Virasoro algebra and conformal field theory

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Lectures in Skoltech, fall 2019. There are mistakes here, if you find some please write to mbersht@gmail.com

Contents

1 Bootstrap in 2d CFT

The main reference in this section is [BP09, Sec. 1], see also [DFMS97],[ZZ90].

1.1 Conformal transformations

Conformal field theory is quantum field theory which has invariance under conformal transformation. First, let us recall the definition of of the conformal group and conformal algebra

Definition 1.1. Let X be a Riemann manifold of dimension d with metric $g_{\mu\nu}$. The transformation $x \mapsto y$ is called *conformal* if the metric tensor is multiplied by a function $\Lambda(x)$, namely

$$g_{\rho\sigma}\frac{\partial y^\mu}{\partial x^\rho}\frac{\partial y^\nu}{\partial x^\sigma} = \Lambda(x)g_{\mu\nu} \tag{1.1}$$

More geometrically this means that conformal transformations preserves angles between curves.

First, let us find corresponding Lie algebra. Consider transformation $y^\mu = x^\mu + \epsilon^\mu$ with small ϵ^μ, then we get

$$g_{\nu\rho}\partial_\mu\epsilon^\rho + g_{\rho\mu}\partial_\nu\epsilon^\rho = K(x)g_{\mu\nu}$$

Here $\Lambda = 1 + \epsilon K$. Assume for simplicity that $g_{\mu\nu}$ is constant. Below we will mainly concentrate on the case $d = 2$, where this condition is not restrictive since any metric is conformally equivalent to constant one [DFN92, Ch. 13]. Then we can lower index and get

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = K(x)g_{\mu\nu}$$

Now exclude K from this system. Taking the trace we get

$$2\partial^\mu\epsilon_\mu = K(x)g_{\mu\nu}g^{\mu\nu} = dK(x)$$

Therefore, finally we get

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{d}(\partial^\sigma\epsilon_\sigma)g_{\mu\nu} \tag{1.2}$$

Problem 1.1. *Deduce the relations*

$$(2-d)\partial_\mu\partial_\nu(\partial^\sigma\epsilon_\sigma) = g_{\mu\nu}(\partial^\sigma\partial_\sigma)(\partial^\rho\epsilon_\rho)$$
$$(d-1)(\partial^\nu\partial_\nu)(\partial^\mu\epsilon_\mu) = 0$$

Problem 1.2. *Find a group of conformal transformation in $d \geq 3$. Find dimension of the group, find isomorphism with some classical group.*

Now take $d = 2$ $g_{\mu\nu} = \delta_{\mu\nu}$. Then the equations (1.2) takes form

$$\partial_0\epsilon_0 = \partial_1\epsilon_1, \quad \partial_1\epsilon_0 = -\partial_0\epsilon_1, \tag{1.3}$$

This are Cauchy–Riemann equations on real and imaginary part of the holomorphic function, if we set $z = x^0 + ix^1$, $\bar{z} = x^0 - ix^1$. Therefore any holomorphic vector field of the form $z^n\partial_z$ is locally conformal. For the corresponding conformal transformation $z \mapsto f(z)$

$$ds^2 = Dzd\bar{z} \mapsto \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}}dzd\bar{z}. \tag{1.4}$$

Denote $L_n = -z^{n+1}\partial_z$. For $n \geq -1$ this is infinitesimal transformation of the neighborhood of 0, for $n < -1$ this can be viewed as infinitesimal transformation of punctured neighborhood. Them satisfy relations

$$[L_n, L_m] = (n-m)L_{n+m}.$$

This Lie algebra is called *Witt algebra*.

In two dimensional conformal field theory, the space of fields form projective representation of this algebra. So actually there is an action of the algebra which has one additional generator C, such that $[C, L_n] = 0$ and

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n+m,0}C. \tag{1.5}$$

This algebra is called *Virasoro algebra*.

Problem 1.3. *Show that any Lie algebra with relations $[L_n, L_m] = (n-m)L_{n+m} + \omega(n,m)C$, $[C, L_n] = 0$ is isomorphic to Virasoro algebra.*

This problem explains that Virasoro algebra is unique central extension of Witt algebra. The relations (1.5) are chosen in such way that L_{-1}, L_0, L_1 form a closed algebra which is isomorphic to $\mathfrak{sl}_2$. This is the Lie algebra of the group $PSL_2(\mathbb{C})$ of global conformal transformations of $\mathbb{CP}^1$, they are fractional linear transformations $z \mapsto (az+b)/(cz+d)$.

For the Riemann surface of genus 1 (torus) the group of global conformal transformations is one dimensional, it consist of shifts $z \mapsto z + a$. For higher genus the group of global conformal transformations is discrete.

1.2 Fields

It turns out to be convenient to consider $z, \bar{z}$ as two independent complex variables. Therefore, the symmetry of the theory will be the product two Virasoro algebras. However, at some point we have to identify $\bar{z}$ with the complex conjugate z^* of z.

Fields only depending on z, i.e. $O(z)$, are called *chiral fields* and fields $O(\bar{z})$ only depending on z are called anti-chiral fields.

Definition 1.2. If a field $O(z, \bar{z})$ transforms under scaling $z \mapsto \lambda z$ according to

$$O(z, \bar{z}) \mapsto \lambda^h \bar{\lambda}^{\bar{h}} O(\lambda z, \bar{\lambda}\bar{z})) \tag{1.6}$$

it is said to have conformal dimensions $(h, \bar{h})$.

Definition 1.3. If a field Φ transforms under conformal transformations $z \mapsto f(z)$ according to

$$\Phi(z, \bar{z}) \mapsto \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})) \tag{1.7}$$

it is called a *primary field* of conformal dimension $(h, \bar{h})$. If this holds only for for global conformal transformations, then Φ is called a *quasi-primary field*.

Infinitesimally $(\partial f/\partial z)^h = (1 + h\partial_z \epsilon)$, and we get for primary field Φ

$$\delta_{\epsilon,\bar{\epsilon}} \Phi(z, \bar{z}) = (h\partial_z \epsilon + \epsilon \partial_z + \bar{h}\partial_{\bar{z}} \epsilon + \epsilon \partial_{\bar{z}})\Phi(z, \bar{z}) \tag{1.8}$$

1.3 Stress-energy-momentum tensor

In Lagrangian formalism correlation functions are defined as

$$\langle O_1(x_1, \bar{x}_1) \dots O_n(x_1, \bar{x}_n) \rangle = \int O_1(x_1, \bar{x}_1) \dots O_n(x_1, \bar{x}_n) \exp(-\mathrm{S}[\varphi]) \mathrm{D}\varphi. \tag{1.9}$$

Here ϕ denotes some fields in our theory, $\mathrm{D}\varphi$ is a measure in the path integral,

$$\mathrm{S}[\varphi] = \int \mathcal{L}(\varphi, \partial^\mu \varphi) d^2 x \tag{1.10}$$

is an action, $\mathcal{L}$ is Lagrangian density[1], $O_i(x_i, \bar{x}_i)$ are some local function of $\varphi(x_i)$ and its derivatives. Typical examples of the local field are $O(z) = \partial^\mu \varphi$ or $O(z) = e^{\alpha\varphi}$.

Now we are going to derive some properties of the space of fields, which follows from Lagrangian description. Many examples of conformal field theories do not have such description. In this case the corresponding properties are postulated.

[1] It is usually assumed that $\mathcal{L}$ do not depend on higher derivatives of φ (for example in mechanics this is a condition that everything is determined by coordinates and momentum or this can stated as principle of least action). But this assumption is not important for our arguments here.

Assume first that correlation functions (1.9). Assume that some coordinate transformation $x^\mu \mapsto x^\mu + \epsilon^\mu$, supplied with an action on fields $\varphi \mapsto \varphi + \delta^\mu_{\epsilon^\mu} O$ is a symmetry of the action

$$\mathrm{S}[\varphi] = \mathrm{S}[\varphi + \delta^\mu_{\epsilon^\mu}\varphi]. \tag{1.11}$$

For any (not necessary symmetry) ϵ we have

$$\mathrm{S}[\varphi + \delta^\mu_{\epsilon^\mu}\varphi] = \mathrm{S}[\varphi] - \int \epsilon_\mu A^\mu[\varphi] d^2x$$

Taking constant ϵ (i.e. assuming translation invariance) we have that $\int A^\mu[\varphi] d^2x = 0$. So $A^\mu[\varphi]$ must be a derivative $A^\mu[\varphi] = \partial_\nu T^{\nu\mu}$. Therefore

$$\delta_\epsilon \mathrm{S} = \int (\partial_\mu \epsilon_\nu) T^{\mu\nu} d^2x \tag{1.12}$$

Remark 1.1. Below we all also assume that $T^{\mu\nu} = T^{\nu\mu}$. This is not obvious (but see [DFMS97]). Using this symmetry one can write

$$\delta_\epsilon \mathrm{S} = \frac{1}{2} \int (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) T^{\mu\nu} d^2x \tag{1.13}$$

The term $(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)$ is a variation of constant metric. Actually, one of the ways to see (1.13) is to consider metric as dynamical variable and get define $T^{\mu\nu}$ as a tensor corresponding to variation of the metric $\delta_g \mathrm{S} = \int \partial_\mu \delta g_{\mu\nu} T^{\mu\nu} d^2x$.

On the equation of motion $\delta_\epsilon \mathrm{S} = 0$ for any ϵ, not necessary corresponding to symmetry of the action. Therefore integrating by parts we have $\partial_\mu T^{\mu\nu} = 0$.

Actually we do not restrict ourselves to the theories defined by (1.10). So we rather postulate then prove properties of stress-energy-momentum tensor. This properties could be viewed as a an application of Noether's theorem.

Recall first this theorem in the classical mechanics. Assume that we have a vector field v on the coordinate space X of the mechanical system. This vector field can be lifted to the vector field on the phase space T^*X, which we denote again by v. Assume that this vector field v represents symmetry of the system, i.e. $L_v H = 0$, where H is a Hamiltonian of the system. Then, there is a Hamiltonian H_v for the vector field v, and this function is an integral of motion $\partial_t H_v = \{H, H_v\} = 0$.

In the field theory, for continuous symmetry ϵ there is current j^ν such that $\partial_\nu j^\nu = 0$ (on the equations of motion). Compare to electric charge, the change of charge in the region is equal to the current flow through the boundary of the region.

In our case there exist (symmetric) tensor $T^{\mu\nu}$ such that for any conformal transformation $x^\mu \mapsto x^\mu + \epsilon^\mu$ we have

$$\partial_\nu (T^{\nu\mu} \epsilon_\mu) = 0 \tag{1.14}$$

Taking ϵ^μ to be constant we get $\partial_\nu T_{\nu\mu} = 0$. Taking to be any conformal transformation we get

$$0 = \partial_\nu (T^{\nu\mu} \epsilon_\mu) = \frac{1}{2} T^{\nu\mu} (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{1}{d} (\partial^\sigma \epsilon_\sigma) T^{\nu\mu} g_{\mu\nu} \tag{1.15}$$

Proposition 1.1. *In CFT we have* $T^\mu_\mu = 0$.

Now restrict ourselves to 2d CFT in complex coordinates. Then the proposition above means that $T_{z\bar{z}} = 0$. Moreover, we have the following property.

Corollary 1.2. *Denote* $T = T_{zz}$, $\bar{T} = T_{\bar{z}\bar{z}}$. *Then equation* $\partial_\nu T_{\nu\mu} = 0$ *means* $\partial_{\bar{z}} T = \partial_z \bar{T} = 0$.

Problem 1.4. *a) Find stress-energy-momentum tensor for the theory with action*

$$\mathrm{S}[\varphi] = \int \left(\partial_\mu \varphi \partial^\mu \varphi - V(\varphi)\right) d^2x \tag{1.16}$$

(free theory with self-action) and the transformations $\varphi \mapsto \varphi + \epsilon^\mu \partial_\mu \varphi(x)$.

b) In case $V = 0$ *find equations of motion. Find* T_{zz}, $T_{z\bar{z}}$, $T_{\bar{z},\bar{z}}$ *on equations of motion.*

1.4 Ward identities

Consider again $z, \bar{z}$ as two different complex numbers. Consider transformation $z \mapsto z + \epsilon(z, \bar{z})$, $\bar{z} \mapsto \bar{z}$, such that ϵ is holomorphic function with respect ot z. Then we have

$$\int O_1(x_1, \bar{x}_1) \dots O_n(x_n, \bar{x}_n) \exp(-\mathrm{S}[\varphi]) \mathrm{D}\varphi =$$
$$\int (1+\delta_\epsilon) O_1(x_1, \bar{x}_1) \dots (1+\delta_\epsilon) O_n(x_n, \bar{x}_n) \exp(-\mathrm{S}[\varphi]) \left(1 + \int d^2 z \bar{\partial}\epsilon T(z)\right) \mathrm{D}(\varphi + \delta_\epsilon \varphi)$$

This equality is just a change of variables. Assume mow that measure in path integral is invariant under the shift $\mathrm{D}(\varphi + \delta_\epsilon \varphi) = \mathrm{D}(\varphi)$. Taking the linear in ϵ term we get

$$\int d^2 z \bar{\partial}\epsilon \langle T(z) O_1(x_1, \bar{x}_1) \dots O_n(x_n, \bar{x}_n)\rangle = \sum_{i=1}^n \langle O_1(x_1, \bar{x}_1) \dots \delta_\epsilon O_i(x_i, \bar{x}_i) O_n(x_n, \bar{x}_n)\rangle$$

Assume now that ϵ vanishes outside the neighborhood $B_1, \dots, B_n$ of $x_1, \dots, x_n$. Therefore the integral on the left side is equal to the of integrals over B_i which due to Stokes theorem equals to contour integral. We get

$$\frac{1}{2\pi i} \sum_{i=1}^n \oint_{C_{x_i}} dz \langle T(z) O_1(x_1, \bar{x_1}) \dots O_n(x_n, \bar{x}_n)\rangle = \sum_{i=1}^n \langle O_1(x_1, \bar{x}_1) \dots \delta_\epsilon O_i(x_i \bar{x}_i) O_n(x_n \bar{x}_n)\rangle$$

Therefore for any point x_i

$$\frac{1}{2\pi i} \oint_{C_{x_i}} dz \langle T(z) O_1(x_1, \bar{x}_1) \dots O_n(x_n, \bar{x}_n)\rangle = \langle O_1(x_1, \bar{x}_1) \dots \delta_\epsilon O_i(x_i, \bar{x}_i) O_n(x_n, \bar{x}_n)\rangle$$

This condition is local, so we have (inside any correlation function)

$$\frac{1}{2\pi i} \oint_{C_x} \epsilon(z) T(z) O(x, \bar{x}) dz = \delta_\epsilon O(x, \bar{x}). \tag{1.17}$$

Assume for simplicity that $x=0$. If we set $T(z)=\sum L_n z^{-n-2}$ then we get

$$L_nO(0)=\frac{1}{2\pi i}\oint_{C_0} z^{n+1}\left(\sum L_n z^{-n-2}\right)O(0)dz=\frac{1}{2\pi i}\oint_{C_0} z^{n+1}T(z)O(0)dz=\delta_{\epsilon=z^{n+1}}O(0), \tag{1.18}$$

i.e. L_n corresponds to the infinitesimal deformation with $\epsilon=z^{n+1}$. Recall that corresponding vector fields satisfy Witt algebra. So it is natural to expect that L_n (Fourier modes of $T(z)$) satisfy Virasoro algebra.

