Itroduction to cluster algebras and varieties

Lecture 2

Seeds. Mutations. Laurent phenomenon

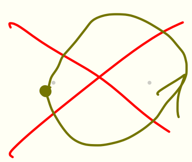
Definition

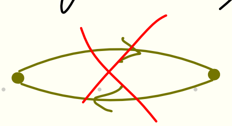
- Symmetric, geometric type
- Seed of rank n is a pair
    - Combinatorial date: antisymm matrix $n \times n$ $b_{ij} \in \mathbb{Z}$

      $\Updownarrow$

      Quiver Q, $b_{ij} = \#\text{edges } i \to j - \#\text{edges } j \to i$

      (without loops and 2-cycles)





    - Algebraic date: n variables $A_1, \dots, A_n$ correspond to vertices

- Mutation μ_k in vertex k

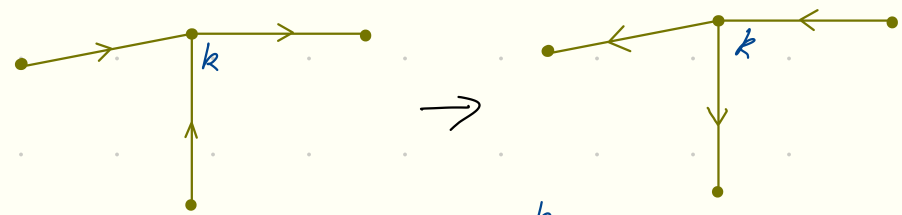
- Combinatorial data

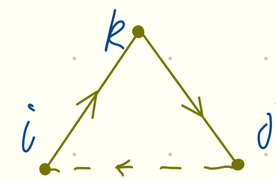
$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i=k \text{ or } j=k \\ b_{ij} + \dfrac{b_{ik}|b_{jk}| - |b_{ik}|b_{jk}}{2} \end{cases}$$

① Inverce arrows at k

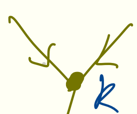
② Complete 3 cycles at k.

③ Delete 2-cycles

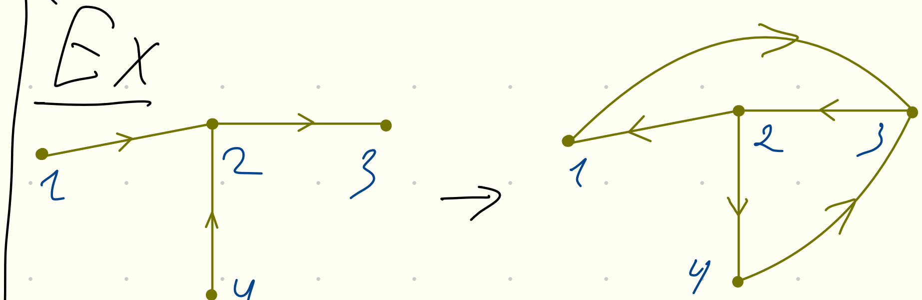




- Algebraic data $A_k' \cdot A_k = \prod_{i \to k} A_i^{b_{ik}} + \prod_{k \to i} A_i^{-b_{ik}}$ $\qquad A_j' = A_j \quad j \neq k$

- Rem If vertex k is sink  then μ_k reverses orientation at edges adjacent to k, $\quad A_k' = \frac{1}{A_k}(1 + \prod A_i)$

Ex



$$A_2' = \frac{A_1 A_4 + A_3}{A_2}$$

- **Lemma** $\mu_k^2 = id$

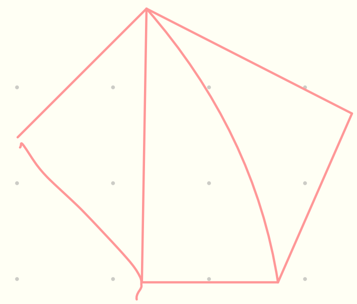
- **Example**

$$\overset{1}{\underset{A_1}{\bullet}} \longrightarrow \overset{2}{\underset{A_2}{\bullet}} \xrightarrow{\mu_1} \overset{1}{\underset{\frac{1+A_2}{A_1}}{\bullet}} \longleftarrow \overset{2}{\underset{A_2}{\bullet}} \xrightarrow{\mu_2} \overset{1}{\underset{\frac{1+A_2}{A_1}}{\bullet}} \longrightarrow \overset{2}{\underset{\frac{1+A_1+A_2}{A_1A_2}}{\bullet}}$$

$$\xrightarrow{\mu_1} \overset{1}{\underset{\frac{1+A_1}{A_2}}{\bullet}} \longleftarrow \overset{2}{\underset{\frac{1+A_1+A_2}{A_1A_2}}{\bullet}} \xrightarrow{\mu_2} \overset{1}{\underset{\frac{1+A_1}{A_2}}{\bullet}} \longrightarrow \overset{2}{\underset{A_1}{\bullet}} \xrightarrow{\mu_1} \overset{1}{\underset{A_2}{\bullet}} \longleftarrow \overset{2}{\underset{A_1}{\bullet}}$$

