Itroduction to cluster algebras and varieties

Lecture 5,6

Cluster Poisson structure. Sklyanin Bracket
Integrable systems.

Positivity revisited

- **Def** $M \in GL_n$ is totally nonnegative if $\Delta_I^J \geq 0$
  Let $G_{\geq 0} \subset GL_n$ set of TNN matrices

Let $G^{u,v}_{\geq 0} = G^{u,v} \cap G_{\geq 0}$. Clearly $G_{\geq 0} = \bigsqcup G^{u,v}_{\geq 0}$

- **Theorem** Let $(u,v) = s_{i_1} \cdots s_{i_k}$ $\quad i_1, \dots, i_k \in \{\bar{1}, \dots, \overline{n-1}, 1, \dots, n-1\}$

Then $\mathbb{E} : \mathbb{R}^{k+n}_{\geq 0} \to G^{u,v}_{\geq 0}$ is biregular bijection

Remark Braid relations $\rightsquigarrow$ mutations $\rightsquigarrow$
no poles since $x_j \neq -1$

Remark $G^{w_0, w_0}_{\geq 0} = G_{\geq 0}$ its $\mathcal{X}$-coordinates $\rightsquigarrow$
another positivity test

- Examplee GL_2 $\quad \bar{S}_2 \times S_2 = \{(e,e), (\bar{s}_1, e), (e, s_1), (\bar{s}_1, s_1)\}$

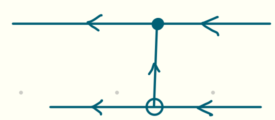
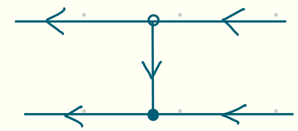
$G^{e,e}$ $\qquad H_1(x_1)H_2(x_2) = \begin{pmatrix} x_1x_2 & 0 \\ 0 & x_2 \end{pmatrix}$

$G^{\bar{s}_1,e}$ $\qquad H_1(x_1)H_2(x_2)F_1H_1(x_3) = \begin{pmatrix} x_1x_2x_3 & 0 \\ x_2x_3 & x_2 \end{pmatrix}$

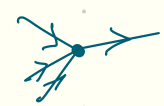
G^{e,s_1} $\qquad H_1(x_1)H_2(x_2)E_1H_1(x_3) = \begin{pmatrix} x_1x_2x_3 & x_1x_2 \\ 0 & x_2 \end{pmatrix}$

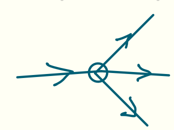
$G^{\bar{s}_1,s_1}$ $\qquad H(x_1)H_2(x_2)F_1H_1(x_3)E_1H_1(x_4) = \begin{pmatrix} x_1x_2x_3x_4 & x_1x_2x_3 \\ x_2x_3x_4 & x_2+x_2x_3 \end{pmatrix}$