Easy integration by parts shows that for any infinitesimal transformation ϵ defined at the vicinity of 0. Then we get from (1.17) that

$$\delta_\epsilon O(0)=\frac{1}{2\pi i}\oint_{C_0}\left(\sum_{k\geq 0}\frac{1}{k!}\partial_z^k\epsilon z^k\right)\sum_{n\in\mathbb{Z}}\frac{L_nO(0)}{z^{n+2}}dz=\sum_{k\geq 0}\frac{1}{k!}\partial_z^k\epsilon L_{k-1}O(0).$$

This means that for any field O we have $L_{-1}O=\partial_z O$. Now assume that field Φ is primary. Then by the infinitesimal formula (1.8) we have by

$$\delta_\epsilon\Phi(0)=(h\partial_z\epsilon+\epsilon\partial)\Phi(0) \tag{1.19}$$

Comparing the last two formulas we get the following proposition.

Proposition 1.3. *Primary fields satisfy highest weight condition*

$$L_k\Phi=0, k>0,\quad L_0\Phi=h\Phi. \tag{1.20}$$

2 Representations of the Virasoro algebra

2.1 Verma modules

Denote by Vir the Virasoro algebra. Let $\mathrm{Vir}_{\geq 0}$ be a subalgebra generated by L_k, $k\geq 0$ and C. Let $\mathrm{Vir}_{<0}$ be a subalgebra generated by L_k, $k<0$.

Definition 2.1. Verma module $\mathrm{V}_{h,c}$ is a module freely generated by $\mathrm{Vir}_{<0}$ from the highest weight vector $v_{h,c}$

$$L_kv_{h,c}=0, k>0,\quad L_0v_{h,c}=hv_{h,c},\quad Cv_{h,c}=cv_{h,c}. \tag{2.1}$$

More formally, one can consider vector space generated by $v_{h,c}$ as one-dimensional representation $\mathbb{C}_{h,c}$ of $\mathrm{Vir}_{\geq 0}$. Then by definition Verma module is an induced representation $\mathrm{V}_{h,c}=U(\mathrm{Vir})\otimes_{U(\mathrm{Vir}_{\geq 0})}\mathbb{C}_{h,c}$

Due to Poincare-Birkhoff-Witt theorem, the module $\mathrm{V}_{h,c}$ has a basis labeled by partitions λ

$$L_{-\lambda}v_{h,c}=L_{-\lambda_1}L_{-\lambda_2}\cdots L_{-\lambda_k}|h\rangle,\ \text{where } \lambda_1\geq\lambda_2\ldots\geq\lambda_k>0.$$

One can define the character of the Virasoro representation by $\operatorname{Tr} q^{L_0}$. It is easy to see that $L_0(L_{-\lambda}v_{h,c})=(h+|\lambda|)L_{-\lambda}v_{h,c}$, where $|\lambda|=\lambda_1+\ldots+\lambda_k$. Then the dimension of the eigenspace of L_0 with eigenvalue $h+N$ is equal to number of partition $p(N)$. Then we have

$$\chi(\mathrm{V}_{h,c}) = q^h \sum_{N\geq 0} p(N)q^N = \frac{q^h}{\prod_{k=1}^{\infty}(1-q^k)}. \tag{2.2}$$

Here q is a formal variable, but of course the analytical properties of the resulting functions could be interesting. Some properties (in particular modular transformations) become better if the character will be defined as $\operatorname{Tr} q^{L_0-c/24}$

Definition 2.2. The vector $w \in \mathrm{V}_{h,c}$ is called singular of the level $N \geq 0$ if it satisfies

$$L_k w = 0, k > 0, \quad L_0 w = (h+N)w. \tag{2.3}$$

Example 2.1. Let $h = 0$. Then $w = L_{-1}v_{h,c}$ is a singular vector. Indeed, obviously $L_k(L_{-1}v_{h,c}) = 0$ for $k \geq 2$ and the only nontrivial computations is

$$L_1(L_{-1}v_{h,c}) = 2hv_{h,c} = 0.$$

Proposition 2.1. *The Verma module $\mathrm{V}_{h,c}$ is reducible if and only if there is a singular vector $w \in \mathrm{V}_{h,c}$.*

Proof. Assume that $w \in \mathrm{V}_{h,c}$ is a singular vector. Then it generates a submodule over the Virasoro algebra, and due to singular vector conditions this submodule does not contain the highest weight vector $v_{h,c}$. So the Verma module $\mathrm{V}_{h,c}$ is reducible.

On the other hand, assume that there exists a proper invariant subspace $W \subset \mathrm{V}_{h,c}$. Since this subspace is invariant under the action of L_0 it should be graded Therefore it contains some vector w with the lowest value of L_0. This vector is annihilated by L_k, for $k > 0$. If w is proportional to $v_{h,c}$, then W coincides with $\mathrm{V}_{h,c}$, that contradicts our assumption. Therefore w is a singular vector. □

It will be convenient to use the following (Liouville) parametrization of the central charge and highest weight

$$c = c(b) = 1 + 6(b^{-1}+b)^2, \quad h = h(P,b) = (\frac{b^{-1}+b}{2})^2 - P^2. \tag{2.4}$$

Theorem 2.2 (Kac-Feigin-Fuchs)**.** *The Verma module $\mathrm{V}_{h,c}$ has a singular vector on the level N is and only if there exist $m, n \in \mathbb{Z}$, $mn > 0$ such that $N = mn$ and $h = h_{mn} = h(P_{mn}, b)$, where $P_{m,n} = (mb^{-1} + nb)/2$.*

Example 2.2. For $P = P_{1,1}$ we have $h = h_{11} = 0$. This agrees with Example 2.1 above.

Problem 2.1. *Find formulas for singular vectors on the level 2. The formulas should depend on b (and do not depend on P).*

By $\mathrm{L}_{h,c}$ we denote the irreducible quotient of $\mathrm{V}_{h,c}$. It exists and unique, it can be defined as a quotient of $\mathrm{V}_{h,c}$ by a sum of all proper submodules.

One of the main goals of representation theory of the Virasoro algebra is to describe $\mathrm{L}_{h,c}$ for all h, c, in particular find their characters. It follows from Theorem 2.2 and Proposition 2.1 that for $h \neq h_{n,m}$ the Verma module is irreducible, $\mathrm{V}_{h,c} \simeq \mathrm{L}_{h,c}$.

Assume that $h = h_{n,m}$. We have a morphism of Virasoro modules $\mathrm{V}_{h_{n,m}+nm,c} \to \mathrm{V}_{h_{n,m},c}$ which sends $v_{h_{n,m}+nm,c}$ to a singular vector $w \in \mathrm{V}_{h_{n,m},c}$. Note that $h_{n,m}+nm = h_{n,-m}$. Due to Theorem 2.2 the Verma module $\mathrm{V}_{h_{n,m}+nm,c}$ is reducible if and only if $P_{n,-m} = P_{r,s}$ for some $r, s \in \mathbb{Z}$, $rs > 0$. Therefore if b^2 is not positive rational number, we see that $\mathrm{V}_{h_{n,m}+nm,c}$ is irreducible. Moreover, under this assumption $\mathrm{V}_{h_{n,m},c}$ has no singular vectors other then w. Therefore we have

$$\chi(\mathrm{L}_{h_{m,n},c}) = \frac{q^{h_{m,n}}(1-q^{mn})}{\prod_{k=1}^{\infty}(1-q^k)}. \tag{2.5}$$

We will call b such $b^2 \notin \mathbb{Q}$ generic. Corresponding central charges are also called generic. The characters of representations for non generic c could have drastically different form.

Example 2.3. Let us take $b^2 = -2/3$. Then the central charge $c = 0$. In this case the Virasoro algebra has trivial representation. Therefore

$$\chi(\mathrm{L}_{0,0}) = 1 = \frac{1 - q - q^2 + q^5 + q^7 - \cdots}{\prod_{k=1}^{\infty}(1-q^k)}.$$

Problem 2.2. ***(Duality** $c \leftrightarrow 26-c$.**)*** *Assume that c is generic. Show there is an embedding of Verma modules $\mathrm{V}_{h_1,c} \to \mathrm{V}_{h_2,c}$ if and only there exist an embedding $\mathrm{V}_{1-h_2,26-c} \to \mathrm{V}_{1-h_1,26-c}$.*

2.2 Shapovalov form

Definition 2.3. *Shapovalov form* is a complex symmetric form on $\mathrm{V}_{h,c}$ such that $(v_{h,c}, v_{h,c}) = 1$ and $L_k^* = L_{-k}$, $C^* = C$.

Introduce notation for the Gramm matrix

$$G_{\lambda,\mu} = (L_{-\lambda}v_{h,c}, L_{-\mu}v_{h,c}).$$

Since the operator L_0 is self adjoint the eigenvectors with different eigenvalues are pairwise orthogonal. Therefore $G_{\lambda,\mu} = 0$ if $|\lambda| \neq |\mu|$. SO the Gramm matrix G has a block form, for the given level N we have a matrix G_N of the size $p(N) \times p(N)$.

Example 2.4. On the level 1 the matrix $G_1 = (G_{\{1\},\{1\}}) = 2h$,

Problem 2.3. *Find the matrix G_2. Compute its determinant in parametrization* (2.4)*. Decompose it into a product of linear functions on P.*

Remark 2.5. There is a similar but different notion for Shapovalov form (see e.g. [KR87, Sec. 3]). Namely, one can consider Hermitian form on $\mathrm{V}_{h,c}$ such that $\langle v_{h,c}, v_{h,c}\rangle = 1$ and $L_n^\dagger = L_{-n}$. Such form exists only if $h, c \in \mathbb{R}$. One can also ask weather this Hermitian form is positive definite, this impose additional constraints on h, c for example from the level 1 calculation one can see that $h > 0$.

Proposition 2.3. *The Verma module* $\mathrm{V}_{h,c}$ *is irreducible if and only the Shapovalov form is non degenerate*

Proof. If the form G is degenerate then its kernel is a submodule of $\mathrm{V}_{h,c}$. Indeed, if $w \in \operatorname{Ker} G$, then for any $k \in \mathbb{Z}$ and $u \in \mathrm{V}_{h,c}$ we have $(L_k w, u) = (w, L_{-k} u) = 0$, so we proved that $L_k w \in \operatorname{Ker} G$.

On the other hand, if $\mathrm{V}_{h,c}$ is reducible then by Proposition 2.1 it contains singular vector w. Then $w \in \operatorname{Ker} G$ since $(L_{-\lambda} v_{h,c}, w) = (v_{h,c}, L_{\lambda_k} \dots L_{\lambda_1} w) = 0$. □

Theorem 2.4 (Kac-Feigin-Fuchs)**.** *Determinant of the Shapovalov form on the level* N *has the form* $\det G_N \sim \prod_{n,m\geq 1}(h-h_{n,m})^{P(N-nm)}$, *where* $\sim$ *means some constant coefficient.*

Note due to this theorem $\det G_n$ vanishes for $h = h_{n,m}$ firstly for $N = nm$. Therefore there is a singular vector in $\mathrm{V}_{h_{n,m},c}$ on the level mn. Hence the Theorem 2.2 follows form the Theorem 2.4. The meaning of power $P(N-nm)$ stands for the dimension of the space descendants of the singular vector, all this descendants belong to the kernel of G.

We will prove the Theorem 2.4 later, but up to now we just show that it gives correct degree of $\det G_N$ as a polynomial in h

Proposition 2.5. *The determinant of the Shapovalov form* $\det G_N$ *has degree in* h *non greater then* $\sum_{|\lambda|=N|} l(\lambda)$.

Here by $l(\lambda)$ we denote a number of parts in the partition λ.

Proof. It is easy to see that degree of each matrix element $G_{\lambda,\mu}$ is not greater that $l(\lambda)$ and $l(\mu)$. □

The following combinatorial statement shows that this bound of degree coincides with the degree in Theorem 2.4.

Proposition 2.6. *For any* $N \geq 1$ *we have* $\sum_{|\lambda|=N|} l(\lambda) = \sum_{m,n\geq 1} p(N-mn)$.

Proof. The proof can be organized as a sequence of identities for generating functions. By $l^{(n)}(\lambda)$ we denote number of parts equal to n in a partition λ. We have

$$\sum_{N\geq 0} q^N \sum_{m,n\geq 1} p(N-mn) = \left(\sum_{m,n\geq 1} q^{mn}\right)\left(\sum_{K\geq 0} p(K) q^K\right) = \sum_{n\geq 1} \frac{q^n}{1-q^n} \prod_{k\geq 1} \frac{1}{1-q^k}$$
$$= \sum_{n\geq 1} \frac{q^n}{(1-q^n)^2} \prod_{k\geq 1, k\neq n} \frac{1}{1-q^k} = \sum_{n\geq 1}\sum_{\lambda} l^{(n)}(\lambda) q^{|\lambda|} = \sum_{\lambda} l(\lambda) q^{|\lambda|}.$$

□

Problem 2.4. *Bound the degree* $\det G_n$ *as a polynomial in* c. *Compare this bound with the Theorem 2.4.*

3 Conformal blocks

3.1 Correlation functions of Primary fields

Using conformal symmetry we can fix two and three point functions to a large extent. Consider two point $\langle\Phi_1(z_1), \Phi_2(z_2)\rangle$ of two quasiprimary chiral fields. It depends on $z_1 - z_2$. Due to rescaling

$$\langle\Phi_1(z_1), \Phi_2(z_2)\rangle \mapsto \langle\lambda^{h_1+h_2}\Phi_1(\lambda z_1), \Phi_2(\lambda z_2)\rangle$$

Therefore $\langle\Phi_1(z_1), \Phi_2(z_2)\rangle = d(z_1 - z_2)^{h_1+h_2}$ for some constant d. Now using inversion

$$\langle\Phi_1(z_1), \Phi_2(z_2)\rangle \mapsto \langle\frac{1}{z_1^{2h_1}}\frac{1}{z_2^{2h_2}}\Phi_1(-1/z_1), \Phi_2(-1/z_2)\rangle$$

So we get $h_1 = h_2$. Therefore the two point function is diagonal on quasi primary fields (of course this is not literally correct if there are more then one quasi primary field of given dimension)

$$\langle\Phi_1(z_1), \Phi_2(z_2)\rangle = \delta_{h_1,h_2}\frac{C_{12}}{(z_1 - z_2)^{2h_1}} \tag{3.1}$$

Similarly

$$\langle\Phi_1(z_1), \Phi_2(z_2), \Phi_3(z_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}z_{23}^{h_2+h_3-h_1}z_{13}^{h_1+h_3-h_2}} \tag{3.2}$$

where $z_{ij} = z_i - z_j$.

For fields which are quasiprimary with respect to anti chiral Virasoro algebra we will get similar dependence on $\bar{z}$. If the fields are quasiprimary with respect to both chiral and anit chiral Virasoro algebras then we have product. For example if for two fields we have $h_1 = \bar{h}_1 = h_2 = \bar{h}_2 = h$ then

$$\langle\Phi_1(z_1, \bar{z}_1), \Phi_2(z_2, \bar{z}_2)\rangle = C|z_1 - z_2|^{-4h}$$

In particular this function has no monodromy on the plane. If $h > 0$ it also decreases at infinity and is integrable over all complex plane.

Four point functions are not determined completely by the conformal invariance. But they can be reduced to the function of the cross-ratio. For example four point function of chiral quasiprimary fields has the form

$$\langle\Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\Phi_4(z_4)\rangle = \prod(z_i - z_j)^{\gamma_{ij}}G(z), \tag{3.3}$$

where $\gamma_{12} = \gamma_{13} = 0$, $\gamma_{14} = -2h_1$, $\gamma_{24} = h_1 + h_3 - h_2 - h_4$, $\gamma_{34} = h_1 + h_2 - h_3 - h_4$, $\gamma_{23} = h_4 - h_1 - h_2 - h_3$, $G(z)$ is some function of cross-ratio

$$z = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}. \tag{3.4}$$

The choice of the exponents γ_{ij} is not canonical, since we can put factors like z^γ or $(1 - z)^\gamma$ either inside the function $G(z)$ or in the prefactor. Here we assumed that prefactor does not have singularities at $z_1 \to z_2$ and $z_1 \to z_3$.

