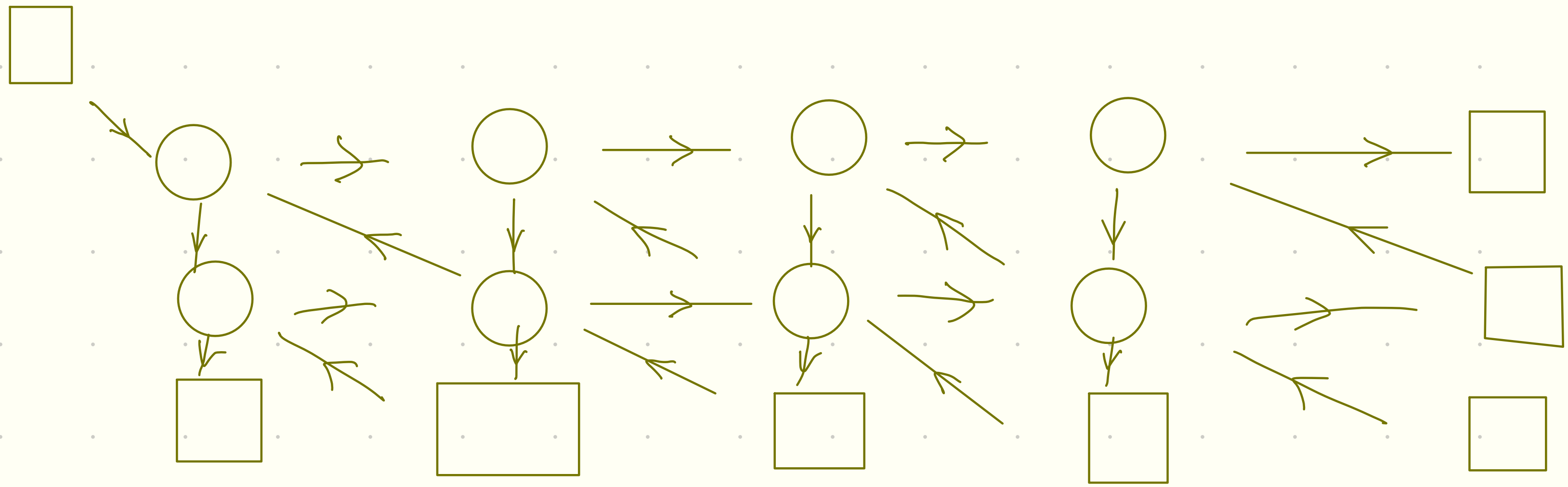


Plucker relation

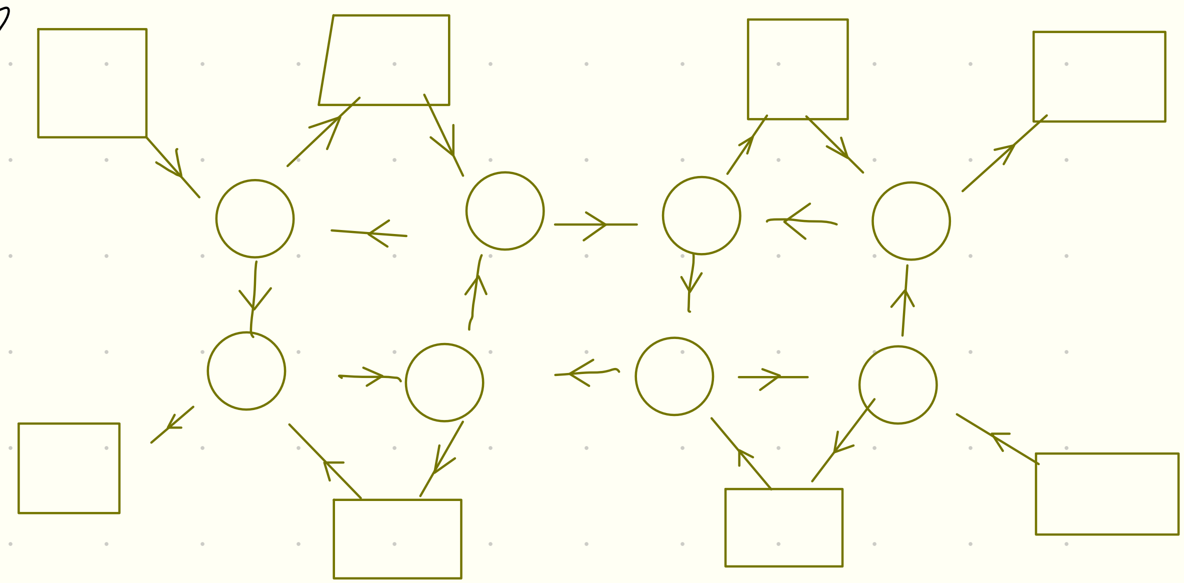
$$\Delta_I \Delta_{I'} = \Delta_{I_1}\Delta_{I_3} + \Delta_{I_2}\Delta_{I_4}$$

