Itroduction to cluster algebras and varieties

Lecture 3

Total positivity. Networks

Laurent phenomenon

- <u>Theorem</u> $\overset{s_0}{\bullet} - \overset{s_1}{\bullet} - \cdots - \overset{s_d}{\bullet}$ Then $\bar{A}(s_d)$ — Laurent polynomials on $A(s_0)$ with integer coefficients.

$$A(t_0) = \underbrace{A_1, \ldots, A_n}_{\text{unfrozen}}, \underbrace{A_{n+1}, \ldots, A_m}_{\text{frozen}}$$

- <u>Theorem</u> $A(s) \in \mathbb{Z}[A_1^{\pm 1}, \ldots, A_n^{\pm 1}, A_{n+1}, \ldots, A_m]$

<u>Pf</u> Induction on d. Consider $A_i(s_d)$ as a function on frozen variable A_r.

<u>Claim</u> Any $A_i(s_d)\big|_{A_r=0}$ has form of substraction free rational function

(i.e. $\frac{P}{Q}$, where P, Q - polynomials with <u>positive</u> coefficients.)

In particular $A_i(s_d) \neq 0$, $A_i(s_d) \neq \infty$

Induction step:

$$A_k' = \frac{M_1 + M_2}{A_k} \Bigg|_{A_r = 0}$$

- A_r cannot appear both in M_1 and M_2
- Sum, product, ratio of substraction free rational functions is substraction free rational function

$\square$

- <u>Remark</u> $A(S_d)$ — Laurent polynomial on $A_1, \dots, A_m$ with coeff in $\mathbb{Z}$

  substraction free rational function on $A_1, \dots, A_m$

- <u>Theorem</u> $A(S_d)$ — Laurent polynomials on $A_1, \dots, A_m$ with coeff in $\mathbb{Z}_{>0}$

<u>Example</u> $$x^2 - xy + y^2 = \frac{x^3+y^3}{x+y}$$ — substraction free rational function

Laurent polynomial, but coeff. are not positive

Totally positive matrices

M — $n\times n$ matrix $\qquad M^{j_1 \dots j_k}_{i_1 \dots i_k}$ — submatrix

— columns

— rows numbers

$$\Delta^J_I = \Delta^{j_1 \dots j_k}_{i_1 \dots i_k} = \det M^{j_1 \dots j_k}_{i_1 \dots i_k} \qquad \begin{matrix} j_1 < \dots < j_k \\ i_1 < \dots < i_k \end{matrix}$$

<u>Def</u> Matrix M is totally positive if $\Delta^J_I > 0$ $\forall\, I, J,\ |I| = |J|$

- <u>Th</u> (Perron–Frobenius) If M s.t $M^j_i > 0$

λ – eigenvalue with maximal $|\lambda|$. Then

ⓐ $\lambda \in \mathbb{R}_{>0}$

ⓑ λ has no degeneracy

ⓒ If $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ is eigen vector $Mv = \lambda v$ then all v_i have the same sign

• **Th** (Gantmaher–Krein) If M is totally positive

ⓐ $\lambda_1 > \lambda_2 > \ldots > \lambda_n > 0$ all eigenvalues of M

ⓑ If $v_j = (v_{1j}, \ldots, v_{nj}) \in \mathbb{R}^n$ is eigen vector $Mv_j = \lambda_j v_j$ then $v_{1j}, \ldots, v_{nj}$ has exactly $j-1$ changes of sign.

Problem Prove ⓐ above.

Example

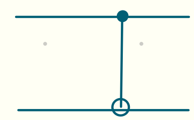
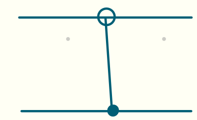
2×2

$$\begin{pmatrix} b & bc \\ ab & abc+d \end{pmatrix}$$

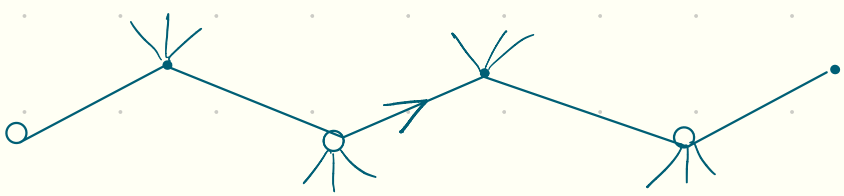
Goals

- Parametrize TP matrices
- Efficient test for TP (much less then $\binom{2n}{n}-1$ minors)

Networks (in form of bypartita graphs)

- n parallel lines
- Between the lines  or 
- Insert vertices to make graph bipartate