Remark 3.1. It is also useful to write confromal invariance of the correlation function of quasiprimary fields in form of invariance under the infinitesimal conformal transformations:

$$z \mapsto z+\epsilon : \qquad \sum_{k=1}^{n} \partial_k \langle \Phi_{h_1}(z_1)\dots\Phi_{h_n}(z_n)\rangle = 0l \tag{3.5}$$

$$z \mapsto z(1+\epsilon) : \qquad \sum_{k=1}^{n} (z_k\partial_k + h_k)\langle \Phi_{h_1}(z_1)\dots\Phi_{h_n}(z_n)\rangle = 0; \tag{3.6}$$

$$z \mapsto \frac{z}{\epsilon z+1} = z - \epsilon z^2 + \dots : \qquad \sum_{k=1}^{n} (z_k^2\partial_k + 2h_k z_k)\langle \Phi_{h_1}(z_1)\dots\Phi_{h_n}(z_n)\rangle = 0. \tag{3.7}$$

3.2 Vertex operators

Now we will work in operator formalism. We consider the correlation function as a matrix element, where fields an 0 and ∞ are in and out Verma modules over chiral and antichiral Virasoro algebra.

And other Primary fields will be considered as a operators between Virasoro representations.

The equation (1.20) can be rewritten in term of operator product expansion. One can consider this as a limit $z \to 0$ and the product $T(z)\Phi(0)$ can have singularities only around 0 (locality condition). And the equation (1.20) means that

$$T(z)\Phi(0) = \frac{h\Phi(0)}{z^2} + \frac{\partial\Phi(0)}{z} + \text{regular terms}. \tag{3.8}$$

This OPE leads to commutation relations

Proposition 3.1. *Commutation relations between L_k and $\Phi_h(z)$ has the form*

$$[L_k, \Phi_h(z)] = \left(z^{k+1}\frac{\partial}{\partial z} + (k+1)z^k h\right)\Phi_h(z). \tag{3.9}$$

Proof. We apply the commutator to any field A located at the origin

$$\begin{aligned}[L_k, \Phi_h(z)]A(0) &= \oint_{C_0} T(w)w^{k+1}\Phi_h(z)A(0)dw - \Phi_h(z)\oint_{C_0} T(w)w^{k+1}A(0)dw \\ &= \oint_{C_z} T(w)w^{k+1}\Phi_h(z)A(0)dw = \oint_{C_z}\left(\frac{h\Phi(z)}{(w-z)^2} + \frac{\partial\Phi(z)}{(w-z)} + \text{reg.}\right)w^{k+1}A(0)dw \\ &= \left(h(k+1)z^k\Phi(z) + z^{k+1}\partial\Phi(z)\right)A(0)\end{aligned}$$

Since the field $A(0)$ is arbitrary we get the desired commutation relation. □

For example, three point correlations function (to be more precise three point conformal block, see below) is just a matrix element $(v_{h_1,c}, \Phi_h(z)v_{h_2,c})$. From the commutation relations one can see that

$$(v_{h_1,c}, \Phi_h(z)v_{h_2,c}) \sim z^{h_1-h-h_2}.$$

This agrees with the formula (3.2).

Proposition 3.2. *Let Φ_h be a vertex operator from $\mathrm{V}_{h_2,c}$ to $\mathrm{V}_{h_1,c}$.*

a) Decompose $\Phi_h(z) = \sum \Phi_h[n] z^{h_1-h-h_2-n}$ We have commutation relation on modes

$$[L_k, \Phi_h[n-k]] = (kh - n + (h_1 - h_2))\Phi_h[n] \tag{3.10}$$

Denote the image of highest vector $v_{h_2,c}$ by $W^{h_1}_{h_2,h}(z) = z^{h_1-h-h_2}\sum W_N z^N$. Then the vectors W_N satisfies

$$L_0 W_N = (h_1 + N)W_N, \quad L_k W_{N+k} = (h_1 + kh - h_2 + N)W_N, \quad k > 0. \tag{3.11}$$

b) Let h_1 be generic. Then the vertex operator Φ_h exists and unique up to (independent on z) normalization.

The vector $W^{h_1}_{h_2,h}(z)$ could be called *Whittaker vector* or *chain vector*.

Proof. The statement a) follows directly from the (3.9).

Recall that condition h_1 is generic if and only if the Shapovalov form on $\mathrm{V}_{h_1,c}$ is non-degenerate. First we prove the uniqueness and existence of the Whittaker vector $W^{h_1}_{h_2,h}(z)$. Indeed, using (3.11) one can reduce the scalar of the form $(L_{-\lambda}v_{h_1,c}, W_N)$ to the scalar product $(v_{h_1,c}, W_0)$, i.e. to normalization of Φ_h. Since the form is non-degenerate this determines W_N uniquely. Denote the vectors defined by this construction by $\widetilde{W}_N$

Problem 3.1 (*). *Prove that obtained vectors $\widetilde{W}_N$ satisfy* (3.11).

Note that in order to determine $\Phi_h v_{h_1,c}$ we used relation (3.10) for $k \geq 0$. Now using this relation for $k < 0$ one determines uniquely the vectors $\Phi_h L_{-\mu} v_{h_1,c}$ $\square$

Remark 3.2. This proof can be rephrased as follows. The Verma module $\mathrm{V}_{h_1,c}$ is free as $\mathrm{Vir}_{<0}$ module, therefore the construction of Φ_h is equivalent to the construction of a map from one dimensional vector space $\langle v_{h_1,c}\rangle$ to $\mathrm{V}_{h_1,c}$, which satisfies (3.9) for $\mathrm{Vir}_{\geq 0}$. For generic h_2 the Verma module $\mathrm{V}_{h_2,c}$ is cofree as $\mathrm{Vir}_{>0}$ module therefore everything reduces to construction of the map from $\langle v_{h_1,c}\rangle$ to $\langle v_{h_2,c}\rangle$ which agrees with (3.9) for $k = 0$. This is given by $z^{h_1-h-h_2}$.

Note that commutation relations (3.9) do not fix normalization of Φ_h. One of standard choices is

$$(v_{h_1,c}, \Phi_h(z)v_{h_2,c}) = z^{h_1-h-h_2}. \tag{3.12}$$

Sometimes it is more convenient to use another normalization, in which this three point function depends on h_1, h_2, h, c.

Problem 3.2. *a) Define representation of the Virasoro algebra on the vector space $\mathcal{F}^{\lambda,\mu}$ with the basis $[n] = z^{\lambda+n}(\partial_z)^\mu$, $n \in \mathbb{Z}$. Equivalently (and probably more standard) one can define basis by the formula $[n] = z^{\lambda+n}(dz)^{-\mu}$.*

b) Show that commutation relations on modes Φ_h are equivalent to the relation on components of intertwining operator $\mathbb{V}_{h_1,c} \otimes \mathcal{F}^{\lambda,\mu} \to \mathbb{V}_{h_2,c}$. Find corresponding λ, μ.

3.3 Conformal blocks

Consider four point correlation function $G(z,\bar{z}) = \langle \Phi_4(\infty)\Phi_3(1)\Phi_2(z,\bar{z})\Phi_1(0)\rangle$. In the operator formalism the target module of Φ_2 (and correspondingly source module for Φ_3) is not specified. This $\mathrm{Vir}\oplus\overline{\mathrm{Vir}}$ module is called *intermediate module* and can be arbitrary from our space of states. Therefore we have

$$G(z,\bar{z}) = \sum_{h,\bar{h}} C_{12}^{h} C_{34}^{h} \left(v_{h_4,c}, \Phi_{h_3}(1)\,{}_{h}\,\Phi_{h_2}(z) v_{h_1,c}\right) \left(v_{\bar{h}_4,\bar{c}}, \Phi_{\bar{h}_3}(1)\,{}_{\bar{h}}\,\Phi_{\bar{h}_2}(\bar{z}) v_{\bar{h}_1,\bar{c}}\right) \tag{3.13}$$

The region of summation depends on the theory. If the space of fields in the theory contains finite number of $\mathrm{Vir}\oplus\overline{\mathrm{Vir}}$ modules then the theory is called rational and the sum in (3.13) is finite. For not rational theories the sum could be viewed as an integral over $h,\bar{h}$. The numbers C_{12}^h, C_{34}^h are responsible for the normalization of Φ_2, Φ_3 and called *structure constants.*

The function $\mathcal{F}(\vec{h},h,c|z) = \left(v_{h_4,c}, \Phi_{h_3}(1)\,{}_h\,\Phi_{h_2}(z) v_{h_1,c}\right)$ is determined by the representation theory of the (chiral) Virasoro algebra. This function is called the *conformal block* of the correlation function. It can be written explicitly inserting all possible descendants of $v_{h,c}$ in intermediate module. Namely

$$\mathcal{F}(\vec{h},h,c|z) = z^{h-h_1-h_2} \sum_{\lambda,\mu,|\lambda|=|\mu|} z^{|\lambda|} \langle h_1|\Phi_{h_2}(1)L_{-\lambda}|h\rangle\langle h|L_{\mu}\Phi_{h_4}(1)|h_3\rangle G^{\lambda,\mu}. \tag{3.14}$$

Here $G^{\lambda,\mu}$ is the inverse of the Shapovalov matrix $G_{\lambda,\mu}$. It is easy to see from (3.10) that the operator $\Phi_h(z^{-1})\colon \mathrm{V}_{h_2,c} \to V_{h_1,c}$ is adjoint to the operator $\Phi_h(z)\colon \mathrm{V}_{h_1,c} \to V_{h_2,c}$. Therefore one can write four point conformal block as scalar product of Whittaker vectors

$$\mathcal{F}(\vec{h},h,c|z) = (W^h_{h_3,h_4}(1), W^h_{h_1,h_2}(z)) \tag{3.15}$$

Formulas (3.14) and (3.15) define conformal block as a power series in z. The first nontrivial term can be easily computed

$$\mathcal{F}(\vec{h},h,c|z) = z^{h-h_1-h_2}\Big(1 + \frac{(h-h_1+h_2)(h-h_4+h_3)}{2h} z + \dots\Big),$$

the further computations although straightforward, they quickly become extremely complex.

3.4 Degenerate field $\Phi_{1,1}$

Quite often the space of fields consist is not a sum of Verma modules but rather the sum of irreducible ones. The physical reason is an exclusion of the states with zero norm. Then the Proposition 3.2 is no longer applicable, and one gets auxiliary constraints.

Recall that Verma modules become reducible for special highest weights, given in the Theorem 2.2. Consider first the case where one fields has highest weight $h_{1,1} = 0$. In the irreducible module the singular vector $L_{-1}v_{h_{1,1},c}$ is excluded.

Consider three point function $\langle \Phi_{h_{1,1}}\Phi_{h_1}\Phi_{h_2}\rangle$. In the operator formalism we have three options where to put degenerate representation.

For example one can put $\Phi_{h_{1,1}}$ to the zero and hence ask for the existence of the vertex operator $\Phi_{h_1}(z)\colon \mathrm{L}_{h_{1,1},c} \to \mathrm{V}_{h_2,c}$. Since we know the existence and uniqueness of the vertex operator $\Phi_{h_1}(z)\colon \mathrm{V}_{h_{1,1}} \to \mathrm{V}_{h_2}$ we need to show that

$$(L_{-\mu}v_{h_2,c}, \Phi_{h_1}[N]L_{-\lambda}L_{-1}v_{h_{1,1},c}) = 0, \text{ for any partitions } \lambda, \mu \text{ and } N \in \mathbb{Z}.$$

Since vectors $v_{h_2,c}$ and $L_{-1}v_{h_{1,1},c}$ are vanish under the action of L_k, $k > 0$ this reduces to the condition $(v_{h_2,c}, \Phi_{h_1}[N]L_{-1}v_{h_{1,1},c}) = 0$, or equivalently

$$0 = (v_{h_2,c}, \Phi_{h_1}(z)L_{-1}v_{h_{1,1},c}) = \partial_z(v_{h_2,c}, \Phi_{h_1}(z)v_{h_{1,1},c}) = (h_2 - h_1 - h_{1,1})z^{h_2-h_1-h_{1,1}-1}.$$

Since $h_{1,1} = 0$ we get that $h_1 = h_2$. In other words the fusion with $\Phi_{h_{1,1}}$ does not change primary field. The field $\Phi_{h_{1,1}}$ can be viewed as an identity operator. Symbolically this can be written as

$$[\Phi_{1,1}][\Phi_h] = [\Phi_h].$$

It is instructive to obtain the same result in another manner. Namely put $\Phi_{h_{1,1}}$ to the infinity and hence ask for the existence of the vertex operator $\Phi_{h_1}(z)\colon \mathrm{V}_{h_2,c} \to \mathrm{L}_{h_{1,1},c}$. Similarly to the proof of Proposition 3.2 it is necessary and sufficient to construct Whittaker vector $W(z) \in \mathrm{L}_{h_{1,1},c}$. Arguing as in the proof of Proposition 3.2 this boils down to the relation that $W(z)$ is orthogonal to the kernel of the Shapovalov form, i.e. we need $(L_{-\mu}L_{-1}v_{h_{1,1},c}, \Phi_{h_1}[N]v_{h_{1,1},c})$. Since Φ_h is adjoint to Φ_h we get the same condition as before: $h_1 = h_2$.

There is a third option to get the same result. Namely one can put $\Phi_{h_{1,1}}$ to the point z an two other fields at zero and infinity. As was shown in Sec. 1.4 the field corresponding to $L_{-1}\Phi_{h_{1,1}}(z)$ is $\partial_z\Phi_{h_{1,1}}(z)$. Therefore we get a condition $(L_{-\mu}v_{h_2,c}, \partial_z\Phi_{h_{1,1}}[N]L_{-\lambda}v_{h_1,c}) = 0$. This boils down to

$$0 = (v_{h_2,c}\partial_z\Phi_{1,1}(z)v_{h_1,c}) = \partial_z(z^{h_2-h_1-h_{1,1}}).$$

Hence we get the same condition as before: $h_1 = h_2$.

3.5 Fusion with $\Phi_{m,n}$

From now on we simplify notations, instead of $\Phi_{h_{m,n}}$ we will write $\Phi_{m,n}$, instead of $v_{h_{m,n},c}$ we will write $v_{m,n}$.

Consider the second nontrivial example $(m, n) = (1, 2)$. The corresponding singular vector equals to $(L_{-1}^2 + b^2L_{-2})v_{1,2}$. One can study this case similar to previous one (see problems) but we prefer to show another way of reasoning [ZZ90].