- <u>Corollary</u> We have a map from $\mathcal{A}$-cluster algebra to $\mathbb{C}[Gr(k,n)]$
- <u>Theorem</u> (Scott) This is an isomorphism.

- Problem (a) Find plabic graph corresp. to seed from previous lecture (triangular seed)



(b) Find plabic graph corresp to square seed of the form



Hint (a) hexagon grid
(b) square grid.

Consider example, say
$Gr(4,9)$

Perfect matchings

- **Def** G finite graph. Perfect matching of G is a collection M of edges of G s.t $\forall$ vertex is contained in precisely one edge of M.

Perfect matchings $\longleftrightarrow$ dimer configurations

- G Bipartite $G \subset \mathbb{D}$, Boundary vertices of G are white

Def Perfect matching with boundary is —//— s.t $\forall$ internal vertex —//—

- Notations

# Black vertices G	$N+k$
# internal white vertices G	N
# Boundary white vertices G	n

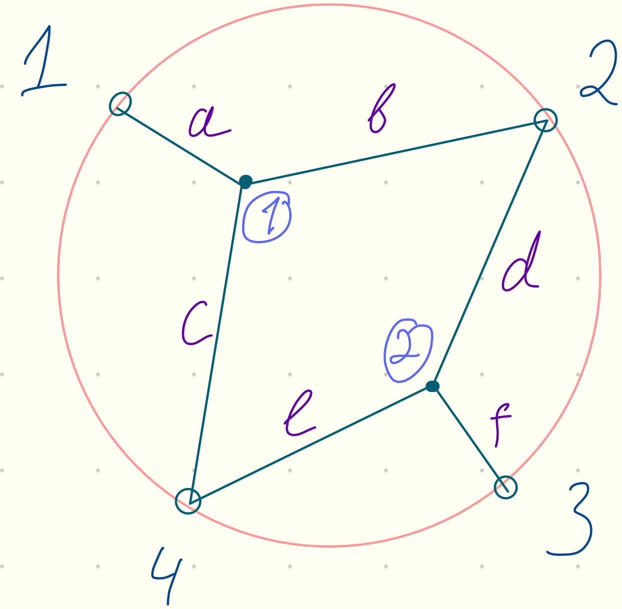
$\forall I \subset \{1, \dots, n\}, \ |I| = k$

$$\Delta_I = \sum_{M} wt(M)$$

$M \in$ Perfect matchings with Boundary k

Example

(For simplicity we dropped condition boundary vertices



$\Delta_{12} = ad \qquad \Delta_{23} = bf$

$\Delta_{13} = af \qquad \Delta_{24} = be + cd$

$\Delta_{14} = ae \qquad \Delta_{34} = cf$

$$\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$$

Theorem $\{\Delta_I\}$ define a point in $\widehat{Gr}(k,n)$

In real case, we have a map $(\mathbb{R}_+)^{\#\text{edges}} \to \widehat{Gr}(k,n)_+$

Example

$K = \begin{pmatrix} a & b & 0 & -c \\ 0 & d & f & e \end{pmatrix}$, minors of K are Δ_I above

- Theorem (Kasteleyn) Let G be planar bipartite graph with black vertices $b_1, \dots b_N$ and white vertices $w_1, \dots, w_N$. Then $\exists K \in Mat(N \times N)$, $K_{ij} = \pm wt(i \to j)$

$$\det K = \sum_{D\text{-perfect matchings}} wt(D)$$

$$K = \begin{array}{c|c} & j \\ \hline i & \begin{pmatrix} & \\ & \pm wt(i,j) \end{pmatrix} \end{array}$$

- Remark If there are no edges between $i \to j$ then $K_{ij} = 0$

$$\det K = \sum_{\sigma} \pm \, wt(1 \to \sigma(1)) \cdot wt(2 \to \sigma(2)) \dots wt(N \to \sigma(N))$$

these signs should be "+" by Th.

