Itroduction to cluster algebras and varieties

Lecture 4

Double Bruhat cells $\mathcal{X}$ variables

Double Weye group

- $W = S_n$ $s_i = (i, i+1)$ $s_i^2 = e$, $\begin{cases} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \\ s_i s_j = s_j s_i \quad |i-j|>1 \end{cases}$

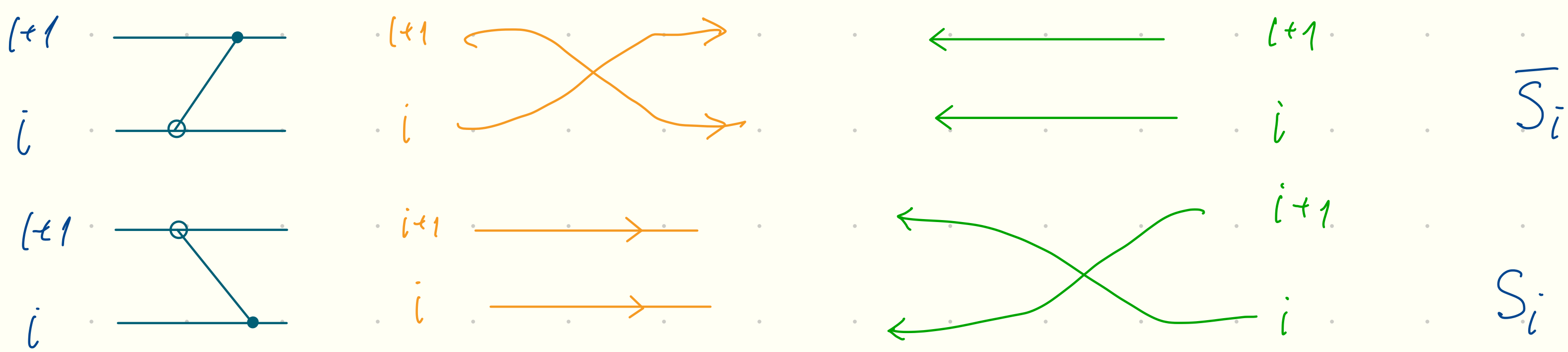
Braid relations

$W \times W = \overline{S}_n \times S_n$

generators $\bar{s}_1, \dots \bar{s}_{n-1}$ — $s_1, \dots, s_{n-1}$ generators

$\bar{s}_1 = s_{\bar{1}}$, $\bar{s}_{n-1} = s_{\overline{n-1}}$

- Recall



- Any $u \in W$ has minimal (reduced) decomposition

$u = s_{i_1} \cdot \ldots \cdot s_{i_\ell}$ $\ell = \ell(u)$

Theorem (Tits) Any 2 reduced decomposition are related by sequence of braid relations