Consider (chiral) four point function $\langle((b^{-2}L_{-1}^2+L_{-2})\Phi_{1,2})(z)\Phi_{h_1}(z_1)\Phi_{h_2}(z_2)\Phi_{h_3}(z_3)\rangle$. The action of L_{-1} on $\Phi_{1,2}$ is a derivation and the action of L_{-2} is given by $\oint_{C_z} T(w)\Phi(z)(z-w)^{-1}dw$. One can deform this contour to the neighborhood of points z_1, z_2, z_3 and using OPE (3.8) get partial differential equation

$$\left(b^{-2}\frac{\partial^2}{\partial z^2} + \sum_{i=1}^{3}\left(\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right)\right)\langle\Phi_{1,2}(z)\Phi_{h_1}(z_1)\Phi_{h_2}(z_2)\Phi_{h_3}(z_3)\rangle = 0. \quad (3.16)$$

If the fields $\Phi_{1,2}$ and Φ_1 are fused to some Φ_h, then due to (3.12) the equation (3.16) has solution with asymptotic behavior $(z-z_1)^X$, where $X = h - h_1 - h_{1,2}$. Substituting this into equation (3.16) we get

$$b^{-2}X(X-1) + h_1 - X = 0. \tag{3.17}$$

Solving this equation we get

$$h = h(P,b), \quad P = P_1 \pm \frac{1}{2}b, \text{ where } h_1 = h(P_1, b).$$

Symbolically this can be written as fusion rule

$$\Phi_{1,2}\Phi_{h(P,b)} = [\Phi_{h(P-b/2,b)}] + [\Phi_{h(P+b/2,b)}]. \tag{3.18}$$

Problem 3.3. *a) Impose vanishing of the matrix element $(v_{h_2}, \Phi_{h_1}(z)(L_{-1}^2 + b^2 L_{-2})v_{1,2})$ and get the same result for fusion.*

b) Define the vertex operator corresponding to the field $L_{-2}v_{h,c}$. Impose vanishing of the vertex operator corresponding to $(L_{-1}^2 + b^2 L_{-2})v_{1,2}$ and get the same result for fusion.

In particular taking the product of two degenerate fields has the form

$$\Phi_{1,2}\Phi_{1,n} = [\Phi_{1,n+1}] + [\Phi_{1,n-1}].$$

This should resemble tensor product of $\mathfrak{sl}_2$ representation, product of 2-dimensional and n-dimensional is a sum of $n+1$-dimensional and $n-1$-dimensional.

In order to find fusion rules with $\Phi_{1,3}$ we use associativity, first fuse with $\Phi_{1,2}$ and then again with $\Phi_{1,2}$. We get

$$\Phi_{1,3}\Phi_{h(P,b)} = [\Phi_{h(P-b,b)}] + [\Phi_{h(P,b)}] + [\Phi_{h(P+b,b)}].$$

Again this should resemble formula for tensor product of 3-dimensional and n-dimensional representations of $\mathfrak{sl}_2$. Arguing in this way we get generic formula for fusion with $\Phi_{1,n}$

$$\Phi_{1,n}\Phi_{h(P,b)} = \sum_{1-n \leq s \leq n-1,\ 2|(n-1-s)} [\Phi_{h(P+sb/2,b)}]. \tag{3.19}$$

The fusion rules for $\Phi_{2,1}$ are the same as for $\Phi_{1,2}$ but with substitution $b \mapsto b^{-1}$

$$\Phi_{2,1}\Phi_{h(P,b)} = [\Phi_{h(P-b^{-1}/2,b)}] + [\Phi_{h(P+b^{-1}/2,b)}].$$

This lead to general formula for fusion rule with degenerate field

$$\Phi_{m,n}\Phi_{h(P,b)} = \sum_{\substack{1-m \leq r \leq m-1,\ 2|(m-1-r) \\ 1-n \leq s \leq n-1,\ 2|(n-1-s)}} [\Phi_{h(P+rb^{-1}/2+sb/2,b)}], \tag{3.20}$$

One can think that here we have to $\mathfrak{sl}_2$, one related to b and another one related to b^{-1}.

3.6 Zamolodchikov's recursion relation

Although it's possible in principle, it is difficult to compute conformal block using just the definition (3.14) or (3.15). There is another way to study this function based on the analytic properties of these functions with respect to h and c, see [Zam84], [Zam87].

In the expansion (3.14) each term is rational function in parameters $h_1, h_2.h_3.h_4, h, c$. The poles of this function can come only from $G^{\lambda,\mu}$ and therefore are related to zeroes of the determinant of the Shapovalov form. Therefore the only possible poles in h are $h = h_{m,n}$, $m, n \in \mathbb{Z}_{>0}$.

Let $\tilde{\mathcal{F}} = \mathcal{F} z^{h_1+h_2-h} = 1 + o(z)$ denotes renormalized conformal block, it will be convenient to write some formulas in for $\tilde{\mathcal{F}}$ instead of $\mathcal{F}$. The pole $h = h_{m,n}$ appears first in $\tilde{\mathcal{F}}$ at the coefficient at z^{mn} and corresponds to singular vector $D_{m,n} v_{h,c}$.

Indeed, one can change the basis on the level mn such that $D_{m,n} v_{h,c}$ will be one of the vectors of the basis. Then the inverse of the Shapovalov form $G^{\lambda,\mu}$ has a pole only for the diagonal element corresponding to $D_{m,n} v_{h,c}$, and this pole is simple (here we used that $G_{\lambda,\mu}$ has only simple zero at $h = h_{m,n}$ on the level mn by theorem 2.4).

Hence the residue of the z^{mn} term at $h = h_{m,n}$ equals to

$$R_{m,n} = (v_{h_4}, \Phi_{h_3}(1) D_{m,n} v_{h,c})(D_{m,n} v_{h,c}, \Phi_{h_2}(1) v_{h_1,c}) \operatorname{Res}_{h=h_{m,n}} \frac{1}{(D_{m,n} v_{h,c}, D_{m,n} v_{h,c})}.$$

Theorem 3.3. *The residue of conformal block at $h = h_{m,n}$ has the form*

$$\operatorname{Res}_{h=h_{m,n}} \tilde{\mathcal{F}}(\vec{h}, h, c; z) = z^{mn} R_{m,n} \tilde{\mathcal{F}}(\vec{h}, h_{m,n} + mn, c; z) \tag{3.21}$$

Moreover, the function $\tilde{\mathcal{F}}$ has only simple pole at $h = h_{m,n}$, (for generic b).

Note that $h_{m,n} + mn$ can be simply written as $h_{m,-n}$. Note also that in the original normalization F the factor z^{mn} in (3.21) disappears.

Proof. Similarly to to the arguments above one can see that singular behavior of $G^{\lambda,\mu}$ comes only from the vectors of the form $L_{-\lambda D_{m,n}} v_{h,c}$.

Lemma 3.4. *Let $G^{(m,n)}_{\lambda,\mu} = (L_{-\lambda D_{m,n}} v_{h,c}, L_{-\mu D_{m,n}} v_{h,c})$. Then we have*

$$G^{(m,n)}_{\lambda,\mu} = (L_{-\lambda} v_{h_{m,-n},c}, L_{-\mu} v_{h_{m,-n},c})(D_{m,n} v_{h,c}, D_{m,n} v_{h,c}) + O((h - h_{m,n})^2). \tag{3.22}$$

In particular, it follows from this lemma and non degeneracy of $(L_{-\lambda} v_{h_{m,-n},c}, L_{-\mu} v_{h_{m,-n},c})$ that $G^{(m,n)}_{\lambda,\mu}$ has only simple poles for $h = h_{m,n}$

For the matrix elements one can easily see that

$$\frac{(v_{h_4,c}, \Phi_{h_3}(1) L_{-\lambda} D_{m,n} v_{h,c})}{(v_{h_4,c}, \Phi_{h_3}(1) D_{m,n} v_{h,c})} = \frac{(v_{h_4,c}, \Phi_{h_3}(1) L_{-\lambda} v_{h+mn,c})}{(v_{h_4,c} \Phi_{h_3}(1) v_{h+mn,c})}, \tag{3.23}$$

Therefore we have

$$Res_{h=h_{m,n}} \mathcal{F}(\vec{h}, h, c; z)$$

$$= Res_{h=h_{m,n}} \sum_{\lambda,\mu,|\lambda=|\mu||} z^{|\lambda|+mn} (v_{h_4}, \Phi_{h_3}(1) L_{-\lambda} D_{m,n} v_{h,c})(L_{-\mu} D_{m,n} v_{h,c}, \Phi_{h_2}(1) v_{h_1,c}) G^{\lambda,\mu}_{(m,n)}$$

$$= z^{mn} R_{m,n} \mathcal{F}(\vec{h}, h_{m,n} + mn, c; z)$$

□

It remains to find the factors $R_{m,n}$. They depend on h_1, h_2, h_3, h_4, c. The residue $R_{m,n}$ vanishes if the fusion condition holds for h_1, h_2, h or for h_3, h_4, h. Therefore the $R_{m,n}$ should be divisible by a factor

$$R^N_{m,n} = \prod_{r,s} (P_1+P_2+r\frac{b^{-1}}{2}+s\frac{b}{2})(P_1-P_2+r\frac{b^{-1}}{2}+s\frac{b}{2})(P_3+P_4+r\frac{b^{-1}}{2}+s\frac{b}{2})(P_3-P_4+r\frac{b^{-1}}{2}+s\frac{b}{2}), \tag{3.24}$$

Here the product goes over the same region as in fusion rules (3.20).

The ratio $R_{m,n}/R^N_{m,n}$ has no more zeroes. It can have some poles but poles could depend only on c. Therefore this ration depends only on c. The meaning of auxiliary is coincidence of $h_{m,n}$ with some other $h_{m',n'}$, for $m'n' < mn$, in this case the determinant of the Shapovalov form on the level mn have higher order of vanishing and therefore conformal block can have higher order poles.

The final answer is $R_{m,n} = R^N_{m,n}/R^D_{m,n}$, where

$$R^D_{mn} = 2 \prod'_{1-m\leq i\leq m, 1-n\leq j\leq n} (ib^{-1} + jb), \tag{3.25}$$

here $'$ means that the factors corresponding to $(i,j) = (0,0)$ and $(i,j) = (m,n)$ are excluded.

Problem 3.4. *On which power $(h - h_{1,2})$ appears in the norm of $(L^2_{-1} + b^2 L_{-2})v_{h,c}$. The answer could depend on the value of central charge.*

Problem 3.5. *Find residue of conformal block for $h = h_{1,2}$ by direct computation using Theorem 3.3. Compare the result with the formulas for R^N_{mn} and R^D_{mn} above.*

Therefore one can write

$$\tilde{\mathcal{F}}(\vec{h}, h, c; z) = \sum_{m,n\in\mathbb{Z}_{>0}} \frac{z^{mn} R_{m,n}}{h - h_{m,n}} \tilde{\mathcal{F}}(\vec{h}, h_{m,n} + mn, c; z) + \lim_{h\to\infty} \tilde{\mathcal{F}}(\vec{h}, h, c; z). \tag{3.26}$$

It remains to find the limit $\lim_{h\to\infty} \tilde{\mathcal{F}}$. This can be done, but we prefer to change point of view and consider the previously found poles and residues as a poles in c. Indeed if $mn > 1$ one can solve equation $h = h_{m,n}(c)$ as $c = c_{m,n}(h)$. The residues will change by the formula

$$R_{m,n}(\vec{h}, h) = R'_{m,n}(\vec{h}, c) \frac{\partial h_{m,n}(c)}{\partial c}.$$

Therefore we get

$$\tilde{\mathcal{F}}(\vec{h},h,c;z)=\sum_{m,n\in\mathbb{Z}_{>0},mn>1}\frac{z^{mn}R'_{m,n}}{c-c_{m,n}}\tilde{\mathcal{F}}(\vec{h},h+mn,c_{m,n};z)+\lim_{c\to\infty}\tilde{\mathcal{F}}(\vec{h},h,c;z). \quad (3.27)$$

In the limit $c\to\infty$ we have $G^{\lambda,\mu}\to 0$ unless $\lambda=\mu=(1)^N$. Therefore the only contribution of the vectors $L_{-1}^N v_{h,c}$ in (3.14) is significant in the limit $c\to\infty$.

Problem 3.6. *Prove that in the limit $c\to\infty$ classical conformal block goes to hypergeometric function ${}_2F_1(h-h_1+h_2,h-h_4+h_3,2h;z)$.*

The recurrence relation (3.27) can be solved in some sense explicitly, see [Per15, eq. (2.28), (2.32)].

Problem 3.7 (*). *One point conformal block $\mathcal{F}(h,h_1,c;z)$ is a trace of the operator $z^{L_0}\Phi_{h_1}(z)$ acting on $\mathrm{V}_{h,c}$. Find the formula for $c\to\infty$ limit of $\mathcal{F}(h,h_1,c;z)$.*

3.7 Singular vector $D_{1,n}$

In the proof of the fusion rules (3.19), (3.20) we used associativity. One can ask for more direct proof. In order to perform it we need to find explicit formulas for singular vectors $D_{m,b}v_{h_{m,n},c}$. This is also important for the proof of the Kac-Feigin-Fuchs theorem.

In the case $(m,n)=(1,n)$ there exists a remarkable formula.

Proposition 3.5 ([BSA88]). *Singular vector in $\mathrm{V}_{h_{1,n},c}$ has the form $D_{1,n}v_{h_{1,n},v}$, where*

$$D_{1,n}=\sum_{k_1+\ldots+k_l=n}\frac{b^{2(n-l)}}{\prod_{i=1}^{l-1}(k_1+\ldots+k_i)(k_{i+1}+\ldots+k_l)}L_{-k_1}L_{-k_2}\ldots L_{-k_l}. \quad (3.28)$$

Two remarks are in order. First, one can write similar formula for $D_{n,1}$ replacing $b\to b^{-1}$. Second, note that in the right side of (3.28) indices $k_1,\ldots,k_l$ are not ordered, so the summands in the right side are linearly dependent.

Example 3.3. The first nontrivial examples of this formula has the form

$$\begin{aligned}
D_{1,1}&=L_{-1},\\
D_{1,2}&=L_{-1}^2+b^2L_{-2}\\
D_{1,3}&=\frac{1}{4}L_{-1}^3+\frac{1}{2}b^2L_{-1}L_{-2}+\frac{1}{2}b^2L_{-2}L_{-1}+b^4L_{-3}
\end{aligned}$$

There is another way to write these formulas (see [BDFIZ91], [DFMS97, Sec. 8.2, 8.4]). Introduce three $n\times n$ matrices J_0,J_+,J_-

$$(J_0)_{i,j}=\frac{n-2i+1}{2}\delta_{i,j},\quad (J_-)_{i,j}=\delta_{i-1,j},\quad (J_+)_{i,j}=i(n-i)\delta_{i+1,j}.$$

These matrices satisfy commutation relations of the Lie algebra $\mathfrak{sl}_2$

$$[J_0,J_+]=J_+,\quad [J_0,J_-]=-J_-,\quad [J_+,J_-]=2J_0.$$

Construct an $n\times n$ matrix with values in the Lie algebra Vir:

$$\mathfrak{D}_{1,n} = -J_- + \sum_{k=0}^{\infty}(b^2 J_+)^k L_{-k-1} \tag{3.29}$$

By rdet of the matrix A with non commutative variables we denote its row determinant

$$\operatorname{rdet} A = \sum_{\sigma\in S_n} (-1)^{l(\sigma)} A_{1\sigma(1)}\cdot\ldots\cdot A_{n\sigma(n)}.$$

Problem 3.8 (*). *Prove that* $\operatorname{rdet}\mathfrak{D}_{1,n}$ *is proportional* $D_{1,n}$*, where* $\mathfrak{D}_{1,n}$*,* $D_{1,n}$ *are given by* (3.29) *and* (3.28) *correspondingly.*

The condition the $D_{1,n}v_{h_{1,n},c}$ is a singular vector can be rewritten in the following way. Introduce the chain of vectors $v_0, v_1, \ldots v_n$ such that

$$\mathfrak{D}_{1,n}\begin{pmatrix} v_{n-1}\\ v_{n-2}\\ \vdots \\ v_0\end{pmatrix} = \begin{pmatrix} v_n\\ 0\\ \vdots\\ 0\end{pmatrix}, \quad v_0 = v_{h_{n,m},c}. \tag{3.30}$$

These relations can be viewed as a system of recursion relations on v_l, the first two of them are $v_1 = L_{-1}v_0$, $v_2 = L_{-1}v_1 + (n-1)L_{-2}v_0$.