- <u>Theorem</u> (Kasteleyn) G bipartite graph with boundary embedded into a disk s.t. all boundary vertices are white. Suppose

#	black vertices G	$N+k$
#	internal white vertices G	N
#	boundary white vertices G	n

Then $\exists\, K \in Mat((N+k)\times(N+n))$, $K_{ij} = \pm wt(i \to j)$ and $\forall\, I \subset \partial G$, $|I| = k$, $\Delta_I = \det K^{1,\dots,N,\,I}_{1,\dots,N+k}$

- <u>Corollary</u> $\{\Delta_I\}$ define a point in $\widehat{Gr}(k,N)$

<u>Pf</u> If all $\Delta_I = 0 \Rightarrow$ O.K.
Otherwise $\exists\, I$, $\det K^{1,\dots,N,\,I}_{1,\dots,N+k} \neq 0$ hence first N columns of K are linearly independent

Applying row operations $K \sim \begin{pmatrix} \mathrm{Id}_N & * \\ 0 & L \end{pmatrix}$

$\Delta_I = \det K^{1,\dots,N,I}_{1,\dots,N+k} = \det L^{I}_{1\dots k}$ — Plucker coordinates of $k \times n$ matrix L

Hence q.e.d. $\square$

- Analogy $\sim$ factorisation scheme $\rightsquigarrow$ Poisson structure

R-matrix bracket

- $\{g \overset{\otimes}{,} g\} = [r, g \otimes g]$ $\qquad g \in (P)GL_n.$

In coordinates $\{g_i^j, g_{i'}^{j'}\} = \frac{1}{2}(\mathrm{sgn}(i'-i) + \mathrm{sgn}(j'-j))\, g_i^{j'} g_{i'}^{j}$

In particular $\{g_1^j, g_1^{j'}\} = \frac{1}{2}\mathrm{sgn}(j'-j)\, g_1^{j'} g_1^{j}$

like "cluster bracket for frozen variables

- Grassmanian $\quad Gr(k,n) = P \backslash GL_n$

$$P = \begin{array}{c|c} & \begin{array}{cc} k & n-k \end{array} \\ \begin{array}{c} k \\ n-k \end{array} & \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \end{array}$$

In coordinates generic

$$g = \begin{pmatrix} w_{11} & 0 \\ w_{12} & w_{22} \end{pmatrix} \begin{pmatrix} 1_k & Y \\ 0 & 1_{n-k} \end{pmatrix}$$

Matrix elements of Y - coordinates on

(open cell of) $Gr(k,n)$

$$y_a^b = \frac{\Delta_{1..k}^{1..k / a \to b}}{\Delta_{1..k}^{1..k}} \qquad 1 \le a \le k \qquad n-k+1 \le b \le n. \qquad \Delta_I^J = \det g_I^J$$

- <u>Theorem</u> P is Poisson-Lie subgroup of G.

<u>Corollary</u> $P\backslash G$ has a natural Poisson structure.

$$\pi : G \to P\backslash G \qquad \pi_* \Pi$$

Equivalently for any $f, g \in \mathbb{C}(P\backslash G)$,

$\{\pi^* f, \pi^* g\} \in \mathbb{C}(G)^P$ hence $\exists$ well defined

$\{f, g\}_{P\backslash G} \in \mathbb{C}(P\backslash G)$ $\qquad \{\pi^* f, \pi^* g\}_G = \pi^* \{f, g\}_{P\backslash G}$

- <u>Problem</u> $\{y_a^b, y_{a'}^{b'}\} = \frac{1}{2}\left(\operatorname{sgn}(a'-a) - \operatorname{sgn}(b'-b)\right) y_a^{b'} y_{a'}^{b}$

<u>Note</u> — sign differs from $\{g_i^j, g_{i'}^{j'}\}$ above

Hint Sufficient to compute $\{\Delta_{1\ldots k}^{1\ldots k\setminus a\cup b}, \Delta_{1\ldots k}^{1\ldots k\setminus a'\cup b'}\}$
Similarly to Corollary below.