totally positive matrix

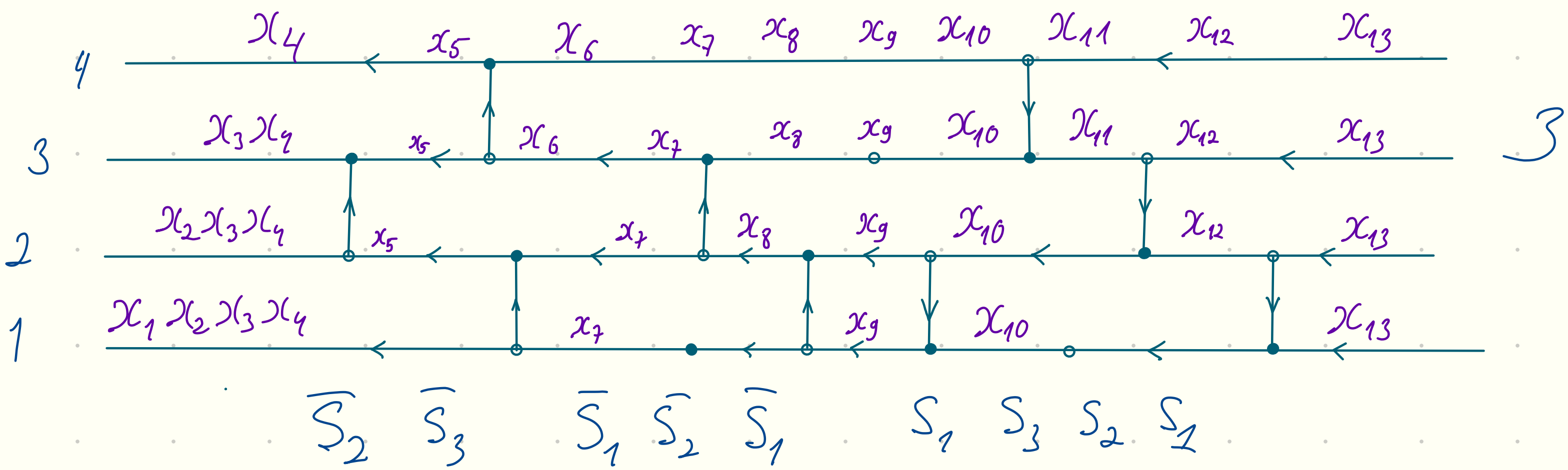
- Orientation   

Postnikov orientation

On black vertices — one outgoing edge 

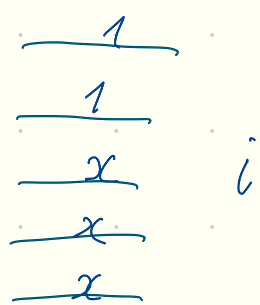
white vertice — one in going edge 

Example



$$\mathbb{E} = H_1(x_1)H_2(x_2)H_3(x_3)H_4(x_4)\, F_2\, H_2(x_5)\, F_3\, H_3(x_6)\, F_1\, H_1(x_7)\, F_2\, H_2(x_8)\, F_1\, H_1(x_9)$$
$$E_1\, H_1(x_{10})\, E_3\, H_3(x_{11})\, E_2\, H_2(x_{12})\, E_1\, H_1(x_{13})$$

$$H_i(x) = \begin{pmatrix} x & & & \\ & \ddots & & \\ & & x & \\ & & & 1 \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \rightsquigarrow$$



Here $\quad wt(path) = \prod_{e \in path} wt(e)$

$$wt\left(\bigsqcup path\right) = \prod wt(path)$$

$wt(\text{vertical edges}) = 1$

Lemma $\quad \mathbb{E}_i^j = \sum_{\text{paths } j \to i} wt(path)$

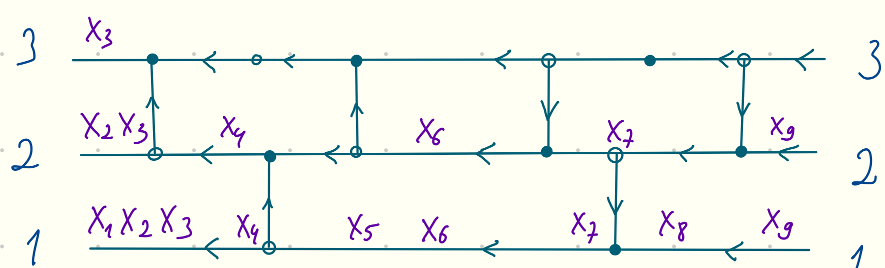
Theorem (LGV) $\mathbb{E}_I^J = \sum_{\substack{\text{non-intersecting} \\ \{paths\} \\ J \mapsto I}} wt(\{paths\})$