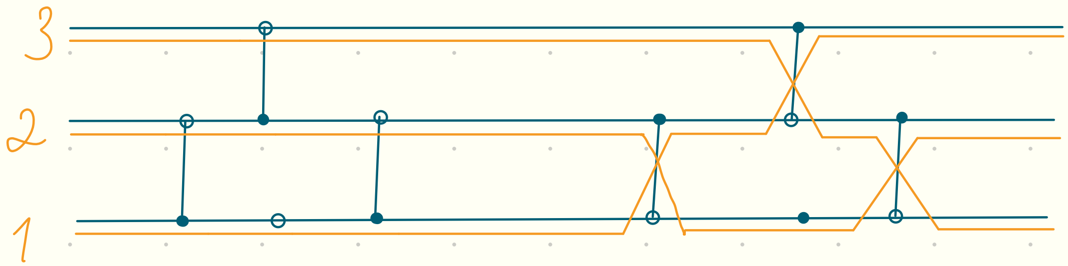
- zig-zag path:



In black vertex move right
In white vertex move left

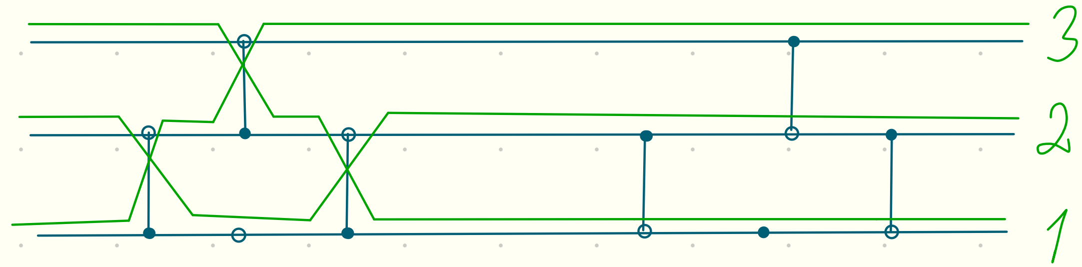
Any edge belongs to 2 zig-zags with opposite direction

Example



zig-zags

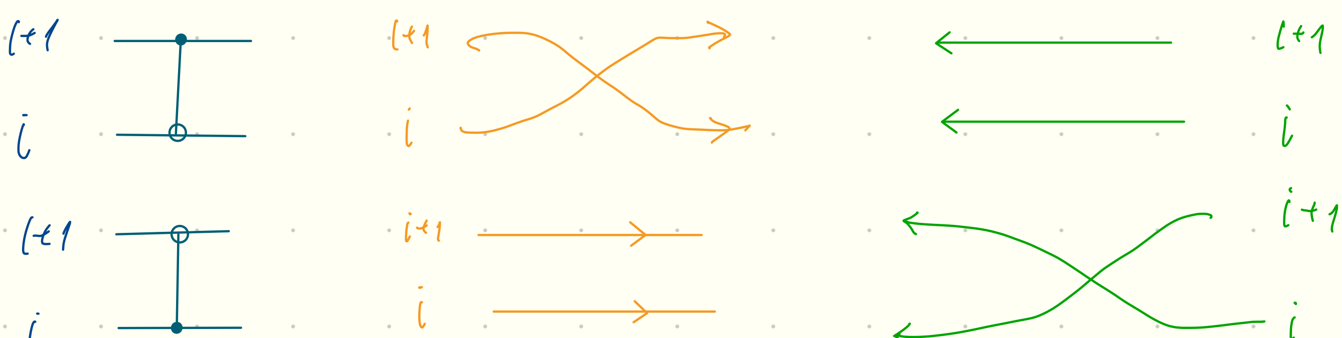
zig-tag



- For our network n zig-zags $\rightarrow$
  n zig-zags $\leftarrow$

We number zig-zags from bottom to top at their beginning

permutation of orange zig-zags



$\bar{S}_i$

S_i

permutation of green zigzags

Networks

- Assume any two zig-zags of the same color intersect exactly once

$\Downarrow$

zig-zags of any color give permutation of the form $w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$

- Equivalently $\bar{s}_i$ form reduced decomposition w_0
  and s_i form reduced decomposition w_0

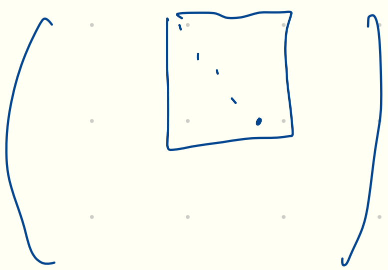
Here $s_i = (i, i+1) \in S_n$ $\qquad$ $w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix} \in S_n$