Example S_3 $\quad w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$

longest element

S_4 $\quad w_0 = s_1 s_2 s_3 s_1 s_2 s_1 = s_1 s_2 s_3 s_2 s_1 s_2 =$

$= s_1 s_3 s_2 s_3 s_1 s_2$

$\parallel$

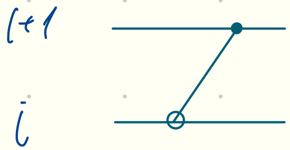
$s_1 s_2 s_1 s_3 s_2 s_1$

$\parallel$

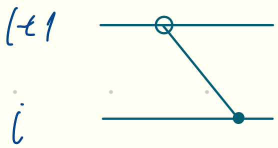
$s_2 s_1 s_2 s_3 s_2 s_1$

- Networks last time: reduced decomposition of $(w_0, w_0) \in S_n \times S_n$

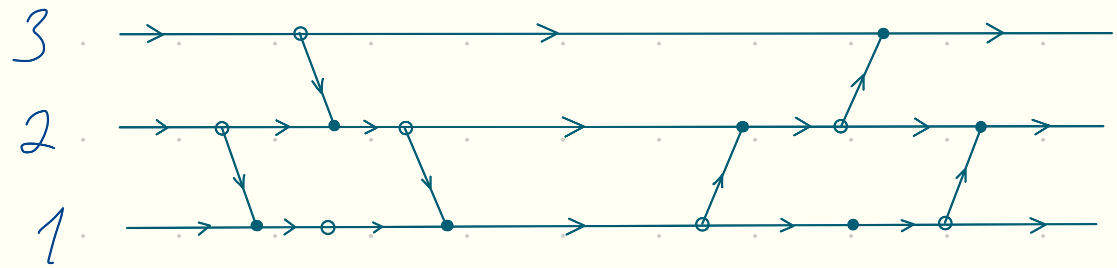
Example:



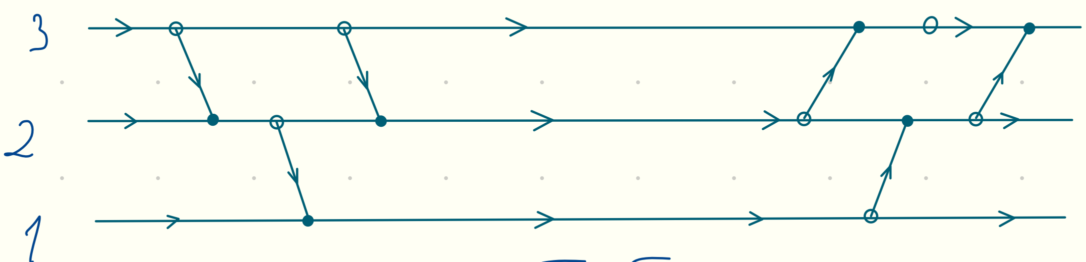
$\bar{s}_i$



s_i



$s_1 \, s_2 \, s_1 \qquad \bar{s}_1 \, \bar{s}_2 \, \bar{s}_1$



$s_2 \, s_1 \, s_2 \qquad \bar{s}_2 \, \bar{s}_1 \, \bar{s}_2$

Braid relations $\longleftrightarrow$ mutations of the seeds

- We can consider more general networks: reduced decomposition of $(u,v) \in S_n \times S_n$

We can construct Q + assign variables Δ

Braid relations $\longleftrightarrow$ mutations of the seeds

Double Bruhat cells

Bruhat cells

- $G = GL_n$, $B = B_+ = \begin{pmatrix} * & & * \\ 0 & \ddots & \\ & & * \end{pmatrix}$ $B_- = \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix}$, $H = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$

$W \simeq S_n$ – Weyl group $\quad s_i = (i, i+1)$ simple reflection

$\simeq N(H)/H$

<u>Theorem</u> $G = \bigsqcup_{w \in W} BwB = \bigsqcup_{w \in W} B_- w B_- = \bigsqcup_{w \in W} B_- w B$

(More accurate to say $\bar{w} \in N(H)$)

<u>Remark</u> ① $w_0 B_- w_0 = B \implies$ If $G = \bigsqcup BwB \implies$

$G = w_0 G = \bigsqcup w_0 BwB =$

$= \bigsqcup B_- w_0 w B$

$= \bigsqcup B_- w B$

② $G = \bigsqcup_W B_- w B$ – Gauss decomposition

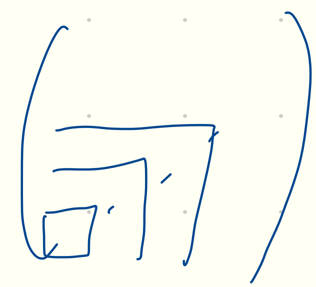
Theorem BwB is determined by conditions

$$\Delta^{1\dots i}_{w(1\dots i)} \neq 0 \qquad \Delta^{1\dots i}_{w(1\dots i-1,j)} = 0 \quad \text{for } i<j,\ w(i)<w(j)$$

Example ① $w=e$ $\qquad BwB = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$

$$\Delta^1_1 \neq 0 \qquad \Delta^1_j = 0 \quad j>1$$

$$\Delta^{12}_{12} \neq 0 \qquad \Delta^{12}_{1j} = 0 \quad j>2$$



② $w = w_0 = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & & 1 \end{pmatrix}$ $\qquad \Delta^{1\dots i}_{n-i+1\dots n} \neq 0$

Problem ⓐ Prove that on BwB these conditions are satisfied. ⓑ Prove the Theorem.