Corollary $\mathbb{E} \in G_{\geq 0}$

Problem Any totally positive matrix can be represented as $\mathbb{E}$ for reduced word of the form

$$(\overline{w_0}, w_0) = (\bar{s}_{n-1}\bar{s}_{n-2} \cdots \bar{s}_1)(\bar{s}_{n-1} \cdots \bar{s}_2) \cdots (\bar{s}_{n-1}\bar{s}_{n-2}) \cdot \bar{s}_{n-1}$$
$$(s_{n-1}s_{n-2} \cdots s_1)(s_{n-1} \cdots s_2) \cdots (s_{n-1}s_{n-2}) \cdot s_{n-1}$$

Example:



$$\mathbb{E} = H_1(x_1)H_2(x_2)H_3(x_3)F_2H_2(x_4)F_1H_1(x_5)$$
$$F_2H_2(x_6)E_2H_2(x_7)E_1H_1(x_8)E_2H_2(x_9)$$

$$= \begin{pmatrix} x_1\cdots x_9 & x_1\cdots x_7\cdot x_9 & x_1\cdots x_7 \\ x_2\cdots x_9 & x_2x_3x_4(1+x_5)x_6x_7x_9 & x_2x_3x_4x_6(1+x_7(1+x_5)) \\ x_3\cdots x_9 & x_3(1+x_4(1+x_5))x_6x_7x_9 & x_3(1+(1+x_4)x_6(1+x_7)+x_4x_5x_6x_7) \end{pmatrix}$$

Hint ⓐ $\forall I, J,\ |I| = |J|$ $\exists$ set non intersecting paths $J \mapsto I$

ⓑ Such set of paths unique for $\{1, \ldots, k\} \to \{i, \ldots, i+k-1\}$ and $\{i, \ldots, i+k-1\} \to \{1, \ldots, k\}$

Hence $\exists$ invertable monomial transf

$$\{\Delta_{1\ldots k}^{i,\ldots i+k-1}, \Delta_{i,\ldots i+k-1}^{1,\ldots,k}\} \Longleftrightarrow \{x_1, \ldots, x_{n^2}\}$$

Cluster varieties

- Combinatorial date b_{ij} $\quad b_{ij}+b_{ji}=0$

frozen variables $\underbrace{1\ldots n}_{\text{unfrozen}}\ \underbrace{n+1 \quad m}_{\text{froze}}$

condition $b_{jk}\in\mathbb{Z}$ if j or k – unfrozen

$\mathcal{X}$ algebraic date $(x_1, \ldots, x_m)$

more geometrically $\rightsquigarrow (\mathbb{C}^*)^m = \mathcal{X}_b$ torus

Mutations $b \rightsquigarrow b'$ $\quad \mathcal{X}_b \sim \mathcal{X}_{b'}$ – birational map

- <u>Def</u> $\mathcal{X}$-cluster variety is a union of $\mathcal{X}_{b'}$ related to b by sequence of mutations glued using birational transform. above

- Def Cluster Poisson Bracket defined by

$$\{x_i, x_j\} = b_{ij} x_i x_j$$

Remark ⓐ b_{ij} - anti symm $\Rightarrow \{\cdot,\cdot\}$ antisymm.

ⓑ $\{\log x_i, \log x_j\} = b_{ij} \Rightarrow$ Jacobi identity

$$\left(\{F, G\} = \sum_{i,j} b_{ij} x_i \frac{\partial F}{\partial x_i} x_j \frac{\partial G}{\partial x_j} = \sum b_{ij} \frac{\partial F}{\partial \log x_i} \frac{\partial G}{\partial \log x_j} \right)$$

$$\Downarrow$$

$$\{\{\log x_i, \log x_j\}, \log x_k\} + \circlearrowleft = 0$$

Problem $\{\cdot,\cdot\}$ is preserved under mutations

$$\Rightarrow \{x_j', x_i'\} = b_{ji}' x_j' x_i'$$

- For any seed $(Q, \bar{A})$ we assign torus $\mathcal{A}_{\mathcal{B}} \simeq (\mathbb{C}^*)^m$
coordinates: $A_1, \ldots, A_m$
mutation: birational map $\mathcal{A}_{\mathcal{B}} \dashrightarrow \mathcal{A}_{\mathcal{B}'}$