- **Example**

$$\underset{A_1}{\bullet} \rightrightarrows \underset{A_2}{\bullet} \qquad \frac{A_2^2+1}{A_1}$$



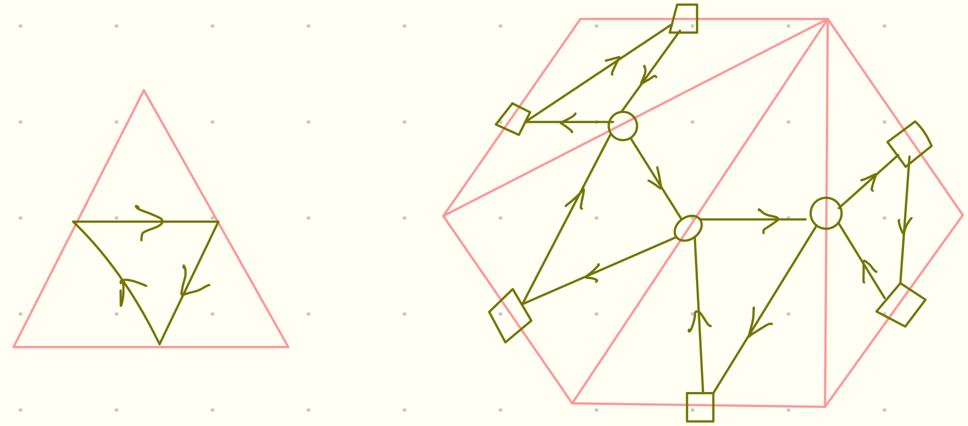
$$z_{k+2} = \frac{z_{k+1}^2+1}{z_k}$$

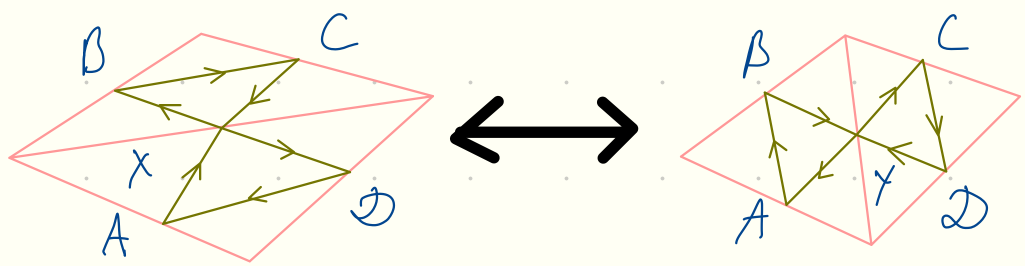
1, 1, 2, 5, 13, 34,

Fibonacci numbers

<u>no periodicity</u> in A-

- Triangulation of polygon





<u>Observation</u>: mutation of the quiver

a —— b $\rightsquigarrow$ $P_{ab} = \begin{vmatrix} x_a & x_b \\ y_a & y_b \end{vmatrix}$ $\quad A_n = \left\{ \begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix} \right\} / SL_2$

$XY = AC + BD$ Plücker relation

- two types of vertices

diagonals — can mutate — unfrozen variables

sides — cannot mutate — frozen variables

- Usually $\underbrace{A_1, \dots, A_n}_{\text{unfrozen}}, \underbrace{A_{n+1}, \dots, A_m}_{\text{frozen}}$

<u>Rk</u> Edges between frozen vertices don't matter

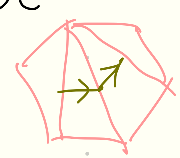
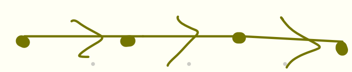
Hence
$$B = n\begin{array}{c} m \\ \left(\qquad\qquad\qquad \right) \end{array}$$