Hint Use $\Delta^{j_1,\dots,j_k}_{i_1\dots i_k}$ is matrix element

$$\Lambda^k M: \quad e_{j_1}\wedge\dots\wedge e_{j_k} \longmapsto e_{i_1}\wedge\dots\wedge e_{i_k}$$

Remark For two sets we say

$\{j_1 < \dots < j_k\} < \{l_1 < \dots < l_k\} \iff \{j_1 \le l_1, \dots, j_k < l_k\}$

Then $\Delta^{1,\dots,i}_{j_1 \dots j_l} = 0$ on BwB if $\{j_1, \dots, j_l\} \not< w(1,\dots,i)$

Hence, there are more vanishing minors.

- **Remark** $(BwB)^t = B_- w^{-1} B_-$

Corollary $B_- w B_-$ is determined by conditions

$\Delta^{w^{-1}(1\dots i)}_{1 \quad i} \neq 0 \quad \Delta^{w^{-1}(1\dots i-1, j)}_{1 \dots i} = 0$ for $i < j$, $w^{-1}(i) < w^{-1}(j)$

- **Def** Double Bruhat cell $G^{u,v} = BuB \cap B_- v B_-$

Determined by equations above $(u,v) \in \bar{S}_n \times S_n$

Factorization scheme

- $E_i(x) = \begin{pmatrix} 1 & & & & \\ & 1 & x & & \\ & & \ddots & & \\ & & & & 1 \end{pmatrix} = \exp(x\, e_i)$ $\qquad E_i = E_i(1)$

  $E_{\bar{i}}(x) = F_i(x) = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & x & \ddots & \\ & & & & 1 \end{pmatrix} = \exp(x f_i)$ $\qquad E_{\bar{i}} = F_i = F_i(1)$

  $H_i(x) = \begin{pmatrix} x & & & & & \\ & x & & & & \\ & & \ddots & & & \\ & & & x & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}$ $\qquad |\bar{i}| = i = |i|$

- For any reduced word $(u,v) = s_{i_1} \cdots s_{i_\ell}$ we assign a product

  $i_1 \ldots i_\ell \in \{1, \ldots, n-1, \bar{1}, \ldots, \overline{n-1}\}$

$$H_1(x_1) \cdots H_n(x_n)\, E_{i_1} H_{|i_1|}(x_{n+1})\, E_{i_2} H_{|i_2|}(x_{n+2}) \cdots = \mathbb{E}_{\mathbf{i}}$$

<u>Example</u> $G = GL_3$ $\quad u = \bar{s}_1 \bar{s}_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad v = e$

$$E_{ii} = H_1(x_1) H_2(x_2) H_3(x_3) F_1 H_1(x_4) F_2 H_2(x_5) =$$

$$= \begin{pmatrix} x_1 x_2 x_3 x_4 x_5 & 0 & 0 \\ x_2 x_3 x_4 x_5 & x_2 x_3 x_5 & 0 \\ 0 & x_3 x_5 & x_3 \end{pmatrix} \in BuB \cap B_-$$

Minors $\Delta_2^1 \neq 0 \quad \Delta_3^1 = 0 \quad \Delta_{23}^{12} \neq 0$

<u>Example</u> $G = GL_3$ $\quad u = \bar{s}_2 \bar{s}_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad v = e$

$$E = H_1(x_1) H_2(x_2) H_3(x_3) F_2 H_2(x_4) F_1 H_1(x_5) =$$

$$= \begin{pmatrix} x_1 x_2 x_3 x_4 x_5 & 0 & 0 \\ x_2 x_3 x_4 x_5 & x_2 x_3 x_4 & 0 \\ x_3 x_4 x_5 & x_3 x_4 & x_3 \end{pmatrix} \in BuB \cap B_-$$