- <u>Def</u> $\mathcal{A}$-cluster variety is a union of $\mathcal{A}_{\mathcal{B}'}$ related to $\mathcal{B}$ by sequence of mutations glued using birational transform. above

- <u>Def</u> Cluster 2-form is $\sum b_{ij} \frac{dA_i}{A_i} \wedge \frac{dA_j}{A_j}$

<u>Problem</u> ⓐ This form is well defined on $\mathcal{A}$-cluster variety

ⓑ* This form lifts to well defined class $\sum b_{ij} \{A_i, A_j\} \in K_2$

[Recall, on chart $\mathcal{U}$, $K_2(\mathcal{U}) = \mathbb{C}(\mathcal{U})^* \otimes_{\mathbb{Z}} \mathbb{C}(\mathcal{U})^* / (\{f, 1-f\})$)

Sklyanin Bracket

- Classical r-matrix

$$r = \frac{1}{2} \sum_{i<j} (e_{ij} \otimes e_{ji} - e_{ji} \otimes e_{ij}) \in \Lambda^2 \mathfrak{gl}_n \subset \mathfrak{gl}_n \otimes \mathfrak{gl}_n$$

- Key property: $[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] \in (\Lambda^3 \mathfrak{g})^G$

Modified Classical Yang Baxter Equation

Instead of $(\Lambda^3 \mathfrak{g})^G$ one can write $\langle e_{ij} \wedge e_{jk} \wedge e_{ki} \rangle$

- <u>Def</u> Sklyanin Poisson structure on GL_n

$$\{g \otimes g\} = [r, g \otimes g]$$

In coordinates $\{g_i^j, g_{i'}^{j'}\} = ([r, g \otimes g])_{i\,i'}^{j\,j'}$

More invariantly $r \in \Lambda^2 \mathfrak{g} = \Lambda^2 T_e G$

$$\Pi = (\rho_g)_* r - (\lambda_g)_* r$$

right shift — left shift

- Theorem (a) Π is Poisson structure

(b) Π defines Poisson-Lie structure on GL_n
i.e $m: G\times G \to G$ is Poisson $m_*(\Pi \oplus \Pi) = \Pi$

(c) Double Bruhat cells are Poisson submanifolds on G (actually unions of sympl. leaves of equal dimension)

- Remark (a) Anti commutativity $\{g_{ij}, g_{i'j'}\} = -\{g_{i'j'}, g_{ij}\}$ follows from $r \in \Lambda^2 \mathfrak{g}$

Jacobi identity follows from MCYBE

(b) Poisson-Lie property for $g = g_1 g_2$

$$\{g \overset{\otimes}{,} g\} = [r, g \otimes g] = [r, g_1 \otimes g_1] g_2 \otimes g_2 + g_1 \otimes g_2 [r, g_2 \otimes g_2]$$

$$= \{g_1 \overset{\otimes}{,} g_1\} g_2 \otimes g_2 + g_1 \otimes g_1 \{g_2 \overset{\otimes}{,} g_2\}$$

Π

$m_* \Pi$

- Problem (a) Compute Sklyanin bracket for $GL_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$ i.e. compute all brackets of the form $\{a, b\}$

(b) Find two algebraically independent Casimir functions

- <u>Theorem</u> (Fock–Gonchorov) $\Pi|_{G^{u,v}}$ – cluster Poisson structure for PGL_n.