Problem 3.9. *a) Prove that*

$$L_0 v_l = (h_{1,n}+l)v_l, \quad L_1 v_l = -l(n-l)\left(1 + b^2\frac{n-2l+3}{2}\right)v_{l-1}.$$

b) Prove that*

$$L_2 v_l = -l(l-1)(n-l)(n-l+1)\left(b^2\frac{7}{4} + b^4\frac{n-2l+6}{2}\right)v_{l-2}.$$

c) Define vectors* w_l *by the formula* $v_l = b^{2l}\prod_{i=1}^{l} i(n-i)w_l$*. Prove that these vectors satisfy chain relations* (3.11)*, find corresponding* h*, is it generic?*

It follows from this problem that $v_n = D_{1,n}v_{h_{1,n},c}$ is a singular vector. So we proved the formula (3.28).

Using this formula one can write differential equations which follows from existence of the singular vector. For each L_{-k} applied to the primary field $\Phi(z)$ in the correlation function $\langle\Phi(z)\Phi_{h_1}(z_1)\ldots\Phi_{h_N}(z_N)\rangle$ we can write a contour integral and then deform the contour

$$L_{-k}\Phi(z) = \oint_{C_z} T(w)(w-z)^{-k+1}\Phi(z)dw \to$$

$$-\sum_{i=1}^{N}\oint_{C_{z_i}} T(w)(w-z)^{-k+1}\Phi_{h_i}(z_i)dw$$

$$=-\sum_{i=1}^{N}\oint_{C_{z_i}}(w-z)^{k+1}\left(\frac{h_i\Phi_{h_i}(z_i)}{(w-z_i)^2}+\frac{\Phi'_{h_i}(z_i)}{(w-z_i)}\right)dw$$

$$=\sum_i \frac{(k-1)h_i}{(z_i-z)^k}-\frac{\partial_{z_i}}{(z_i-z)^{k-1}}.$$

Problem 3.10. *a)Deduce the fusion rules for $\Phi_{1,3}$ from the explicit formula for $D_{1,3}$.*
b) Do the same for $\Phi_{1,4}$.
(Calculations performed in computer algebra systems are welcome)
c) Do the same for any $\Phi_{1,n}$.*

4 Three point function

4.1 Hypergeometric equation

We will need some standard properties of hypergeometric functions and hypergeometric equation. For the reference see [WW96, Ch. 14].

The hypergeometric equation has the form

$$z(1-z)\frac{d^2}{dz^2}F+(C-(A+B+1)z)\frac{d}{dz}F-ABF=0. \tag{4.1}$$

This is the linear second order linear differential equation with three regular singularities at $0,1,\infty$. The asymptotic behavior of solutions near these points has the form

0	1	∞
z^0, z^{1-C}	$(z-1)^0, (z-1)^{C-A-B}$	$(1/z)^A, (1/z)^B$.

Any second order linear differential equation with three regular singularities at $0,1,\infty$ can be transformed to (4.1) by $F\mapsto(1-z)^\alpha z^\beta F$. The particular form (4.1) is convenient since here we have solutions with constant leading term z^0 at 0 and solution with constant leading term $(z-1)^0$ at 1.

The expansions of solutions of (4.1) near singularities can be found term by term. The answer for the solution with constant leading term is the hypergeometric function

$${}_2F_1(A,B;C;z)=\sum_{n=0}^{\infty}\frac{A^{\uparrow n}B^{\uparrow n}}{C^{\uparrow n}n!}z^n, \tag{4.2}$$

where $X^{\uparrow n}=X(X+1)\cdots(X+n-1)$. Another solution near zero is

$$z^{1-C}{}_2F_1(A-C+1,B-C+1;2-C;z).$$

These two solutions have simple monodromy for the path encircling 0. One can similarly write solutions expanded at $(z-1)$ and $1/z$ which have simple monodromy for the path encircling corresponding singularities. There are formulas which connects these different bases in the spaces of solutions, we will need the following one

$$\begin{aligned} F(A,B;C;z) &= \frac{\Gamma(C)\Gamma(B-A)}{\Gamma(B)\Gamma(C-A)}(-z)^{-A}F(A,1-C+A;1-B+A;1/z) \\ & \frac{\Gamma(C)\Gamma(A-B)}{\Gamma(A)\Gamma(C-B)}(-z)^{-B}F(B,1-C+B;1-A+B;1/z). \quad (4.3) \end{aligned}$$

It is usually proven using integral presentation for ${}_2F_1$.

Now let us return to conformal field theory. If we consider four point conformal block $\langle\Phi_{1,2}(z)\Phi_{h_1}(z_1)\Phi_{h_2}(z_2)\Phi_{h_3}(z_3)\rangle$ with degenerate field $\Phi_{1,2}$. It satisfies differential equation (3.16). If we put other three fields to $0, 1, \infty$ as in (3.3) then, the resulting equation will be just second order differential equation in z. One can write this equation using conformal invariance (3.5)-(3.7). Another way is to study asymptotic of the solutions. Namely we shown in Sec. 3.5 that this conformal block can have asymptotic behavior at $z \to z_i$ as $(z-z_i)^{h(P_1+sb/2,b)-h(P_1,b)-h_{1,2}}$, for $s = \pm 1$.

Problem 4.1. *Let $\mathcal{F}_s$, $s \pm 1$ denotes the conformal block*

$$\mathcal{F}_s(P_1,P_3,P_4,b;z) = \Big(v_{h_4,c}, \Phi_{h_3}(1) {}_{h(P_1+sb/2,b)} \Phi_{1,2}(z) v_{h_1,c}\Big) \quad (4.4)$$

where $s = \pm 1$. Then we have

$$\mathcal{F}_s(z) = z^{D_s}(1-z)^{E_s}F(A_s,B_s;C_s;z) \quad (4.5)$$

where

$$\begin{aligned} A_s &= -sbP_1 - bP_4 - bP_3 + 1/2 && (4.6) \\ B_s &= -sbP_1 + bP_4 - bP_3 + 1/2 && (4.7) \\ C_s &= 1 - 2sbP_1, && (4.8) \\ D_s &= h(P_1 + sb/2, b) - h_1 - h_{1,2} && (4.9) \\ E_s &= h(P_3 + b/2, b) - h_3 - h_{1,2}. && (4.10) \end{aligned}$$

As the consequence of this Problem and formula (4.3) we get a connection formula for the degenerate conformal blocks

$$\mathcal{F}_s(P_1,P_3,P_4)(z) = z^{-2h_2} \sum_{t=\pm 1} B_{st}\, \mathcal{F}_t(P_4,P_3,P_1)(1/z). \quad (4.11)$$

4.2 DOZZ formula

Dorn, Otto and Zamolodchikov, Zamolodchikov found a formula for the three point function in the Liouville theory [DO94], [ZZ96]. Recall that dependence on the position of the fields is fixed by the conformal invariance see the formula (3.2). So the only function to determine is the dependence on the conformal dimensions. Note also that this functions serve as normalization of vertex operators which are not fixed by commutation relations (3.9). In this section we mainly follow [Tes95].

The Liouville theory is defined by the local density of the Lagrangian $\mathcal{L} = \frac{1}{2\pi}(\partial_a\varphi)^2 + \mu e^{2b\varphi}$. Here b parametrizes the central charge of the theory via (2.4), and parameter μ is usually called the cosmological constant. The field $\varphi(z\bar{z})$ has both holomorphic and and anti-holomorphic parts.

The primary fields in the theory has the from $V_\alpha = \exp(\alpha\varphi)$. Them has conformal dimensions $h = \bar{h} = \alpha(b^{-1} + b - \alpha)$. The coincidence of chiral and antichiral conformal dimensions lead to absence of the monodromy in the correlation functions, which we will use below. By $C(\alpha_1, \alpha_2, \alpha_3)$ we denote the three point function of fields V_{α_1}, V_{α_2}, V_{α_3}.

The parametrizations by α and P are related by $\alpha = \frac{1}{2}(b^{-1} + b) - P$. For example for the degenerate field $\Phi_{1,2}$ we have $P = (b^{-1} + 2b)/2$ and $\alpha = -b/2$.

Consider four point function with one degenerate field. We have

$$\langle V_{\alpha_1}(0)V_{-b/2}(z,\bar{z})V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle = \sum_{s=\pm1} C(\alpha_1, -b/2, \alpha_1 + sb/2)C(\alpha_1 + sb/2, \alpha_3, \alpha_4) \mathcal{F}_s(P_1, P_3, P_4)(z)\mathcal{F}_s(P_1, P_3, P_4)(\bar{z}) \quad (4.12)$$

Now we tend z to ∞ and use transformation formula (4.11). The result should be a linear combination of terms $\mathcal{F}_t(P_4, P_3, P_1)(z^{-1})\mathcal{F}_t(P_1, P_3, P_4)(\bar{z}^{-1})$, the crossed terms $\mathcal{F}_t(P_4, P_3, P_1)(z^{-1})\mathcal{F}_{-t}(P_1, P_3, P_4)(\bar{z}^{-1})$ should cancel. This cancellation follows from the fact fusion of the fields V_{α_4} and $V_{-b/2}$ consist of two fields $V_{\alpha_4+tb/2}$, $t = \pm1$. In other terms on this cancellation follows from the absence of the monodromy in the correlation function $\langle V_{\alpha_1}(0)\Phi_{-b/2}(z,\bar{z})V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle$.

Therefore we get a difference relation

$$\frac{C(\alpha_4, \alpha_3, \alpha_1 + b/2)}{C(\alpha_4, \alpha_3, \alpha_1 - b/2)} = -\frac{B_{-1,+1}\bar{B}_{-1,-1}}{B_{+1,+1}\bar{B}_{+1,-1}}\frac{C(\alpha_1, -b/2, \alpha_1 - b/2)}{C(\alpha_1, -b/2, \alpha_1 + b/2)}.$$

The last fact depends only on α_1 and is related to normalization of the field V_α. So up to this factor we get a shift relation

$$\frac{C(\alpha_1 + b, \alpha_2, \alpha_3)}{C(\alpha_1, \alpha_2, \alpha_3)} \sim \frac{\gamma(b(\alpha_2 + \alpha_3 - \alpha_1 - b))}{\gamma(b(\alpha_1 + \alpha_2 + \alpha_3 - Q))\gamma(b(\alpha_1 + \alpha_2 - \alpha_3))\gamma(b(\alpha_1 + \alpha_3 - \alpha_2))}, \quad (4.13)$$

where $\gamma(x) = \Gamma(x)/\Gamma(1 - x)$.

Inserting another degenerate field $V_{-b^{-1}/2}$ one can get similar shift relation for the shift by b^{-1}. These system of shift relations can be solved using certain analog of a multiple gamma function.

Quite often the answer is written in terms of the function $\Upsilon(x)$ which is defined by the integral presentation (for $\operatorname{Re} b > 0$, $\operatorname{Re}(b + b^{-1}) > \operatorname{Re} x > 0$,)

$$\log \Upsilon_b(x) = \int_0^\infty \frac{dt}{t}\left(\left(\frac{b^{-1} + b}{2} - x\right)^2 e^{-t} + \frac{e^{-\frac{b^{-1}+b}{2}t}(1 - e^{(-x+\frac{b^{-1}+b}{2})t})(1 - e^{(x-\frac{b^{-1}+b}{2})t})}{(1 - e^{-b^{-1}t})(1 - e^{-bt})}\right),$$

an by analytic continuation to the other regions. This function satisfies difference identities

$$\Upsilon(x+b) = \gamma(bx) b^{1-2bx} \Upsilon(x)$$
$$\Upsilon(x+b^{-1}) = \gamma(b^{-1}x) b^{2x/b-1} \Upsilon(x)$$

Problem 4.2. *The double gamma function is defined by the formula*

$$\log \Gamma_{\epsilon_1,\epsilon_2}(x) = \frac{d}{ds}|_{s=0} \sum_{m,n=0}^{+\infty} (x+\epsilon_1 m+\epsilon_2 n)^{-s} = \frac{d}{ds}|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s \frac{e^{-tx}}{(1-e^{-\epsilon_1 t})(1-e^{-\epsilon_2 t})}. \tag{4.14}$$

Find the ratio $\Gamma_{\epsilon_1,\epsilon_2}(x+\epsilon_1)/\Gamma_{\epsilon_1,\epsilon_2}(x)$. *Show that*

$$\Upsilon_b(x) = \frac{\Gamma_{b,b^{-1}}(\frac{b^{-1}+b}{2})^2}{\Gamma_{b,b^{-1}}(x)\Gamma_{b,b^{-1}}(b^{-1}+b-x)}. \tag{4.15}$$

Putting all things together one can show that

$$C_{\alpha_1,\alpha_2,\alpha_3} \sim \frac{1}{\Upsilon_b(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_3+\alpha_1-\alpha_2)}.$$

Using the normalization $V_\alpha = e^{\alpha\varphi}$ one can find the full answer, see [ZZ96, eq. (3.14)]

Remark 4.1. In terms of the AGT relation the parameters b, b^{-1} which appears here in shift relation correspond to Nekrasov parameters ϵ_1, ϵ_2 which are responsible for the rotation of the lines in $\mathbb{C}^2$.

5 Minimal models

Before we had formulas for characters of Verma modules (2.2) and irreducible ones (2.5)

$$\chi(\mathrm{V}_{h,c}) = \frac{q^h}{\prod_{k=1}^\infty(1-q^k)}, \qquad \chi(\mathrm{L}_{h_{m,n},c}) = \frac{q^{h_{m,n}}(1-q^{mn})}{\prod_{k=1}^\infty(1-q^k)}.$$

But the second formula works only for generic central charges. In terms of Liouville parametrization this means that $b^2 \notin \mathbb{Q}$. In this section we consider the special cases, with $b^2 \in \mathbb{Q}$.

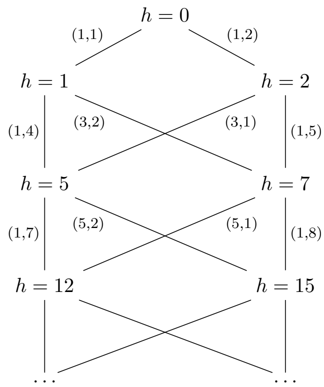
5.1 Example $c = 0$

Let us start from the simplest case of $c = 0$. In this case we have trivial representation $\mathbb{C}$. Its character is just 1 but can be rewritten in the form similar to (2.5)

$$1 = \frac{1-q-q^2+q^5+q^7-q^{12}+\dots}{\prod_{k=1}^\infty(1-q^k)} = \frac{\sum_{n\in\mathbb{Z}}(-1)^n q^{(3n^2-n)/2}}{\prod_{k=1}^\infty(1-q^k)} \tag{5.1}$$

The numbers $(3n^2-n)/2$ are called pentagonal numbers and the identity which we used is called Euler pentagonal theorem.

In Liouville parametrization we have $(b^{-1}+b)^2 = -3$, one can take as a solution $b = \sqrt{-2/3}$. Under this choice we have $2b^{-1} + 3b = 0$. This leads to additional symmetries of the exceptional highest weights $h_{m,n} = h_{m+2,n+3}$, besides standard one $h_{m,n} = h_{-m,-n}$.



The formula (5.1) gives the character for the module with $h = 0$. The right side (namely the numerator $1-q-q^2$) suggests that corresponding Verma module has singular vectors on the level 1 and 2. Indeed

$$h = 0 = h_{1,1}, \quad h = 0 = h_{-1,-1} = h_{1,2}.$$

The submodules generated by these two singular vectors necessary intersect This is clear from the comparison of the characters

$$\frac{1}{\prod_{k=1}^{\infty}(1-q^k)} - \frac{q}{\prod_{k=1}^{\infty}(1-q^k)} - \frac{q^2}{\prod_{k=1}^{\infty}(1-q^k)} = 1 - q^5 - q^6 - 3q^7 - \dots$$

Therefore these submodiles intersect, at least at the level 5.