- For $\widehat{Gr(2,n)}$ we have n frozen, $n-3$ unfrozen

\#seeds – finite C_n – nth Catalan number

\#variables – finite $\frac{n(n-1)}{2}$

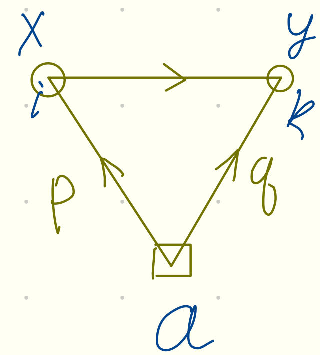
quiver of finite type

Quiver for $\widehat{Gr(2,n)}$   oriented Dynkin diagram A_{n-3}

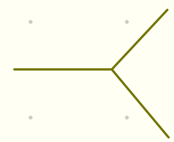
- <u>Problem</u> All orientations of tree are mutation equivalent to each other via mutations on sinks or sources

- **Problem** (a) If $b_{ik}=0$ then $\mu_i\mu_k=\mu_k\mu_i$

(b) If $b_{ik}=1$ then $(\mu_i\mu_k)^5=id$



Hint Sufficient to check for one frozen variable

- **Problem** For quiver 

(a) $\#\,\text{variables} = \#\,\Phi_+ + \#-\Pi$

positive roots — negative simple roots

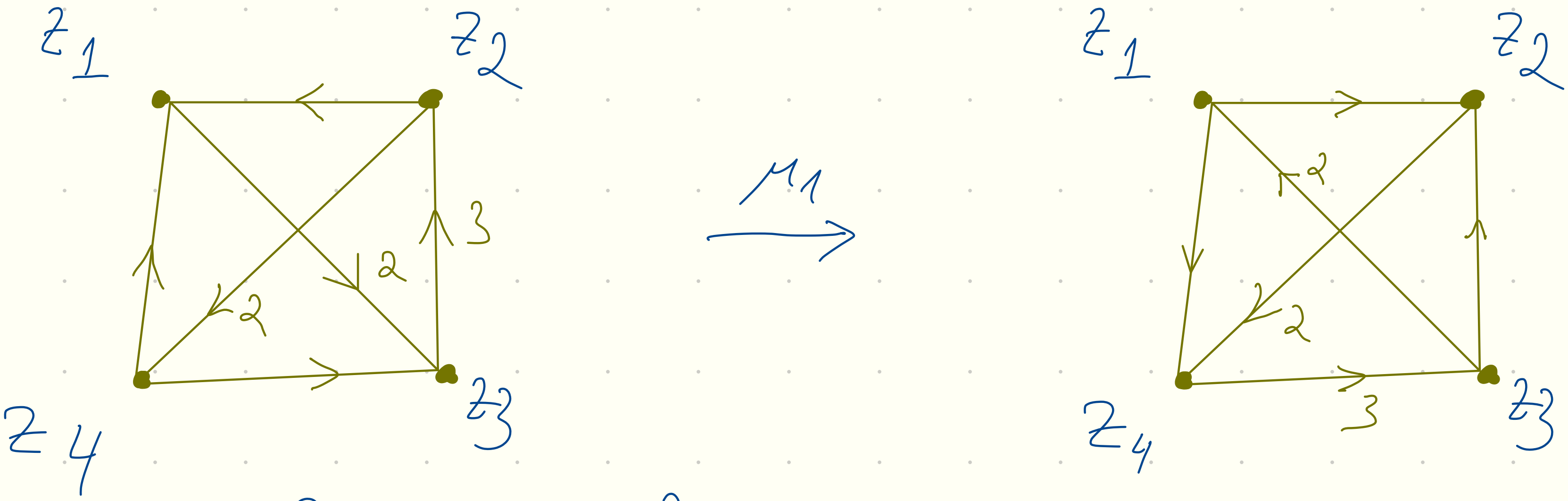
Check natural bijection

$\beta=\sum m_{\beta i}\alpha_i \longleftrightarrow A_\beta$ variable with denominator $\prod A_i^{m_{\beta,i}}$

simple roots

(b) Check this for another finite root system

- Somos sequence $z_{k+4} = \dfrac{z_{k+1} z_{k+3} + z_{k+2}^2}{z_k}$



$z_5 z_1 = z_2 z_4 + z_3^2$

$z_6 z_2 = z_5 z_3 + z_4^2$

- <u>Problem</u> Find quiver for Somos 5

- **Notation** $s \overset{k}{\text{———}} s'$ if seeds s, s' are connected by μ_k

$\Downarrow$

n-regular graph

(Fomin-Zelevinsky)

- **Theorem** $s_0 \text{ ——— } s_1 \text{ ——— } \cdots \text{ ——— } s_d$ Then $\bar{A}(s_d)$ — Laurent polynomials on $A(s_0)$ with integer coefficients.