- Below we will do computations for SL_n

$$H_i(x) = \begin{array}{c} 1 \\ \vdots \\ i \\ i+1 \\ \vdots \\ n \end{array} \begin{pmatrix} x^{\frac{n-i}{n}} & & & & \\ & \ddots & & & \\ & & x^{\frac{n-i}{n}} & & \\ & & & x^{-\frac{i}{n}} & \\ & & & & \ddots \\ & & & & & x^{-\frac{i}{n}} \end{pmatrix} =_{PGL_n} H_i^{old}(x) = \begin{pmatrix} x & & & & \\ & \ddots & & & \\ & & x & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$$

In particular $H_n(x) = 1$

- Due to Poisson-Lie property it sufficient to prove only for $(u,v) = s_i$ $i \in \{\bar{1}, \dots, \overline{n-1}, 1, \dots, n-1\}$

- $\Pi|_{G^{e,e}} = 0$

- **Problem*** For $G^{s_i,e}$ $\mathbb{E} = H_1(x_1)\cdots H_{n-1}(x_{n-1})\,F_i\,H_i(x_n)$ the Sklyanin structure has a form

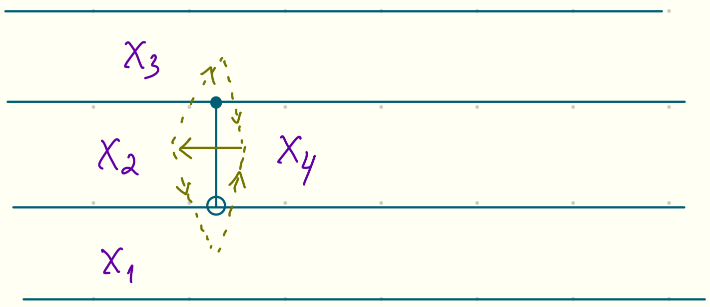
$$\{x_n, x_i\} = x_n x_i$$

$$\{x_i, x_{i-1}\} = \frac{1}{2} x_i x_{i-1} \qquad \{x_n, x_{i-1}\} = -\frac{1}{2} x_n x_{i-1}$$

$$\{x_i, x_{i+1}\} = \frac{1}{2} x_i x_{i-1} \qquad \{x_n, x_{i+1}\} = -\frac{1}{2} x_n x_{i+1}$$

all other brackets are 0.

Example $G^{s_1,e} \in PGL_4$



Dashed line stands for $b_{ij} = \frac{1}{2}$

Lesson $b_{ij} \notin \mathbb{Z}$ if i,j — frozen

Lemma For G^{e,s_i} $\mathbb{E} = H_1(x_1)\cdots H_{n-1}(x_{n-1})E_i H_i(x_n)$ the Sklyanin structure has a form

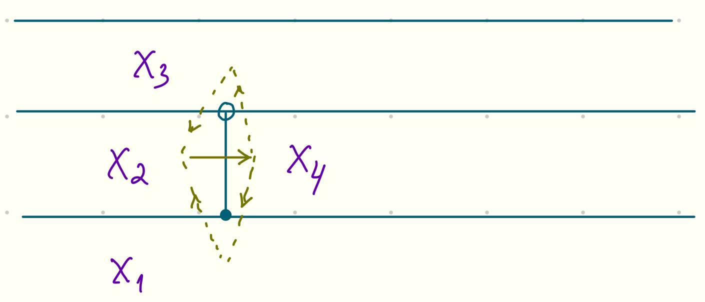
$$\{x_n, x_i\} = -x_n x_i$$

$$\{x_i, x_{i-1}\} = -\frac{1}{2} x_i x_{i-1} \qquad \{x_n, x_{i-1}\} = \frac{1}{2} x_n x_{i-1}$$

$$\{x_i, x_{i+1}\} = -\frac{1}{2} x_i x_{i-1} \qquad \{x_n, x_{i+1}\} = \frac{1}{2} x_n x_{i+1}$$

all other brackets are 0.