Indeed, one can show the existence of such singular vectors. We will often use a formula $h_{m,n} + mn = h_{m,-n} = h_{-m,n}$. Therefore we get

$$h = 1 = h_{1,1} + 1 = h_{-1,1} = h_{1,3}, \text{ and, } h = 2 = h_{1,2} + 2 = h_{1,-2} = h_{3,1}.$$

So we showed the existence of the singular vector on the level 5. Similarly we get

$$h = 1 = h_{1,1} + 1 = h_{1,-1} = h_{3,2}, \text{ and, } h = 2 = h_{1,2} + 2 = h_{-1,2} = h_{1,5}.$$

This corresponds to the singular vector on the level 7.

Arguing in similar manner one can get the whole diagram of embedding. This can be summarized in the following theorem

Theorem 5.1. *There is an exact sequence of the* Vir *modules*

$$0 \longleftarrow \mathrm{L}_{0,0} \longleftarrow \mathrm{V}_{0,0} \longleftarrow \mathrm{V}_{1,0} \oplus \mathrm{V}_{2,0} \longleftarrow \mathrm{V}_{5,0} \oplus \mathrm{V}_{7,0} \longleftarrow \mathrm{V}_{12,0} \oplus \mathrm{V}_{15,0} \longleftarrow \dots$$

This theorem can be viewed as materialization of the character formula (5.1). Of course we did not prove it, we just gave some arguments in support.

Remark 5.1. This exact sequence means in particular that Verma module $\mathrm{V}_{0,0}$ contains singular vectors on the levels $1, 2, 5, 7, 12, 15, \dots$. We noticed the first two above. Similarly one can get

$$h = 0 = h_{1,1} = h_{3,4}, \quad h = 0 = h_{-1,-1} = h_{1,2} = h_{3,5}$$

so we get singular vectors on the level $12, 15$ but missed levels $5, 7$. The existence of these vectors do not follow from the standard $h_{m,n}$ formula. [2]

[2] Note that $h = 0 = h_{1,1} = h_{2,5/2}, \quad h = 0 = h_{-1,-1} = h_{1,2} = h_{2,7/2}$.

Problem 5.1. *a) Prove that any Verma module* $\mathrm{V}_{h_{m,n},0}$ *is a submodule of one of the following modules*

$$\mathrm{V}_{h_{1,1},0}, \mathrm{V}_{h_{1,3},0}, \mathrm{V}_{h_{2,1},0}, \mathrm{V}_{h_{2,2},0}, \mathrm{V}_{h_{2,3},0}, \mathrm{V}_{h_{2,6},0}. \tag{5.2}$$

b) Find the character of the modules* $\mathrm{L}_{h_{m,n},0}$.
Hint: find first characters of the modules (5.2),

5.2 Minimal models

Definition 5.1. Chiral part of minimal model (p,p'), where $p'>p>1$ are coprime integer numbers is a chiral conformal theory with central charge given by $b=\sqrt{-p/p'}$ and fields $\Phi_{m,n}$, $1\leq m\leq p-1$, $1\leq n\leq p'-1$ with identification $\Phi_{m,n}=\Phi_{p-m,p'-n}$.

Here as usual the word chiral means that we consider only holomorphic Virasoro algebra. The honest minimal models are products of chiral and anti chiral parts.

The parameter b above satisfies relation $pb^{-1}+p'b=0$. Therefore we have

$$h_{m,n}=h_{-m,-n}=h_{p-m,p'-n}.$$

Hence fields $\Phi_{m,n}$ and $\Phi_{p-m,p'-n}$ have equal conformal dimension and in "minimal" theory should be identified. The number of primary fields in (p,p') minimal model is $(p-1)(p'-1)/2$.

The first example of the minimal model is $(p,p')=(2,3)$. The central charge is equal to $c=0$ and the theory has only one primary field of conformal dimension $h_{1,1}=h_{1,2}=0$. This example was discussed in the previous section. The only representation is one-dimensional, hence the theory contains only identity operator. This theory is called empty. Since the stress-energy-momentum tensor $T(z)$ in this theory vanishes it is usually excluded from the list of minimal models.

Proposition 5.2. *The fields* $\Phi_{m,n}$ *of the minimal model are closed under the fusion.*

Proof. We know that all these fields can be obtained by the fusion of the simplest degenerate fields $\Phi_{1,2}$ and $\Phi_{2,1}$. So it is sufficient to show that fusion of $\Phi_{1,2}$ and $\Phi_{m,n}$, with $1\leq m\leq p-1$, $1\leq n\leq p'-1$ satisfies the same restrictions. This is clear from the fusion rule (3.18) if $n\leq p'-2$. If $n=p'-1$ one can first use symmetry $\Phi_{m,n}=\Phi_{p-m,p'-n}$ and then fusion rule (3.18). □

The singular vectors can be found similarly to the previous section. For example for the module $h=0=h_{1,1}=h_{-1,-1}=h_{p-1,p'-1}$ we have two singular vectors on the level 1 and level $(p-1)(p'-1)$. The corresponding highest weights also corresponds to degenerate representations $h_{-1,1}=h_{p-1,p'+1}=h_{1,-1}=h_{p+1,p'-1}$, and $h_{p-1,1-p'}=h_{2p-1,1}=h_{1-p,p'-1}=h_{1,2p'-1}$, and so forth. This leads to the explicit formulas for characters.

Introduce notation for the minimal model character $\chi^{p/p'}_{m,n}=\chi(\mathrm{L}_{h_{m,n},c(b)})$, where $b=\sqrt{-p/p'}$. Then it was proven in [FF84], [RC85] (see also [FF90] and [IK11]) that

$$\chi^{p/p'}_{m,n}=\left(\sum_{k\in\mathbb{Z}}x^{\frac{(p'(2kp+m)-pn)^2-(p'-p)^2}{4pp'}}-x^{\frac{(p'(2kp+m)+pn)^2-(p'-p)^2}{4pp'}}\right)\prod_{k=1}^{\infty}\left(1-x^k\right)^{-1} \tag{5.3}$$

Problem 5.2. *For minimal models* $(2, 2p+1)$ *express characters* $\chi^{p/p'}_{m,n}$ *as an infinite product.*

Problem 5.3. *The minimal model* $(3, 4)$ *has central charge* $1/2$ *and usually called Ising minimal model.*

a) Find the formulas for the characters $\chi^{3/4}_{1,1} + \chi^{3/4}_{1,3}$*,* $\chi^{3/4}_{1,1} - \chi^{3/4}_{1,3}$*,* $\chi^{3/4}_{1,2}$ *in the form of a product (without denominator).*

b) Prove these formulas from representation theory of real fermion.*

5.3 Unitarity.

Recall that we defined above two Shapovalov form, one of them is complex linear form $(\cdot, \cdot)$ and another one is sesquilinear $\langle \cdot, \cdot \rangle$, see Remark 2.5. Note that the existence of sesquilinear form with $L_0^\dagger = L_0$ and $C^\dagger = C$ requires $h, c \in \mathbb{R}$. Therefore one can consider the Virasoro algebra defined over the field $\mathbb{R}$, here these forms $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ coincide.

One can ask whether this sesquilinear form $\langle \cdot, \cdot \rangle$ is positive-definite. Of course this question makes sense only for irreducible modules $\mathrm{L}_{h,c}$, $h, c \in \mathbb{R}$. This property is equivalent to the fact that on the Verma module $\mathrm{L}_{h,c}$ every vector has non negative norm.

The following theorem was proven by Friedan, Qiu, Shenker [FQS84],[FQS86], another exposition is given by Langlands [Lan88], see also [IK11, Ch. 11]

Theorem 5.3. *The Verma module* $\mathrm{V}_{h,c}$ *has no vectors of negative norm if and only if one of the following conditions hold:*

- $h \geq 0$, $c \geq 1$
- $c = 1 - 6/m(m+1)$, $h = h_{r,s}$, $m \in \mathbb{Z}_{\geq 2}$, $1 \leq r \leq s < m$.

The proof of this theorem can be decomposed into several steps. We will do the first ones.

Step 1

Lemma 5.4. *If* $\mathrm{V}_{h,c}$ *has no vectors of negative norm then* $c \geq 0$, $h \geq 0$.

Proof. Since $0 \leq (L_{-1}v_{h,c}, L_{-1}v_{h,c}) = 2h$ we get $h \geq 0$. Since $0 \leq (L_{-n}v_{h,c}, L_{-n}v_{h,c}) = 2nh + c(n^3 - n)/12$ we get $c \geq 0$. □

Step 2

Lemma 5.5. *The limit* $\lim_{h\to\infty} h^{-l(\lambda)} \langle L_{-\mu}v_{h,c}, L_{-\lambda}v_{h,c}\rangle = \alpha_\lambda \delta_{\lambda,\mu}$*, where* $\alpha_\lambda \in \mathbb{Z}_{\geq 1}$*.*

Proof. In the proof of Proposition 2.5 that degree of $\langle L_{-\mu}v_{h,c}, L_{-\lambda}v_{h,c}\rangle$ is non greater the $l(\lambda)$. Moreover, this degree can equal to $l(\lambda)$, if any commutator of the form $L_{\mu_i}, L_{-\lambda_j}$ gives L_0. This is possible only if $\lambda = \mu$. If the partition $\lambda = (1^{m_1}, 2^{m_2} \ldots)$ (i.e. has m_1 parts equal to 1. m_2 parts equal to 2, and so on) then the leading coefficient is equal to $\alpha_\lambda = \prod_i (2i)^{m_i} m_i!$. □

Corollary 5.6. *Determinant of the Shapovalov form is equal to* $\alpha_N \prod_{r,s\geq 1}(h-h_{r,s})^{P(N-rs)}$*, where* $\alpha_N \in \mathbb{Z}_{\geq 0}$.

Proof. The roots and their multiplicities follows from the Theorem 2.2. The leading coefficient equals to $\alpha_N = \prod_{\lambda,|\lambda|=N} \alpha_\lambda$ □

Lemma 5.7. *If* $c \geq 1$ *and* $h \geq 0$ *then* $\mathrm{V}_{h,c}$ *has no vectors of negative norm.*

Proof. It is sufficient to proof the statement for strict inequalities $c > 1$ and $h > 0$ and then use continuity. First we prove that under these conditions Verma module $\mathrm{V}_{h,c}$ is irreducible.

Consider two cases. First assume that $c \geq 25$. In Liouville parametrization $c = 1 + 6(b + b^{-1})^2$, so we get $b \in \mathbb{R}$, we can assume that $b \geq 1$. Therefore for any $r, s \in \mathbb{Z}_{\geq 1}$ we have

$$h_{r,s} = (\frac{b^{-1}+b}{2})^2 - (\frac{rb^{-1}+sb}{2})^2 < 0.$$

Now assume that $1 < c < 25$. Then $b^2 \notin \mathbb{R}$ but $b^{-2}+b^2 \in \mathbb{R}_{>0}$. Therefore $rb^{-2}+sb^2 \in \mathbb{R}$ only if $r = s$. Hence $h_{r,s} \in \mathbb{R}$ only if $r = s$. We have $h_{r,r} = (1-r^2)(b^{-1}+b)^2/4 \leq 0$.

Now note that for $h \gg 0$ the Shapovalov form is positive definite due to Lemma 5.5. Since this form is nondegenerate for $c \geq 1$ and $h \geq 0$ we see that it is positive definite in this region (by continuity). □

Step 3 Due to previous two steps it remains to consider case $0 \leq c < 1$, $h \geq 0$. The Theorem 5.3 means that in this region unitarity holds only for special points corresponding to minimal models.

The idea can be explained as follows. If for example $(h-h_{1,2})(h-h_{2,1}) < 0$ then on the level 2 determinant of the Shapovalov form becomes negative. Therefore in this region the form on $\mathrm{V}_{h,c}$ is not positive definite Similarly one can the region $(h-h_{1,3})(h-h_{3,1}) < 0$ is also forbidden. One can see the corresponding regions in the Fig. 1.

One can make this argument another way. Namely the norm of the vector $D_{1,2}v_{h,c}$ vanishes for $h = h_{1,2}$. But it follows from the Corollary 5.6 the multiplicity of this root is equal to 1. Therefore the norm of this vector changes the sign after having crossed the line $h = h_{1,2}(c)$. Since for $h \gg 0$ the form is positive definite therefore in this region we have vector of the negative norm

Now we are going to prove that regions of the form $(h-h_{r,s})(h-h_{s,r}) < 0$ covers the strip $0 \leq c < 1$, $h \geq 0$.

Step 4 It is convenient to use new parametrization

$$c = 1 - \frac{6}{m(m+1)}, \quad h = \frac{\rho^2 - 1}{4m(m+1)}.$$

Comparing to Liouville parametrization we have $m = -1/(b^2+1)$, $\rho = 2P/(b^{-1}+b)$.

Since $0 \leq c < 1$ we can take $2 \leq m < \infty$. Since $h \geq 0$ we can assume that $\rho \geq 1$. For singular dimension $h_{r,s}$ we have $\rho_{r,s} = \pm(-mr + (1+m)s)$.

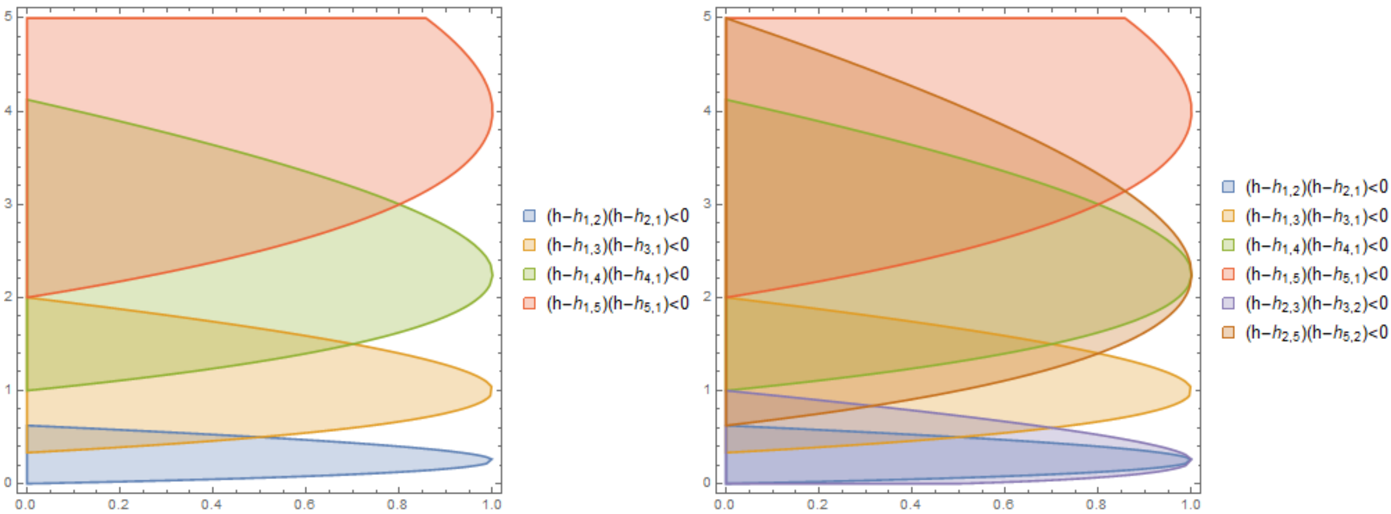


Figure 1:

Assume that $(h-h_{r,s})(h-h_{s,r})<0$. We can assume that $r<s$, then $|\rho_{r,s}|<|\rho_{s,r}|$. Therefore we have $h_{r,s}<h<h_{s,r}$ and $|\rho_{r,s}|<\rho<|\rho_{s,r}|$. This leads to three inequalities, which we label by A,B,C

$$\begin{cases} \rho < -mr + (m+1)s & \text{(B)} \\ \rho > -sm + (1+m)r & \text{(C)} \\ \rho > sm - (1+m)r & \text{(A)} \end{cases} \tag{5.4}$$

Denote the open region bounded by these lines by D. This region is drawn in Fig. 2/ We proved the following lemma

Lemma 5.8. *$(h-h_{r,s})(h-h_{s,r})<0$ if and only if the point (s,r) belongs to D.*

Step 5

Lemma 5.9. *There exists an integer point (s,r) inside the closure of D.*

Proof. Taking s sufficiently large we need to find point between two parallel lines (A) and (C). This means that we need to have integer point in the segment $[s\frac{m}{m+1}-\frac{-\rho}{m+1}, s\frac{m}{m+1}+\frac{\rho}{m+1}]$, $s\in\mathbb{Z}_{>>0}$.