- Minors of $y_{a_1\ldots a_l}^{b_1\ldots b_l} = \det Y_{a_1\ldots a_l}^{b_1\ldots b_l} = \pm \dfrac{\Delta_{1\ldots k}^{1\ldots k\setminus\{a_i\to b_i\}}}{\Delta_{1\ldots k}^{1\ldots k}}$

- Notations
  - $a' < \{a_1,\ldots,a_l\}$ if $a' < a_i \ \forall i$
    $a' > \{a_1,\ldots,a_l\}$ if $a' > a_i \ \forall i$

$A = \{a_1 \ldots a_k\}$

  - $$\operatorname{sgn}(a' - \{a_1,\ldots,a_l\}) = \begin{cases} 1 & \text{if } a' > \{a_1,\ldots,a_l\} \\ -1 & \text{if } a' < \{a_1,\ldots,a_l\} \\ 0 & \text{if } a' \in \{a_1,\ldots,a_l\} \\ \text{undefined} & \text{otherwise} \end{cases}$$

- Lemma If $\operatorname{sgn}(a'-A)$, $\operatorname{sgn}(b'-B)$ are defined and $|\operatorname{sgn}(a'-A) - \operatorname{sgn}(b'-B)| \le 1$ then

$$\{y_A^B, y_{a'}^{b'}\} = \frac{1}{2}(\operatorname{sgn}(a'-A) - \operatorname{sgn}(b'-B))\, y_A^B\, y_{a'}^{b'}$$

Pf Assume $b' \in B = \{b_1, \dots, b_l\}$ $\quad a' > A = \{a_1, \dots, a_l\}$

$$\{y_A^B, y_{a'}^{b'}\} = \sum_{p,q=1}^{l} (-1)^{p+q} \{y_{a_p}^{b_q}, y_{a'}^{b'}\}\, y_{A\setminus a_p}^{B\setminus b_q}$$

$$= \frac{1}{2}\sum_{p,q} (-1)^{p+q} \Big(\operatorname{sgn}(a'-a_p)\, y_{a'}^{b_q}\, y_{a_p}^{b'}\, y_{A\setminus a_p}^{B\setminus b_q}$$

$$+ \operatorname{sgn}(b'-b_q)\, y_{a'}^{b_q}\, y_{a_p}^{b'}\, y_{A\setminus a_p}^{B\setminus b_q}\Big)$$

$$= \frac{1}{2}\sum_{p} (-1)^{p} \operatorname{sgn}(a'-a_p)\, y_{a_p}^{b'}\, y_{A\setminus a_p \cup a'}^{B}$$

$$+ \sum (-1)^{q} \operatorname{sgn}(b'-b_q)\, y_{a'}^{b_q}\, y_{A}^{B\setminus b_q \cup b'}$$

- If $b_q \neq b' \Rightarrow y_A^{B\setminus b_q \cup b'} = 0$
- If $b_q = b' \Rightarrow \operatorname{sgn}(b'-b_q) = 0$

$\Rightarrow$ second term $= 0$

- First term $\operatorname{sgn}(a'-a_p)=1$

$$0 = y^{B\cup b'}_{A\cup a'} = y^{b'}_{a'}\, y^{B}_{A} + \sum (-1)^{p+1}\, y^{b'}_{a_p}\, y^{B}_{A\setminus a_p \cup a'}$$

Hence $\{y^B_A, y^{b'}_{a'}\} = \frac{1}{2} y^B_A\, y^{b'}_{a'}$ $\square$

- <u>Corollary</u> Let $A=\{a_1,\dots,a_l\}$, $B=\{b_1,\dots,b_l\}$, $A'=\{a'_1,\dots,a'_k\}$, $B'=\{b'_1,\dots,b'_k\}$. Assume that for any $1\le i\le k$, $1\le j\le k$ conditions of the Lemma are satisfied for A, B, a'_i, b'_j. Then

$$\{y^B_A, y^{B'}_{A'}\} = \sum_i \frac{1}{2}\left(\operatorname{sgn}(a'_i - A) - \operatorname{sgn}(b'_i - B)\right) y^B_A\, y^{B'}_{A'}$$

logarithmically constant Poisson bracket.

- Consider a set of minors $F_{p,q}$ $\quad 1\le p\le k,\quad 1\le q\le n-k$

If $p<q$ $\qquad F_{p,q} = y^{k-p+q+1,\dots,k+q}_{k-p+1 \qquad k}$

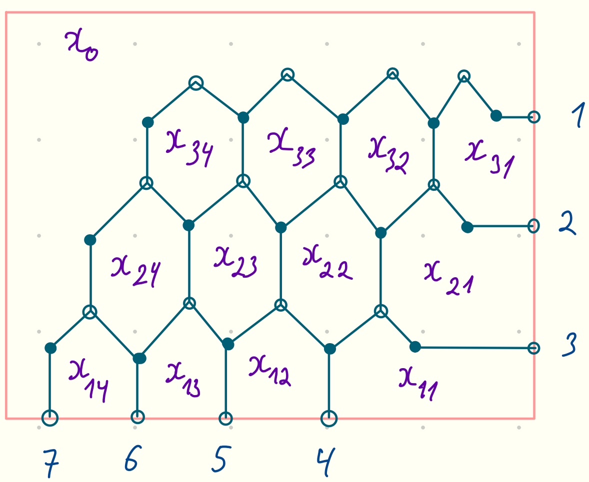
If $p>q$ $\qquad F_{p,q} = y^{k+1,\dots,k+q}_{k-p+1,\dots,k-p+q}$