Example $G^{e,s_1} \in PGL_4$



Dashed lines stand for $b_{ij} = \frac{1}{2}$

Amalgamation

- Seed $(I, I_f, B^I, \bar{x}^I)$

set of vertices (for I) set of frozen vertices (for I_f)

Seed $(J, J_f, B^J, \bar{x}^J)$

Assume we have $L \rightarrow I_f$, $L \rightarrow J_f$

<u>Def</u> Amalgamation $(K, K_f, B^K, \bar{x}^K)$

$K = I \sqcup_L J \qquad K_f = I_f \sqcup_L J_f$

$$b^K_{ij} = \begin{cases} 0 & \text{if } i \in I \setminus L,\ j \in J \setminus L \text{ or versa} \\ b^I_{ij} & \text{if } i \in I \setminus L \text{ and } j \in L, \text{ or versa} \\ b_{ij} & \text{- - - - -} \\ b^I_{ij} + b^J_{ij} & \text{if } i, j \in L \end{cases}$$

$$x_i^K = \begin{cases} x_i^I & \text{if } i \in I \setminus L \\ x_i^J & \text{if } i \in J \setminus L \\ x_i^I x_i^J & \text{if } i \in L \end{cases}$$

Remark If for some $i \in L$ all $b_{ij} \in \mathbb{Z}$ we can unfroze variable x_i^K

- Lemma (a) Amalgamation is Poisson map.
  (b) Amalgamation commutes with mutation

- $G^{u,v} \sim (u,v) = s_{i_1} \cdots s_{i_N}$

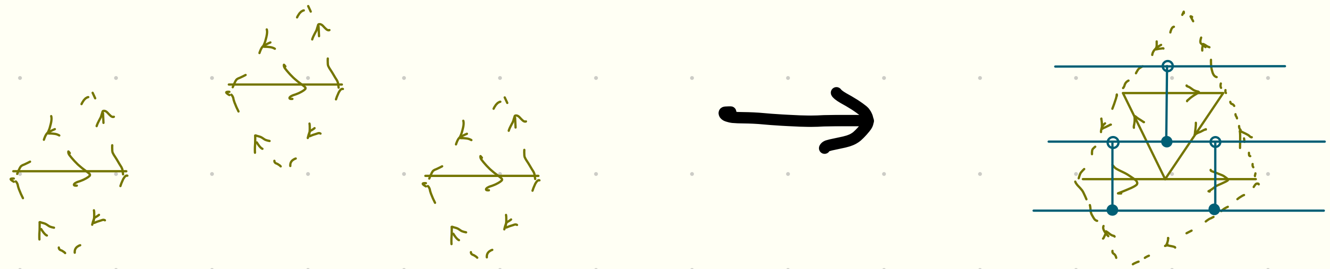
take amalgamation of G^{s_i} and unfroze vertices corresp to closed faces

- Examples of amalgamation & unfrozen

If we have

$s_i \bar{s}_i$

$s_i s_{i+1} s_i$



Poisson-Lie property and Lemmas for $C^{s_i,e}$ and C^{e,s_i}
$\Rightarrow$ cluster bracket $\Rightarrow$ Sklyanin bracket $\square$

- Remark We get the same quiver as above on unfrozen vertices: edges clockwise around black vertices
counter-clockwise around white vertices

- **Def** $c \in W$ - Coxeter element if $c = s_{i_1} \cdot \ldots \cdot s_{i_{n-1}}$ where $i_1 \ldots i_{n-1}$ is permutation of $\{1, \ldots, n-1\}$

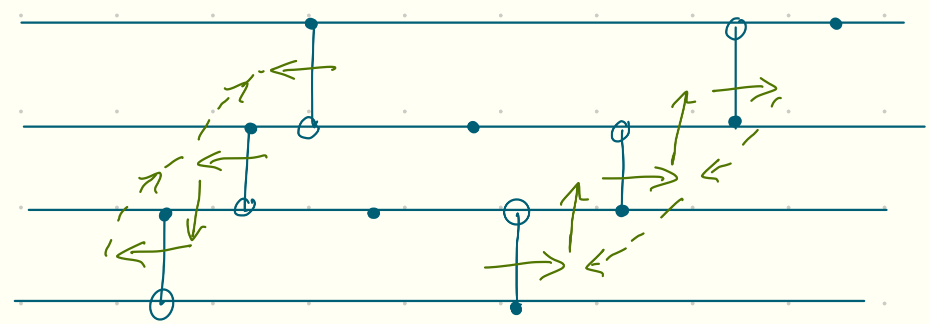
Theorem All Coxeter elements are conjugated
Order of c is h - Coxeter number