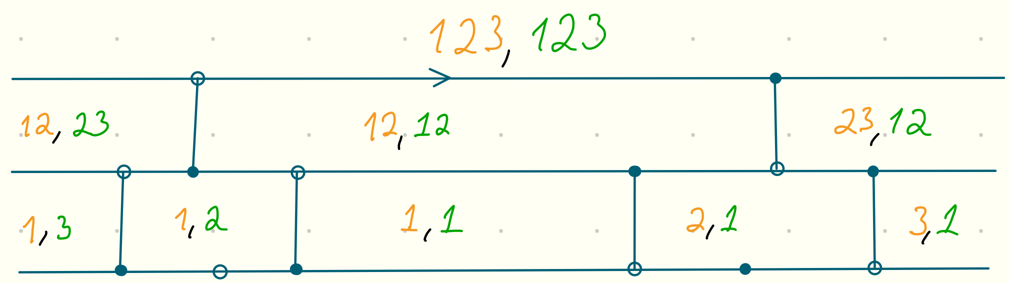
Face variables

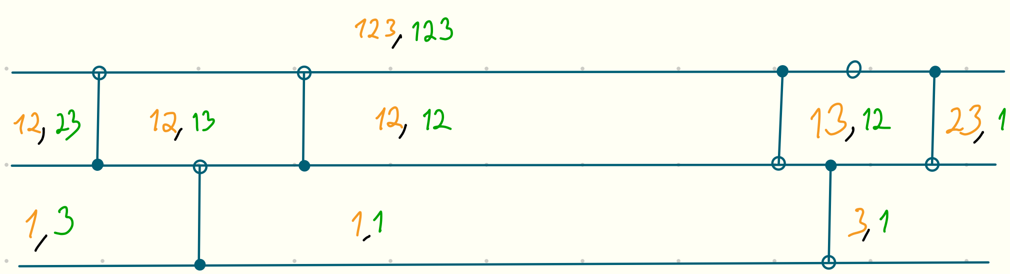
- **Def** To any face we assign $\triangle$ zig-zags below
  zig-zags below

Examples

initial minors







- # number of faces $= n^2 = n + \binom{n}{2} + \binom{n}{2}$

number of slanted edges

- The minors on the boundary do not depend on network $\Delta^{n-i+1,\dots,n}_{1,2,\dots,i}$ $\quad \Delta^{1,2,\dots,i}_{n-i+1,\dots,n}$

$\underline{123,\ n-2\ n-1\ n}$ $\qquad$ $\underline{n-2\ n-1\ n,\ 123}$

$\underline{12,\ n-1\ n}$ $\qquad$ $\underline{n-1\ n\ \ 12}$

$\underline{1,\ n}$ $\qquad$ $\underline{n,\ 1}$

nti-diagonal principle minors

These will be <u>frozen</u> variables

Quiver

- Vertices – faces of graph

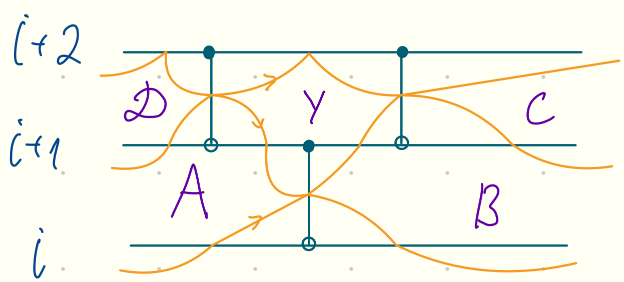
Edges – clockwise around black vertices
conter-clockwise around white vertices

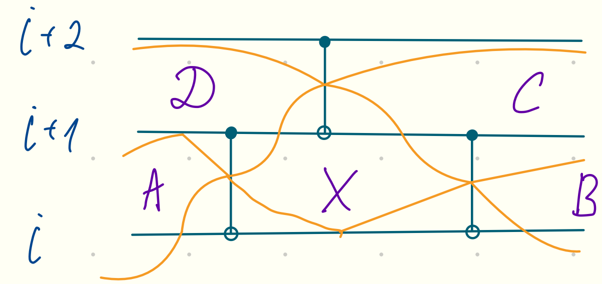
- To any network we assigned a seed (Q, Δ^J_I)

Theorem ⓐ Any two networks are connected by sequence of the following transf

ⓑ These transformations correspond to mutations of the seeds

1

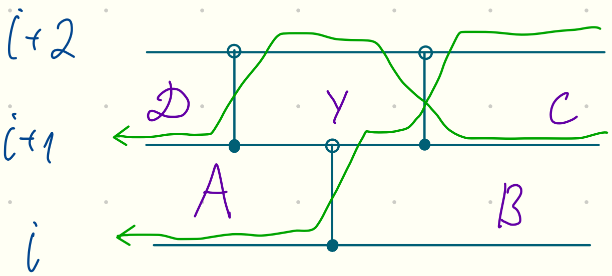


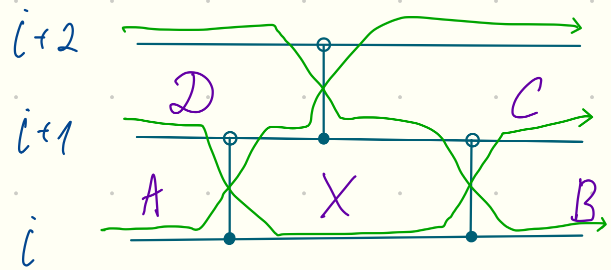


relation on face variables $XY = AC + BD$

In terms reflections $\bar{S}_{i+1}\bar{S}_i\bar{S}_{i+1} = \bar{S}_i\bar{S}_{i+1}\bar{S}_i$