Minors $\Delta_3^1 \neq 0, \quad \Delta_{13}^{12} \neq 0, \quad \Delta_{23}^{12} = 0$

- **Theorem** For any reduced decompos of $(u,v) \in W \times W$ the map

$$(\mathbb{C}^*)^{\ell(u,v)+n} \longrightarrow G^{u,v}$$

is birational isomorphism

Pf Let us prove that $\mathbb{E}_{ii} \in BuB$. Since $\forall\ E_i, H_j \in B$ it is sufficient to prove that for $u = s_{i_1} \cdot \ldots \cdot s_{i_\ell}$

$$H_1(x_1) \cdot \ldots \cdot H_n(x_n) F_{i_1} H_{i_1}(x_{n+1}) \cdot \ldots \cdot F_{i_k} H_{i_k}(x_{n+k}) \in BuB.$$

Note that $F_i \in B\bar{s}_i B$. Indeed $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Problem* If $\ell(us_i) = \ell(u)+1$ then $Bus_iB = BuB \cdot Bs_iB$

Hint Use description of BuB through minors

We proved existance of map to $G^{u,v}$. It remains to check that map is injective and compare dimensions. ∎

Relations

$$H_i(x)\,H_j(y) = H_j(y)\,H_i(x)$$

- $H_i E_j = E_j H_i \qquad H_i F_j = F_j H_i \qquad i \neq j$
- $H_i(x)\,H_i(y) = H_i(xy)$
- $E_i E_j = E_j E_i \qquad F_i F_j = F_j F_i \qquad |i-j|>1 \qquad E_i F_j = F_j E_i \qquad i \neq j$

  $S_i S_j = S_j S_i \qquad \bar{S}_i \bar{S}_j = \bar{S}_j \bar{S}_i \qquad S_i \bar{S}_j = \bar{S}_j S_i$

- $E_i E_{i+1} H_i(x) E_i = H_i(1+x)\, H_{i+1}\left(\frac{1}{1+x^{-1}}\right) E_{i+1} E_i H_{i+1}(x^{-1}) E_{i+1} H_i\left(\frac{1}{1+x^{-1}}\right) H_{i+1}(1+x)$

  $S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}$

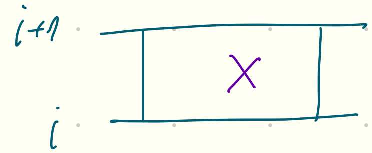
- $F_i F_{i+1} H_i(x) F_i = H_{i+1}(1+x)\, H_i\left(\frac{1}{1+x^{-1}}\right) F_{i+1} F_i H_{i+1}(x^{-1}) F_{i+1} H_{i+1}\left(\frac{1}{1+x^{-1}}\right) H_i(1+x)$

  $\bar{S}_i \bar{S}_{i+1} \bar{S}_i = \bar{S}_{i+1} \bar{S}_i \bar{S}_{i+1}$

- $F_i H_i(x) E_i = H_i\left(\frac{1}{1+x^{-1}}\right) E_i H_i(x^{-1}) F_i H_i\left(\frac{1}{1+x^{-1}}\right) H_{i-1}(1+x) H_{i+1}(1+x)$