$$h = \frac{\dim \mathfrak{g}}{\dim \mathfrak{h}} - 1$$

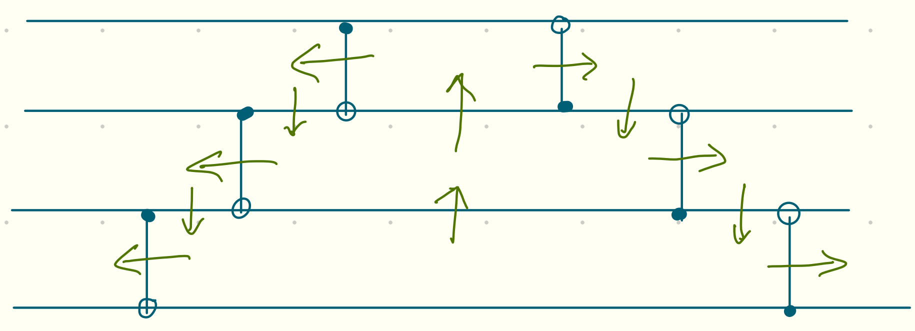
Coxeter cells: $\mathcal{C}^{c,c'}$, c, c' - Coxeter elements

Example PGL_4

$\bar{s}_1 \bar{s}_2 \bar{s}_3 \bar{s}_4 \, s_1 s_2 s_3 s_4$



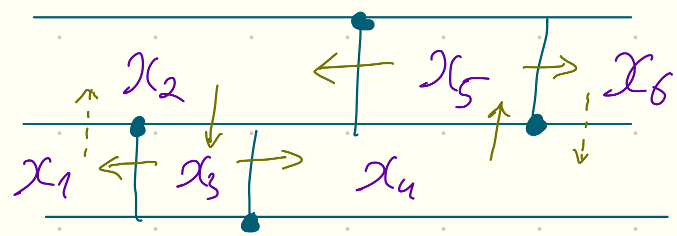
$\bar{s}_1 \bar{s}_2 \bar{s}_3 \bar{s}_4 \, s_4 s_2 s_2 s_1$



Integrable systems

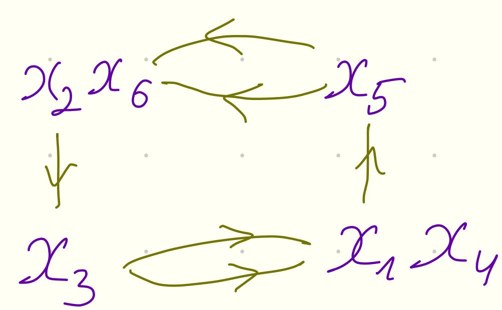
- $G^{u,v} / \mathrm{Ad}\,H$
  - we glue (amalgamate) frozen variables on each border
  - then we unfroze them

<u>Example</u> $G^{s_1 s_2 ,\, s_1 s_2} \in PGL_3$



$$H_1(x_1)\, H_2(x_2)\, F_1\, H_1(x_3)\, E_1\, H_1(x_4)\, F_2\, H_2(x_5)\, E_2\, H_2(x_6)$$

$$\underset{\mathrm{Ad}H}{\sim} F_1\, H_1(x_3)\, E_1\, H_1(x_4 x_1)\, F_2\, H(x_5)\, E_2\, H_2(x_2 x_6)$$