<u>Remark</u> Any $F_{p,q}$ and $F_{p',q'}$ satisfy conditions of Corollary. Hence $\{F_{p,q}, F_{p',q'}\} = \# F_{pq} F_{p'q'}$.

<u>Problem</u> Compute $\{F_{pq}, F_{p'q'}\}$.

- Consider the following bipartite graph

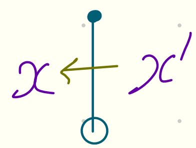
Example $Gr(3,7)$



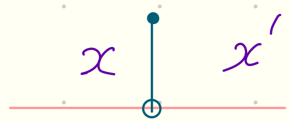
Hexagonal $(k-1)\times(n-k-1)$ inside

x_{ij} — monodromy of the face

counter-clockwise edges are oriented 

Cluster Poisson bracket 

$\{x', x\} = x'x$

Near the boundary 

$\{x', x\} = \frac{1}{2} x'x$

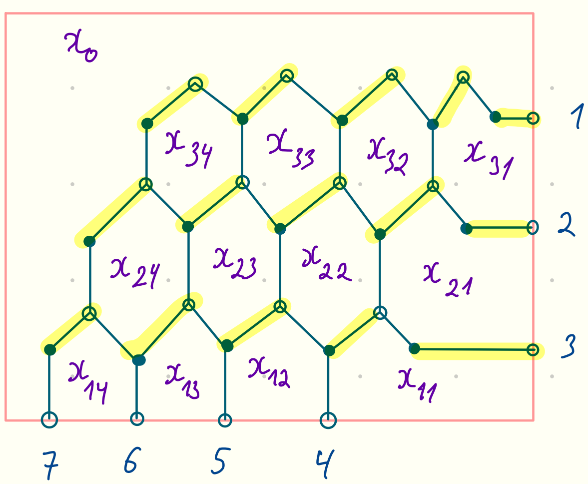
frozen variables

<u>Th</u> The cluster bracket coinsides with Sklyanin bracket on the $Gr(k,n)$

- <u>Remark</u> Δ_I - is not function on $Gr(k,n)$, but on $\widehat{Gr}(k,n)$. But ratios $\Delta_I / \Delta_{I'}$ are functions on $Gr(k,n)$. On the other hand, these ratios are functions on face variables $x_{i,j}$.

- We prove for certain bipartite graph but since any two are connected by moves and mutation preserves Poisson bracket Theorem holds $\forall G$, $\pi_G = \pi_{n-k,n}$.

<u>Pf</u>



$\Delta_{12..k}$ is monomial since $\exists$ only one perfect matching.

Moreover $\Delta_{1,..,k-p,\; k-p+q+1,..,k+q}$ monomial

$1 \le p \le k \qquad 1 \le q \le n-k$

We have $$\frac{\Delta_{1,\dots,k-p,\ k-p+q+1,\dots,k+q}}{\Delta_{1,\dots k}} = \prod_{i=1}^{p} \prod_{j=1}^{q} x_{i,j}^{1+\min(p-i,\,q-j)} \qquad (*)$$

Examples $\frac{\Delta_{124}}{\Delta_{123}} = x_{11}$, $\frac{\Delta_{125}}{\Delta_{123}} = x_{11}x_{12}$, $\frac{\Delta_{134}}{\Delta_{123}} = x_{11}x_{21}$, $\frac{\Delta_{145}}{\Delta_{123}} = x_{11}^2 x_{12} x_{21} x_{22}$