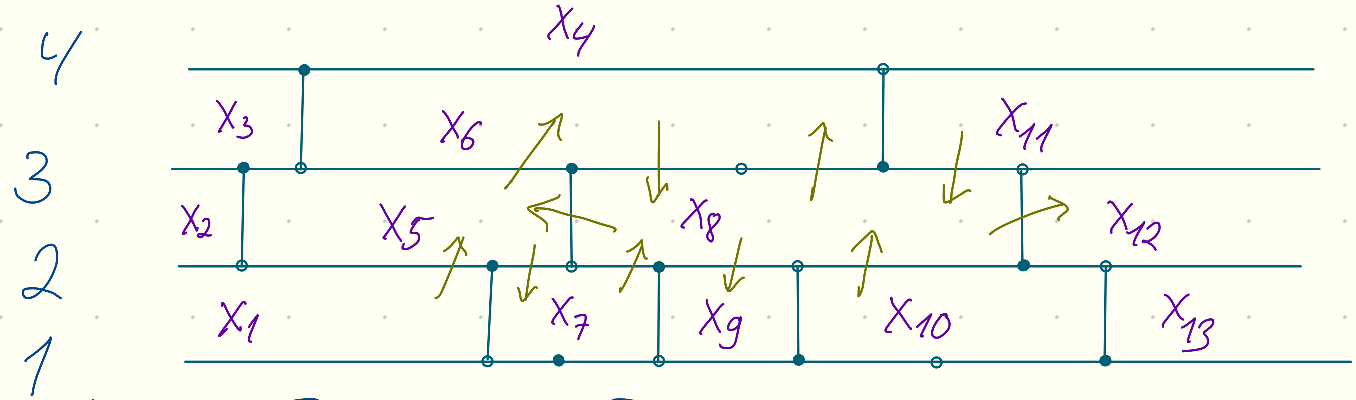
  $\bar{S}_i S_i = S_i \bar{S}_i$

- Assign x variables to the cell

$H_i(x) \rightsquigarrow$ 

from left to right.

- Example

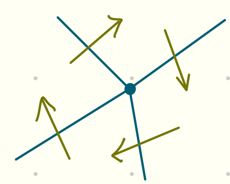


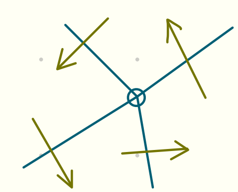
$\bar{S}_2 \quad \bar{S}_3 \quad \bar{S}_1 \quad \bar{S}_2 \quad \bar{S}_1 \quad S_1 \quad S_3 \quad S_2 \quad S_1$

$$\mathbb{E}_{ii} = H_1(x_1)H_2(x_2)H_3(x_3)H_4(x_4)\,F_2\,H_2(x_5)\,F_3\,H_3(x_6)\,F_1\,H_1(x_7)\,F_2\,H_2(x_8)\,F_1\,H_1(x_9)$$
$$E_1\,H_1(x_{10})E_3\,H_3(x_{11})\,E_2\,H_2(x_{12})\,E_1\,H_1(x_{13})$$

- Recall quiver

Vertices – faces of graph





Edges – clockwise around black vertices
conter-clockwise around white vertices

X variables

- Pair $Q, x_1 \dots x_n$ assigned to vertices

Mutation μ_k : $Q \rightsquigarrow Q'$ $\quad x_j' = \begin{cases} x_k^{-1} & j=k \\ x_j\left(1+x_k^{\operatorname{sgn} b_{jk}}\right)^{b_{jk}} & j \neq k. \end{cases}$

- Problem For cluster seed $(Q, \bar{A})$ let $x_j = \prod A_i^{b_{ij}}$

Mutation of seed $\rightsquigarrow$ mutation of X variables

ensemble map

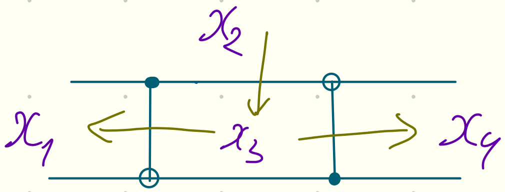
- Properties (a) $\mu_k^2 = id$ (b) $\varepsilon_{jk}=0 \Rightarrow \mu_k \mu_j = \mu_j \mu_k$

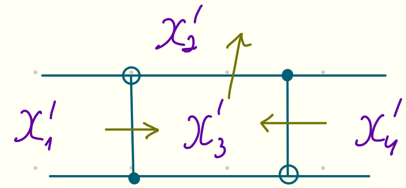
(c) $\varepsilon_{jk} = -1$ $\quad \mu_k \mu_j \mu_k \mu_j \mu_k = (j,k)$

- Theorem Braid moves in reduced decomposition corresponds to x cluster mutations.

Problem Prove theorem.