- <u>Lemma</u> $I_k = \frac{1}{k} \operatorname{Tr} g^k \qquad \{I_k, I_l\} = 0$

<u>Proof</u> $\{\operatorname{Tr} g^k, \operatorname{Tr} g^l\} = \operatorname{Tr}\{g^k \overset{\otimes}{,} g^l\} =$

$$= \operatorname{Tr} \sum_{i=1}^{k} \sum_{j=1}^{l} (g^{i-1} \otimes g^{j-1}) \{g \overset{\otimes}{,} g\} (g^{k-i-1} \otimes g^{l-j-1})$$

$$= \operatorname{Tr} \sum_{i=1}^{k} \sum_{j=1}^{l} (g^{i-1} \otimes g^{j-1}) [r, g \otimes g] (g^{k-i-1} \otimes g^{k-j-1}) = \operatorname{Tr} [r, g^k \otimes g^l] = 0$$

- Counting: $G^{u,v} \subset SL_n \qquad \dim G^{u,v} = n-1 + \ell(u) + \ell(v)$

$\dim G^{u,v}/AdH = \ell(u) + \ell(v)$ (if u, v contains s_i or $\bar{s}_i$ for any $i \in \{1, .., n-1\}$)

For Coxeter cells $\dim G^{c,c'}/AdH = 2(n-1)$

$n-1$ commuting I_k $\Rightarrow$ Integrable systems

Coxeter-Toda integrable system

Remark Quivers for $G^{u,v}/\mathrm{Ad}H$ depends only on cyclic order of simple reflections i.e quivers for $s_{i_1}\cdots s_{i_N}$ and $s_{i_2}\cdots s_{i_N}s_{i_1}$ coinside

Problem (a) Show that Coxeter element depends only on orientation of Dynkin diagram i.e. for any i what is earlier s_i or s_{i+1}.

(b) Show that using transformations preserving quiver $G^{c,c'}/\mathrm{Ad}H$ (i.e cyclic permutations

$$s_i\bar{s}_j=\bar{s}_j s_i \quad i\neq j \qquad s_i s_j = s_j s_i, \quad \bar{s}_i\bar{s}_j=\bar{s}_j\bar{s}_i \quad |i-j|>1$$
)

the word reduces to $s_{i_1}\cdots s_{i_{r-1}}\bar{s}_{j_1}\cdots\bar{s}_{j_{r-1}}$

(c) There are 3^{n-1} different quivers for $G^{c,c'}/\mathrm{Ad}H$

All of them are mutation equivalent

- **Problem*** Consider cell $G^{c,c}/AdH$. Take decomposition

$$g = H_1(x_1)F_1H_1(y_1)E_1 \cdot \ldots \cdot H_{n-1}(x_{n-1})F_{n-1}H_{n-1}(y_{n-1})E_{n-1}$$

(a) Compute Poisson brackets of $x_1,\ldots,x_{n-1},y_1,\ldots,y_{n-1}$.

(b) Compute $I_1 = \operatorname{tr} g$.

(c) Let $\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_n$ exponential Darboux coordinates

$\{\eta_i,\xi_j\} = \delta_{ij}\eta_i\xi_j$, $\{\xi_i,\xi_j\} = \{\eta_i,\eta_j\} = 0$.

Show that $x_i = \eta_i/\eta_{i+1}$ $y_i = \xi_{i+1}/\xi_i$ is Poisson map.

(d) Let $L_i = \begin{pmatrix} \mu\,\xi_i^{1/2}\eta_i^{-1/2} + \xi_i^{-1/2}\eta_i^{1/2} & \mu\,\xi_i^{-1/2}\eta_i^{-1/2} \\ \xi_i^{1/2}\eta_i^{1/2} & 0 \end{pmatrix}$ – Lax matrix

$L_1 \cdot L_2 \cdot \ldots \cdot L_n = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Let $A = \sum \tilde{I}_k \mu^{n-k}$

show that $\tilde{I}_1 \sim I_1$ (proportional by monomial commuting with $x_1,\ldots,x_{n-1},y_1,\ldots,y_{n-1}$).

(e) Show the same for $\tilde{I}_k$ and $I_k = \operatorname{Tr}\Lambda^k g$.

References

- Fock Goncharov Cluster $\mathcal{X}$-varieties, amalgamation and Poisson-Lie groups.