In terms of y-coordinates we have

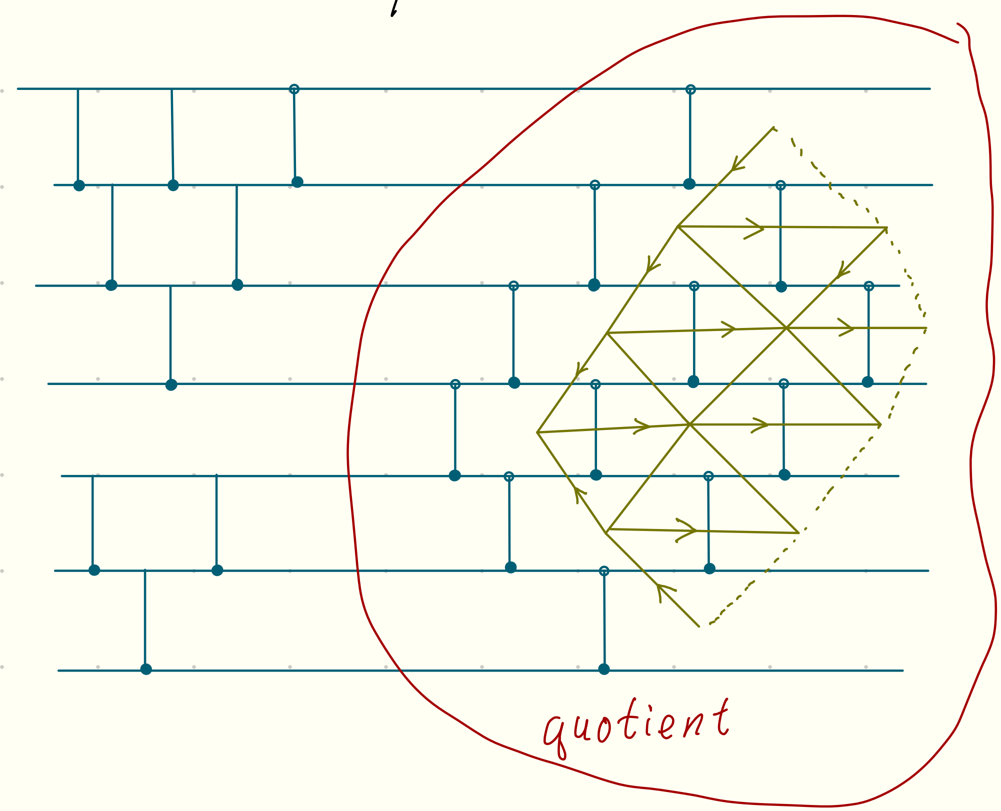
$$\frac{\Delta_{1,\dots,k-p,k-p+q+1,\dots,k+q}}{\Delta_{1,\dots,k}} = \begin{cases} y_{k-p+1,\dots,k}^{k-p+q+1,\dots,k+q} & \text{if } p \le q \\ y_{k-p+1,\dots,k-p+q}^{k+1,\dots,k+q} & \text{if } p > q \end{cases} = F_{pq}$$

Problem Compute $\{F_{pq}, F_{p'q'}\}$ using $(*)$ and cluster Poisson bracket $\square$

- <u>Remark</u> There is one more construction of cluster structure on Grassmanian (Fock–Goncharov)

The group $\mathcal{U} = \left\{ \begin{array}{c} \\ k \end{array}\!\!\begin{array}{c} \begin{array}{cc} & n-k \end{array} \\ \begin{pmatrix} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & * \\ & \begin{matrix} 1 & & 0 \\ & \ddots & \\ & & 1 \end{matrix} \end{pmatrix} \end{array} \right\}$ acts on $Gr(k,n)$ with open orbit. On the other hand $\mathcal{U}$ is a quotient of Borel subgroup $B_n = B(PGL_n)$ by subgroup $B_k \times B_{n-k} = \left\{ \begin{pmatrix} \begin{matrix} 1 & \ddots & * \\ & & 1 \end{matrix} & 0 \\ 0 & \begin{matrix} 1 & \ddots & * \\ & & 1 \end{matrix} \end{pmatrix} \right\}$

Recall $B = G^{e, w_0}$. We pick reduced decomp. of $w_0 \in W(PGL_n)$ s.t. it starts from reduced decomp. of $w_0 \times w_0 \in W(PGL_k) \times W(PGL_{n-k})$ and perform quotient.



- <u>Example</u> $Gr(3,7)$ $\quad B_7 \big/ B_3 \times B_4$

References

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- Scott Grassmanians and clucter algebras
- Speyer Variations on a theme Kasteleyn with application to the totally nonnegative grassmanian
- Gekhtman Shapiro Vainshtein Poisson geometry of Directed Networks in a Disk
- Gekhtman Shapiro Vainshtein Cluster algebras and Poisson Geometry.