Example $\quad G = GL_2 \quad u = \bar{S}_1 \quad v = S_1$





$$\bar{S}_1 S_1 \quad = \quad S_1 \bar{S}_1$$

$$H_1(x_1) H_2(x_2) F_1 H_1(x_3) E_1 H_1(x_4) =$$

$$= H_1(x_1) H_2(x_2) H_1\left(\frac{1}{1+x_3^{-1}}\right) H_2(1+x_3) E_1 H_1(x_3^{-1}) F_1 H_1\left(\frac{1}{1+x_1^{-1}}\right) H_1(x_4)$$

$$= H_1\left(\frac{x_1}{1+x_3^{-1}}\right) H_2(x_2(1+x_3)) E_1 H_1(x_3^{-1}) F_1 H_1\left(\frac{x_4}{1+x_1^{-1}}\right)$$

$$= H_1(x_1') H_2(x_2') E_1 H_1(x_3) F_1 H_1(x_4')$$

- $\exists$ simple relation between x-coord of $g \in G^{u,v}$ and minors of $g' \in G^{u^{-1},v^{-1}}$

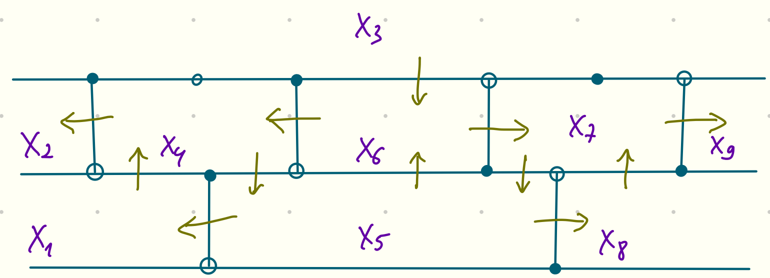
twist map

$$\zeta^{u,v}: G^{u,v} \to G^{u^{-1},v^{-1}}$$

$$\zeta^{u,v}(g) = g'$$

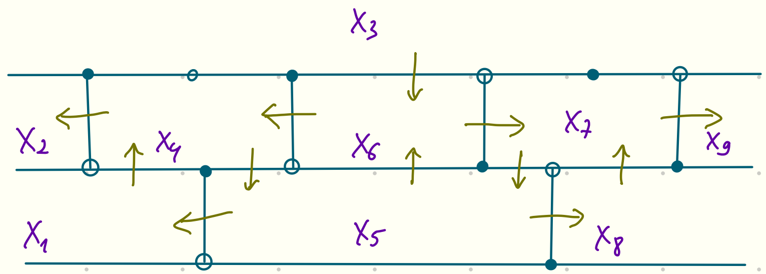
Example $G = GL_3$

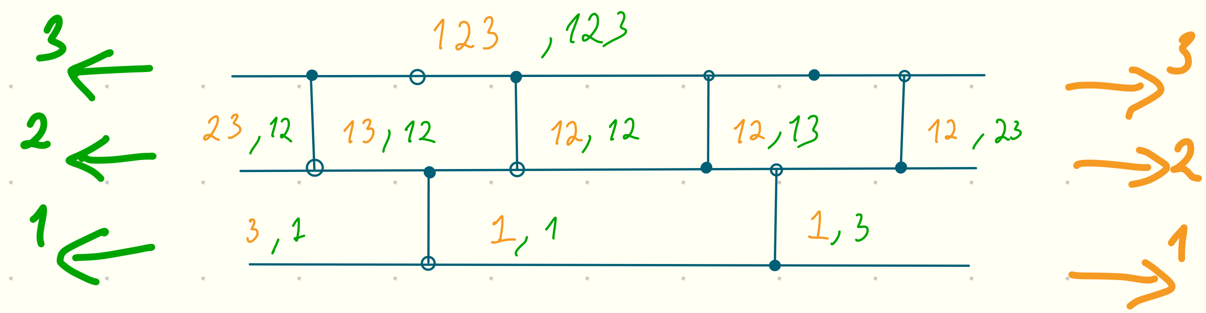
G^{w_0,w_0}



$$g = H_1(x_1)H_2(x_2)H_3(x_3)F_2H_2(x_4)F_1H_1(x_5)F_2H_2(x_6)E_2H_2(x_7)E_1H_1(x_8)E_2H_2(x_9)$$