2

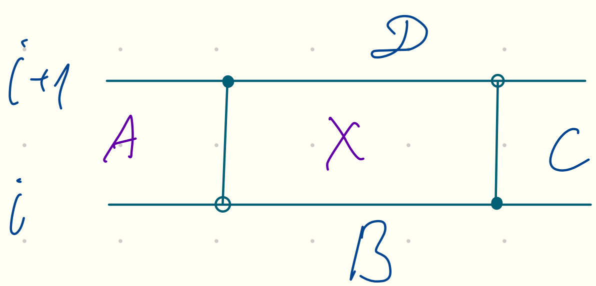


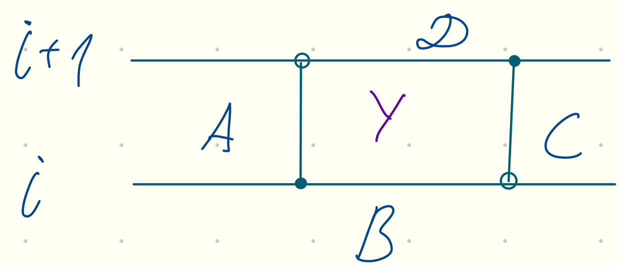


$XY = AC + BD$

$S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}$

3



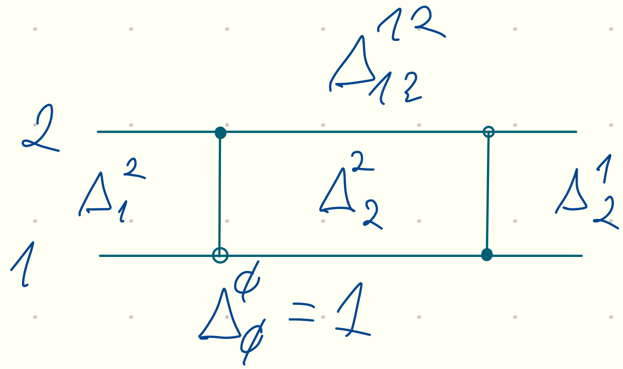


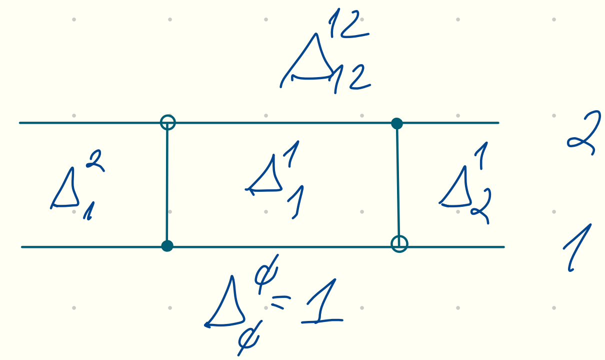
$XY = AC + BD$

$S_i \bar{S}_i = \bar{S}_i S_i$

Example

$n=2$





Relation on face variables: $\Delta^1_1 \Delta^2_2 = \Delta^{12}_{12} + \Delta^2_1 \Delta^1_2$

Problem Prove (B) part of Theorem.

Hint For relations on minors use Jacobi's relation on minors of inverse matrix

$$\det(A)^J_I = (-1)^{\sum i_\ell + \sum j_\ell - 2\binom{k+1}{2}} \det A \ \det(A^{-1})^{J^c}_{I^c},$$ where $I^c = \{1,\dots,n\}\setminus I$

Remark Relations (1) and (2) are particular cases of Plucker relation

Relation (3) is Desnanot-Jacobi identity.

Used in Dodgson condensation (Lewis Carroll identity)

- <u>Theorem</u> Any Δ^I_J appears in seed for some network

- <u>Problem*</u> Prove this theorem.

- <u>Problem</u> (a) For $n=3$ show that unfrozen part of quiver is equival to D_4 quiver

  (b) Find two cluster variables which are not minors.

- <u>Corollary</u> If for given network all $\Delta^I_J > 0$ Then all $\Delta^I_J > 0$

Test for TP consisting on n^2 minors

References

- Fomin Zelevinsky Total positivity tests and paremetrization