Notation $\bar{A}(s_0) = (A_1, \dots, A_m)$

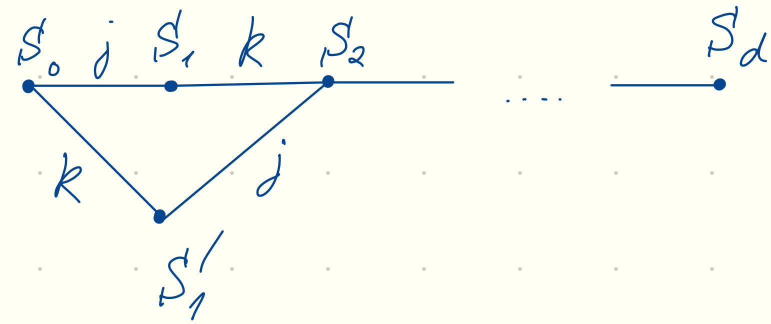
Pf Induction on d.

Base $d=1$ $\quad s_0 \overset{j}{\text{———}} s_1 \quad A_j' = \dfrac{M_1 + M_2}{A_j}$ trivial

$d=2$ $\quad s_0 \underset{A_j'}{\overset{j}{\text{———}}} s_1 \underset{A_k'}{\overset{k}{\text{———}}} s_2 \quad j=k \quad \bar{A}(s_2) = \bar{A}(s_0)$

$j \neq k \quad A_k' = \dfrac{1}{A_k}(M_3 + M_4)$

Step $<d \Rightarrow d$



Case 1 $\quad b^0_{jk} = 0$

$$A(t_1) = (A_1, \dots, A_j', \dots, A_k, \dots, A_m) \qquad A_j' = \frac{M_1+M_2}{A_j}$$
$$A(t_1') = (A_1, \dots, A_j, \dots, A_k', \dots, A_m) \qquad A_k' = \frac{M_3+M_4}{A_k}$$

By induction $A(S_d)$ Laurent polynom on $A(S_1)$ and on $A(S_1')$

If M_1+M_2 and M_3+M_4 are coprime in $\mathbb{C}[A_1, \dots, A_m]$ then we are done

Complication It could be not true e.g. $M_1+M_2 = M_3+M_4$
or $M_1+M_2 = M_3^3+M_4^3$

Solution Add new frozen variable

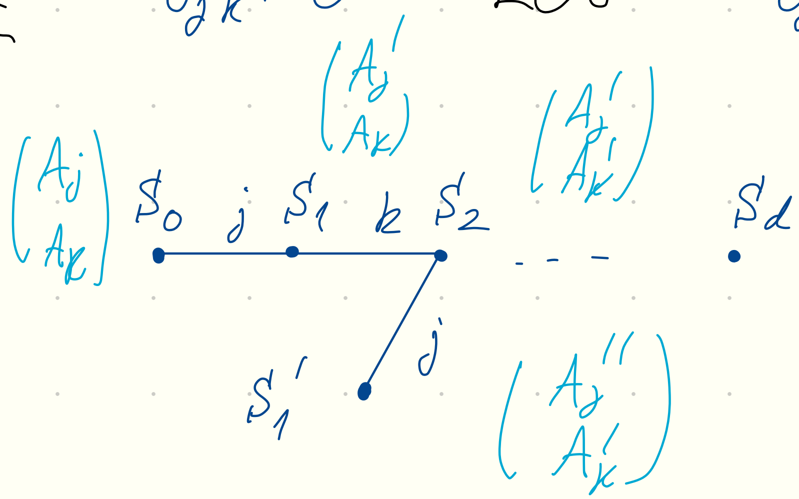
$$j \longleftarrow \boxed{m+1} \qquad k$$

Then $M_1 + M_2 \rightsquigarrow M_1 A_{m+1} + M_2$ now

$M_3 + M_4 \rightsquigarrow M_3 + M_4$ coprime!

Laurent for extended quiver $\Rightarrow$ $A_{m+1} = 1$,
Laurent for original quiver.

Case 2 $b^0_{jk} \neq 0$ Let $b^0_{jk} = -b < 0$



Lemma 1 A_j'' is Laurent polynomial in $A_1, \dots, A_n$

Pf computation $\square$

By induction $A(s_d)$ Laurent polynom on $A(s_1)$ and on $A(s_1')$

Add two frozen vars

$$j \longleftarrow \boxed{m+1} \qquad k \longrightarrow \boxed{m+2}$$

Lemma A_j' is coprime with A_j'' and A_k'

Pf Computation $\square$ $\square$

- **Problem** $z_{k+4} z_k = a z_{k+3} z_{k+1} + b z_{k+2}^2$, prove that z_k are integer if $a, b \in \mathbb{Z}$, $z_1 = z_2 = z_3 = z_4 = 1$

References

- Fomin Williams Zelevinsky Introduction to cluster algebras I