If m is irrational number then by Kronecker's Approximation theorem there exist s such that $\{-\frac{sm-\rho}{m+1}\}<\frac{2\rho}{m+1}$, i.e. there is integer point in the segment $[\frac{sm-\rho}{m+1},\frac{sm+\rho}{m+1}]$.

Now assume that m is rational, let $m=\frac{a}{b}$, then $\frac{m}{m+1}=\frac{a}{a+b}$, where a,b are coprime integers. Then there exists s such that $\{-\frac{sm-\rho}{m+1}\}<\frac{1}{a+b}$. On the other hand the size of the segment is $\frac{2\rho}{m+1}=\frac{2\rho b}{a+b}>\frac{2}{a+b}$, therefore there is an integer point in the segment. $\square$

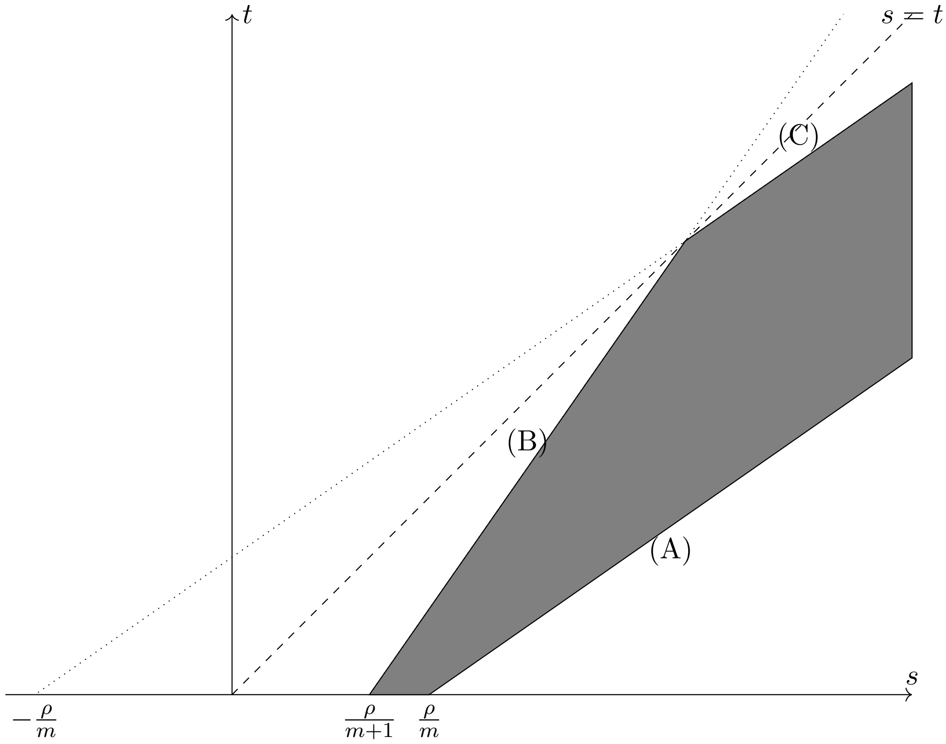


Figure 2:

Step 6 Let $s_0 = \min\{s|(s,r) \in \bar{D}.\}$ and $r_0 = \min\{r|(s,r) \in \bar{D}.\}$. It is easy to see geometrically that $(s_0, r_0) \in \bar{D}$.

Lemma 5.10. *If $(s_0, r_0) \in D$ then there are vectors in $\mathrm{V}_{h,c}$ with negative norm.*

Proof. It is sufficient to prove that determinant of the Shapovalov form on certain level is negative. For this determinant we will use the formula from Corollary 5.6. By the Lemma 5.8 the determinant contains negative term $(h - h_{r_0,s_0})(h - h_{s_0,r_0})$ and all other terms of the form $(h - h_{r,s})(h - h_{s,r})^{p(r_0 s_0 - rs)}$ should be positive. Therefore either the determinant is negative or there exists r such that $h - h_{r,r} < 0$ and $r^2 < r_0 s_0$. Let r_1 be minimal r with such property, then the determinant on the level r_1^2 contains only one negative term $(h - h_{r_1,r_1})$. $\square$

Due to this lemma we have shown that if the Verma module $\mathrm{V}_{h,c}$ has no vectors of negative norm and $0 \le c < 1$, $h \ge 0$ then $h = h_{r,s}$. It remains to show that c should correspond to $(m/m+1)$ minimal model and also impose certain constraints on r, s. For these steps we refer to the papers mentioned above.

It is also remains to show "if" part, namely that for $c = 1 - 6/m(m+1)$, $h = h_{r,s}$, $m \in \mathbb{Z}_{\ge 2}$, $1 \le r \le s < m$ there is no vectors of negative norm. We will do this in Theorem 6.12 below.

6 Affine $\widehat{\mathfrak{sl}}(2)$

6.1 Affine $\widehat{\mathfrak{sl}}(2)$

We denote the standard generators of $\widehat{\mathfrak{sl}}(2) = \mathfrak{sl}(2) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ by $e_n = e \otimes t^n$, $f_n = f \otimes t^n$, $h_n = h \otimes t^n$, and the central element by K. with the relations

$$[e_n, e_m] = [f_n, f_m] = 0, \quad [e_n, f_m] = h_{n+m} + n\delta_{n+m}K, \tag{6.1}$$
$$[h_n, e_m] = 2e_{n+m}, \quad [h_n, f_m] = -2f_{n+m}, \quad [h_n, h_m] = 2n\delta_{n+m}K.$$

Sometimes we will use notations $J_+ = e$, $J_- = f$, $J_0 = h$. It is convenient to introduce currents

$$e(z) = \sum_{n\in\mathbb{Z}} e_n z^{-n-1}, \quad h(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n-1}, \quad f(z) = \sum_{n\in\mathbb{Z}} f_n z^{-n-1}. \tag{6.2}$$

We denote by $\mathcal{V}_{l,k}$ the Verma module. This is module generated by the highest weight vector $v_{l,k}$ such that

$$e_n v_{l,k} = 0, \ n \geq 0 \ \ h_n v_{l,k} = f_n v_{l,k} = 0, \ n > 0 \quad h_0 v = lv, \ Kv = kv.$$

The value k of the central element is called the level of the representation. We denote by $\mathcal{L}_{l,k}$ the irreducible quotient of $\mathcal{V}_{l,k}$.

On can define action of the Virasoro algebra on these modules. The stress–energy–momentum tensor is given by the Sugawara formula

$$T_{\mathrm{Sug}}(z) = \frac{1}{2(k+2)} : \left(\frac{1}{2}h^2(z) + e(z)f(z) + f(z)e(z)\right) : . \tag{6.3}$$

Proposition 6.1. *Define L_n by the expansion $T_{\mathrm{Sug}}(z) = \sum L_n z^{-n-2}$.*
a) We have a relation $[L_n, J_{a,m}] = -mJ_{a,n+m}$.
b) L_n satisfy Virasoro algebra with central charge $c = \frac{3k}{k+2}$.

The vector $v_{l,k}$ is the highest weight vector for the Virasoro algebra:

$$L_n v_{l,k} = 0, n > 0, \quad L_0 v_{l,k} = \frac{l(l+2)}{4(k+2)} v_{l,k}.$$

One can define the character of the representation V of $\widehat{\mathfrak{sl}}(2)$ as $\chi(V)(x, q) = \operatorname{Tr} q^{L_0} x^{h_0}|_V$.

Lemma 6.2. *We have*

$$\chi(\mathcal{V}_{l,k}) = \frac{x^l q^{l(l+2)/4(k+2)}}{\prod_{j=1}^{\infty}(1 - x^2 q^j)(1 - q^j)(1 - x^{-2} q^{j-1})} \tag{6.4}$$

Proof. The Verma module $\mathcal{V}_{l,k}$ has a basis

$$v_{\vec{n}} = \prod_{j=0}^{\infty} f_{-j}^{n_{f,j}} \prod_{j=1}^{\infty} h_{-j}^{n_{h,j}} \prod_{j=1}^{\infty} e_{-j}^{n_{e,j}} v_{l,k},$$

where all numbers $n_{f,j}, n_{h,j}, n_{e,h} \in \mathbb{Z}_{\geq 0}$ and only finite number of them are nonzero. We have

$$h_0 v_{\vec{n}} = \left(l + 2\sum_{j=1}^{\infty} n_{e,j} - 2\sum_{j=0}^{\infty} n_{f,j} \right) v_{\vec{n}}, \quad L_0 v_{\vec{n}} = \left(\frac{l(l+2)}{4(k+2)} + \sum_{j=1}^{\infty} (jn_{e,j} + jn_{h,j} + jn_{f,j}) \right) v_{\vec{n}}$$

Therefore we have

$$\begin{aligned}\chi(\mathcal{V}_{l,k}) &= \sum_{\vec{n}} x^{l+2\sum_{j=1}^{\infty} n_{e,j} - 2\sum_{j=0}^{\infty} n_{f,j}} q^{\frac{l(l+2)}{4(k+2)} + \sum_{j=1}^{\infty}(jn_{e,j}+jn_{h,j}+jn_{f,j})} \\ &= x^l q^{\frac{l(l+2)}{4(k+2)}} \left(\prod_{j=0}^{\infty} \sum_{n_{f,j}=0}^{\infty} x^{-2n_{f,j}} q^{jn_{f,j}} \right) \left(\prod_{j=1}^{\infty} \sum_{n_{h,j}=0}^{\infty} q^{jn_{h,j}} \right) \left(\prod_{j=1}^{\infty} \sum_{n_{e,j}=1}^{\infty} x^{2n_{e,j}} q^{jn_{e,j}} \right) \\ &= \frac{x^l q^{l(l+2)/4(k+2)}}{\prod_{j=1}^{\infty}(1-x^2q^j)(1-q^j)(1-x^{-2}q^{j-1})}\end{aligned}$$

□

Of course the more delicate question is the character of irreducible module $\mathcal{L}_{l,k}$ For generic l, k the Verma module $\mathcal{V}_{l,k}$ is irreducible. The precise statement is given in the following theorem.

Theorem 6.3 ([KK79])**.** *The Verma module $\mathcal{V}_{l,k}$ is irreducible unless $l + n(k+2) = m$ or $k - l + n(k+2) = m$ for some $m, n \in \mathbb{Z}_{\geq 0}$.*

Example 6.1. Let $l = m \in \mathbb{Z}_{\geq 0}$. Then the vector $f_0^{m+1} v_{l,k}$ is singular, namely annihilated by e_n, $n \geq 0$, h_n, f_n, $n > 0$. The only nontrivial check is for e_0, which reduces to the case of finite dimensional algebra $\mathfrak{sl}_2$.

Problem 6.1. *For $k - l = m \in \mathbb{Z}_{\geq 0}$ show that vector $e_{-1}^{m+1} v_{l,k} \in \mathcal{V}_{l,k}$ is singular.*

For $k \notin \mathbb{Q}$ the only one of the conditions in Theorem 6.3 can hold. Therefore the irreducible module is a quotient of the Verma module by submodule generated by one singular vector. This submodule itself is isomorphic to Verma module, hence we get the formula for character.

Lemma 6.4. *For generic k and $l \in \mathbb{Z}_{\geq 0}$*

$$\chi(\mathcal{L}_{l,k}) = \frac{(x^l - x^{-l-2}) q^{l(l+2)/4(k+2)}}{\prod_{j=1}^{\infty}(1-x^2q^j)(1-q^j)(1-x^{-2}q^{j-1})} \tag{6.5}$$

Remark 6.2. It is instructive to compare this formula with the formula for the character of the finite-dimensional $\mathfrak{sl}_2$ module:

$$\chi_l = \frac{x^l - x^{-l-2}}{1 - x^{-2}} = \frac{x^{l+1} - x^{-l-1}}{x - x^{-1}}$$

Note also that the character has symmetry $\chi_l = -\chi_{-l-2}$.

6.2 Integrable modules

Definition 6.1. Shapovalov form $\langle \cdot, \cdot \rangle$ on $\mathcal{V}_{l,k}$ is sesquilinear form such that $\langle v_{l,c}, v_{l,c} \rangle = 1$ and operators are conjugated by

$$e_n^\dagger = f_{-n}, f_n^\dagger = e_{-n}, h_n^\dagger = h_{-n}, K^\dagger = K. \tag{6.6}$$

As before one can also define complex symmetric Shapovalov form, but we do not use it here. The following proposition can be proven similarly to Proposition 2.3.

Proposition 6.5. *The kernel of the Shapovalov form coincides with the largest nontrivial submodule in $\mathcal{V}_{l,k}$.*

Therefore one can define Shapovalov form on the irreducible quotient $\mathcal{L}_{l,c}$, where this form is nondegenerate. Similarly to the Virasoro case one can ask whether this form is positive definite. Ore, equivalently, weather Verma module $\mathcal{V}_{l,k}$ has no vectors of negative norm. It appears that this question for $\widehat{\mathfrak{sl}}(2)$ is simpler.

Theorem 6.6. *The module $\mathcal{L}_{l,k}$ has positive definite Shapovalov form if and only if $l, k \in \mathbb{Z}$, $0 \le l \le k$.*

Proof. Assume that $\mathcal{V}_{l,k}$ has no vectors of negative norm. Since operators K, h_0 are self adjoint their eigenvalues should be real. Therefore it is necessary that $l, k \in \mathbb{R}$.

It is easy to see that

$$\langle f_0^m v, f_0^m v \rangle = l^{\downarrow m} m!.$$

These numbers are non negative only if $l \in \mathbb{Z}_{\ge 0}$. On the other hand

$$\langle e_{-1}^m v, e_{-1}^m v \rangle = (k-l)^{\downarrow m} m!.$$

Therefore $k - l \in \mathbb{Z}_{\ge 0}$. Hence we get $l, k \in \mathbb{Z}$, $0 \le l \le k$.

It remains to prove that for such l, k the module $\mathcal{L}_{l,k}$ has positive definite Shapovalov form. This can be shown using induction by k. For $k = 0$ we have $l = 0$ and $\mathcal{L}_{0,0} = \mathbb{C}$ is trivial representation. For $k = 1$ the modules $\mathcal{L}_{0,1}$ and $\mathcal{L}_{1,1}$ have explicit construction see Problem 6.2 below. It follows from this construction that Shapovalov form is positive definite on these modules.