$$g' = \begin{pmatrix} \dfrac{g_{11}}{g_{31}g_{13}} & \dfrac{M_{12,13}}{g_{31}M_{12,23}} & \dfrac{1}{g_{31}} \\[2ex] \dfrac{M_{13,12}}{g_{13}M_{23,12}} & \dfrac{g_{3,3}M_{12,12} - \det x}{M_{23,12}\,M_{12,23}} & \dfrac{g_{3,2}}{M_{23,12}} \\[2ex] \dfrac{1}{g_{13}} & \dfrac{g_{23}}{M_{12,23}} & \dfrac{M_{23,23}}{\det g} \end{pmatrix}$$





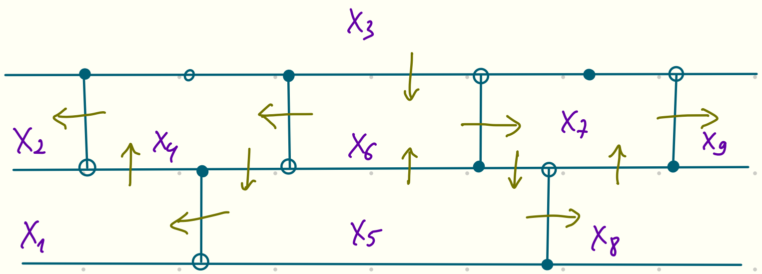
$$x_j = \prod A_i^{b_{ij}}$$

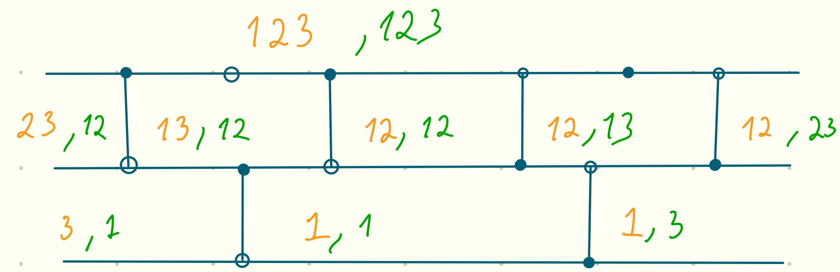
(zig-zags are numbered by their ends from bottom to top)

$$x_4 = \frac{M'^{12}_{12} \cdot M'^{1}_{3}}{M'^{12}_{23} \; M'^{1}_{1}} \qquad x_5 = \frac{M'^{13}_{12} \; M'^{12}_{13}}{M'^{1}_{3} \; M'^{12}_{12} \; M'^{3}_{1}}$$

$$x_6 = \frac{M'^{1}_{1} \; M'^{123}_{123}}{M'^{12}_{13} \; M'^{13}_{12}} \qquad x_7 = \frac{M'^{12}_{12} \; M_{1,3}}{M'_{1,1} \cdot M'^{23}_{12}}$$

for boundary small corr is needed





For minors of g :

$$x_4 = \frac{M^{12}_{12} \cdot M^{1}_{3}}{M^{12}_{23} \, M^{1}_{1}} \qquad \frac{(1+x_4^{-1})(1+x_7^{-1})}{x_5} = \frac{M^{13}_{12} \, M^{12}_{13}}{M^{1}_{3} \, M^{12}_{12} \, M^{1}_{1}}$$

$$\frac{1}{(1+x_4)x_6(1+x_7)} = \frac{M^{1}_{1} \, M^{123}_{123}}{M^{12}_{13} \, M^{13}_{12}} \qquad x_7 = \frac{M^{12}_{12} \, M_{1,3}}{M_{1,1} \cdot M^{23}_{12}}$$

Mutation in x_4 and x_7 + inversion

References

- Fomin Zelevinsky Double Bruhat cells and total positivity