For generic $l, k \in \mathbb{Z}$, $0 \le l \le k$ we consider tensor product $\mathcal{L}_{0,1}^{\otimes l} \otimes \mathcal{L}_{1,1}^{\otimes k-l}$. This module has positive definite Shapovalov form, therefore it is isomorphic to direct sum of irreducible modules which are orthogonal with respect to this form. The highest weight vector has the weight (l, k), therefore it generates irreducible module $\mathcal{L}_{l,k}$. Hence we proved that $\mathcal{L}_{l,k}$ has positive definite Shapovalov form. □

Problem 6.2. *Let a_n, $n \in \mathbb{Z}$, $\hat{Q}$ generators of the Heisenberg algebra with commutation relations $[a_n, a_{-n}] = n$, $[a_0, \hat{Q}] = 1$. Introduce the field*

$$\varphi(z) = \sum_{n \in \mathbb{Z} \setminus 0} \frac{1}{-n} a_n z^{-n} + a_0 \log z + \hat{Q}.$$

Prove that formulas

$$e(z) = {}^*_*\exp(\sqrt{2}\varphi(z))^*_*, \quad h(z) = \sqrt{2}\partial\varphi(z), \quad f(z) = {}^*_*\exp(-\sqrt{2}\varphi(z))^*_*$$

satisfies relations $\widehat{\mathfrak{sl}}_2$ on the level 1.

b) By F_μ denote Fock representation of the Heisenberg algebra with generators a_n and highest weight μ. Show that sums

$$\mathcal{L}_{0,1} = \bigoplus_{m\in\mathbb{Z}} \mathrm{F}_{m\sqrt{2}}, \quad \mathcal{L}_{1,1} = \bigoplus_{m\in\mathbb{Z}+1/2} \mathrm{F}_{m\sqrt{2}}$$

form integrable representations $\widehat{\mathfrak{sl}}_2$ on the level 1.

c) Show that $\mathcal{L}_{0,1}$ and $\mathcal{L}_{1,1}$ have positive definite Shapovalov form.

Using this construction one can get formulas for the characters:

$$\chi(\mathcal{L}_{0,1}) = \frac{\sum_{m\in\mathbb{Z}} x^{2m}q^{m^2}}{\prod_{j=1}^{\infty}(1-q^j)} = 1 + (x^2{+}1{+}x^{-2})q + (x^2{+}2{+}x^{-2})q^2 + (2x^2{+}3{+}2x^{-2})q^4 + \dots, \tag{6.7}$$

$$\chi(\mathcal{L}_{1,1}) = \frac{\sum_{m\in\mathbb{Z}+1/2} x^{2m}q^{m^2}}{\prod_{j=1}^{\infty}(1-q^j)} = (x^{-1}{+}x)q^{1/4} + (x^{-1}{+}x)q^{5/4} + (x^{-3}{+}2x^{-1}{+}2x{+}x^3)q^{9/4} + \dots, \tag{6.8}$$

Problem 6.3. *a) Find the formula for Sugawara Virasoro algebra on $\mathcal{L}_{0,1}$, $\mathcal{L}_{1,1}$ in terms of a_n and without exponents.*

b) Decompose $\mathcal{L}_{0,1}$, $\mathcal{L}_{1,1}$ as representations of this Virasoro algebra.

Since Lie algebra $\widehat{\mathfrak{sl}}(2)$ is infinitedimensional the definition of the Lie group it is a delicate question. But for any $m \in \mathbb{Z}$ we have a finite dimensional $\mathfrak{sl}(2)$ subalgebra generated by $e_m, f_{-m}, h_0 + mK$. The $\widehat{\mathfrak{sl}}(2)$ module V is called *integrable* the action of any such $\mathfrak{sl}(2)$ subalgebra integrates to the action of the group $SL(2)$.

Theorem 6.7. *The module $\mathcal{L}_{l,k}$ for $l, k \in \mathbb{Z}$, $0 \le l \le k$ is integrable.*

Proof. For any $m \ge 0$ the action of e_m on $\mathcal{L}_{l,k}$ is locally nilpotent. For $m < 0$ the action of f_{-m} on $\mathcal{L}_{l,k}$ is locally nilpotent. Therefore, for any $m \in \mathbb{Z}$ the module $\mathcal{L}_{l,k}$ is highest weight module with respect to subalgebra generated by $e_m, f_{-m}, h_0 + mK$. Since this module has positively definite form it is isomorphic to a direct sum of finite dimensional modules with respect to this subalgebra. It remains to note that any finite dimensional $\mathfrak{sl}(2)$ integrates to the $SL(2)$ module. $\square$

The integrable modules are analogues of finite-dimensional representations of simple Lie algebras. In particular, there is Weyl-Kac formula for the character. It has the form

$$\chi(\mathcal{L}_{l,k}) = \frac{\mathrm{Num}_{l,k}}{\prod_{j=1}^{\infty}(1-x^2q^j)(1-q^j)(1-x^{-2}q^{j-1})} \tag{6.9}$$

where the numerator is given by the formula

$$\mathrm{Num}_{l,k} = q^{l(l+2)/4\kappa} \sum_{n\in\mathbb{Z}} q^{\kappa n^2+(l+1)n}(x^{l+2\kappa n} - x^{-l-2\kappa n-2}) \tag{6.10}$$

and we used notation $\kappa = k + 2$. It is also convenient to rewrite the numerator as

$$\mathrm{Num}_{l,k} = q^{-1/4\kappa} \sum_{n\in\mathbb{Z}} q^{\frac{1}{4\kappa}(2\kappa n+l+1)^2}(x^{l+2\kappa n} - x^{-l-2\kappa n-2}) \tag{6.11}$$

In this form it is easy to see an affine Weyl group symmetry in the form

$$\mathrm{Num}_{l+2\kappa,k} = \mathrm{Num}_{l,k}, \quad \mathrm{Num}_{-l-2,k} = -\mathrm{Num}_{l,k}. \tag{6.12}$$

6.3 Coset construction

Consider the tensor product of two $\widehat{\mathfrak{sl}}(2)$ modules $\mathcal{L}_{i,1} \otimes \mathcal{L}_{h,k}$, where $i = 1, 2$. There is an action of the algebra $\widehat{\mathfrak{sl}}(2) \otimes \widehat{\mathfrak{sl}}(2)$ on this space, we denote by $e_n^{(1)}, h_n^{(1)}, f_n^{(1)}$ the generators of the first factor and by $e_n^{(2)}, h_n^{(2)}, f_n^{(2)}$ the generators of the second factor.

The space $\mathcal{L}_{i,1} \otimes \mathcal{L}_{h,k}$ becomes a level $k + 1$ representation under the diagonal action of $\widehat{\mathfrak{sl}}(2)$: $e_n^{\Delta} = e_n^{(1)} + e_n^{(2)}, h_n^{\Delta} = h_n^{(1)} + h_n^{(2)}, f_n^{\Delta} = f_n^{(1)} + f_n^{(2)}$.

Lemma 6.8 ([GKO86]). *Denote*

$$T_{\mathrm{Coset}}(z) = T^{(1)}_{\mathrm{Sug}}(z) + T^{(2)}_{\mathrm{Sug}}(z) - T^{\Delta}_{\mathrm{Sug}}(z)$$

Then modes of $T_{\mathrm{Coset}}(z)$ satisfies Virasoro algebra with $c = 1 - 6/(k + 2)(k + 3)$. This coset Virasoro algebra commutes with operators $e_n^{\Delta}, h_n^{\Delta}, f_n^{\Delta}$

Proof. Using Proposition 6.1 we get

$$[L_n^{\mathrm{Coset}}, e_m^{\Delta}] = [L_n^{(1)}, e_m^{(1)} + e_m^{(2)}] + [L_n^{(2)}, e_m^{(1)} + e_m^{(2)}] - [L_n^{\Delta}, e_m^{\Delta}] = -me_m^{(1)} - me_m^{(2)} + me_m^{\Delta} = 0.$$

Using this commutativity we obtain

$$\begin{aligned}(m-n)L^{(1)}_{m+n} + \frac{n^3-n}{12}\delta_{m+n}c^{(1)} + (m-n)L^{(2)}_{m+n} + \frac{n^3-n}{12}\delta_{m+n}c^{(2)} \\ = [L_n^{(1)} + L_n^{(2)}, L_m^{(1)} + L_m^{(2)}] = [L_n^{\mathrm{Coset}} + L_n^{\Delta}, L_m^{\mathrm{Coset}} + L_m^{\Delta}] \\ = [L_n^{\mathrm{Coset}}, L_m^{\mathrm{Coset}}] + (m-n)L^{\Delta}_{m+n} + \frac{n^3-n}{12}\delta_{m+n}c^{\Delta}.\end{aligned}$$

Therefore, operators L_n^{Coset} satisfy Virasoro algebra with central charge

$$c^{\mathrm{Coset}} = c^{(1)} + c^{(2)} - c^{\Delta} = 1 + \frac{3k}{k+2} - \frac{3k+3}{k+3} = 1 - \frac{6}{(k+2)(k+3)}.$$

□

Note that for $k \in \mathbb{Z}_{\geq 0}$ the central charge $c = 1 - 6/(k+2)(k+3)$ corresponds to the minimal model $(k+2, k+3)$. The parameter b corresponding to this central charge is equal to $b_{k+2/k+3} = \sqrt{-(k+2)/(k+3)}$.

Proposition 6.9. *Let (l,k) be generic, $P = \frac{1}{2}\frac{(1+l)}{\sqrt{-(k+2)(k+3)}}$. Then we have a decomposition of the $\mathcal{L}_{i,1} \otimes \mathcal{L}_{l,k}$ as a* $\mathrm{Vir} \oplus \widehat{\mathfrak{sl}}(2)$ *module:*

$$\mathcal{L}_{0,1} \otimes \mathcal{L}_{l,k} = \bigoplus_{m\in\mathbb{Z}} \mathrm{L}_{h(\mathrm{P}+mb),c} \otimes \mathcal{L}_{l+2m,k+1}, \quad \mathcal{L}_{1,1} \otimes \mathcal{L}_{h,k} = \bigoplus_{m\in\mathbb{Z}+1/2} \mathrm{L}_{h(\mathrm{P}+mb),c} \otimes \mathcal{L}_{l+2m,k+1},$$

here $b = b_{k+2/k+3}$, $c = c(b)$.

Proof. Since all modules on the right side are generic Verma modules it is sufficient to show equality of the characters. The denominators on the left and right sides are equal to $\prod_{j=1}^{\infty}(1-x^2q^j)(1-q^j)^2(1-x^{-2}q^{j-1})$, so it is sufficient to look to numerators. Using the formula (6.7) we have for the first isomorphism

$$(\sum_{m\in\mathbb{Z}} x^{2m}q^{m^2})x^l q^{\frac{l(l+2)}{4(k+2)}} = \sum_{m\in\mathbb{Z}} x^{l+2m} q^{\frac{(l+2m)(l+2m+2)}{4(k+3)}} q^{h(\mathrm{P}+mb)}. \tag{6.13}$$

Calculation for the second isomorphism is similar. $\square$

Theorem 6.10. *Let $\mathcal{L}_{l,k}$ be integrable representation. Then we have a decomposition of the $\mathcal{L}_{i,1} \otimes \mathcal{L}_{l,k}$ as a* $\mathrm{Vir} \oplus \widehat{\mathfrak{sl}}(2)$ *module*

$$\mathcal{L}_{i,1} \otimes \mathcal{L}_{l,k} = \bigoplus_{l' \equiv l+i \bmod 2} \mathrm{L}_{h_{2l+1,2l'+1},c} \otimes \mathcal{L}_{l',k+1}. \tag{6.14}$$

Proof. Again it is sufficient to show relations on characters. Since denominators are equal we concentrate on numerators. Using the formulas (6.7) and (6.10)

$$\begin{aligned}
&(\sum_{m\in\mathbb{Z}} q^{m^2}x^{2m})\mathrm{Num}_{l,k} \\
&\quad = (\sum_{m\in\mathbb{Z}} q^{m^2}x^{2m}) \left(q^{\frac{l(l+2)}{4\kappa}} \sum_{n\in\mathbb{Z}} q^{\kappa n^2+(l+1)n}(x^{l+2\kappa n} - x^{-l-2\kappa n-2}) \right) \\
&\quad = q^{\frac{l(l+2)}{4\kappa}} \sum_{m,n} q^{\kappa n^2+(l+1)n+m^2}(x^{l+2\kappa n+2m} - x^{-l-2\kappa n-2-2m}) \\
&\quad = q^{\frac{l(l+2)}{4\kappa}} \sum_{l',n} q^{(\kappa+1)n^2+(l'+1)n+\frac{1}{4}(l-l')^2}(x^{2(\kappa+1)n+l'} - x^{-2(\kappa+1)n-l'-2}) \\
&\quad = \sum_{l'} q^{\frac{l(l+2)}{4\kappa}-\frac{l'(l'+2)}{4(\kappa+1)}+\frac{1}{4}(l-l')^2}\mathrm{Num}_{l',k+1},
\end{aligned} \tag{6.15}$$

where we have substituted $l' = l+2m-2n$, and used following relations

$$l' = l+2m-2n,$$

$$\kappa n^2 + (l+1)n + m^2 = (\kappa+1)n^2 + (l'+1)n + \frac{1}{4}(l'-l)^2.$$

Now using (6.12) we represent the right side as a sum of $\mathrm{Num}_{l',k+1}$, with $0 \le l' \le k+1$ and $l' \equiv l \bmod 2$. Calculating the coefficients we get the numerators from (5.3). $\square$

Remark 6.3. The coset constrution is not symmetric under $b \leftrightarrow b^{-1}$ replacement. One can see from decomposition (6.14) that $\widehat{\mathfrak{sl}}(2)$ on the level k is corresponds to b^{-1} but $\widehat{\mathfrak{sl}}(2)$ on the level $k+1$ is corresponds to b.

Theorem 6.11. *Determinant of the Shapovalov form on the Verma module $\mathrm{V}_{h_{r,s},c}$ vanishes on the level rs*

Proof. Take $k \in \mathbb{Z}_{\ge 1}$, such that $(k+2-r)(k+3-s) > rs$. It follows from the calculation of characters in the proof of Theorem 6.10 that certain multiplicity space for $\widehat{\mathfrak{sl}}(2)$ has the character which equals (5.3). One of these characters has the form $q^{h_{r,s}}(1-q^{rs}+\ldots)/\prod_j(1-q^j)$. Coset Virasoro algebra with central charge $c = 1-6/(k+2)(k+3)$ acts on this multiplicity space. Hence, the representation of the Virasoro algebra with this central charge and highest weight $h_{r,s}(c)$ has singular vector on the level rs.

So we proved that for any large integer k the determinant of the Shapovalov form on Verma module $\mathrm{V}_{h_{r,s},c}$, $c = 1-6/(k+2)(k+3)$ vanishes on the level rs. Since this determinant is polynomial in k we see that it vanishes for any k. $\square$

Combining with $h \to \infty$ arguments we proved Theorem 2.4 and hence Theorem 2.2.

Theorem 6.12. *The representations $\mathrm{L}_{h,c}$ for $c = 1-6/m(m+1)$, $h = h_{r,s}$, $m \in \mathbb{Z}_{\ge 2}$, $1 \le r \le s < m$ have positive definite Shapovalov form.*

Remark 6.4. Note that precisely these modules appears as exceptional cases in the Theorem 5.3.

Proof. Due to Theorem 6.6 the tensor product $\mathcal{L}_{i,1} \otimes \mathcal{L}_{l,k}$ has positive definite form with conjugation of $J^{(1)}_{a,n}, J^{(2)}_{a,n}$ given by (6.6). Therefore the conjugation of the Virasoro generators $L^{(1)}_n$, $L^{(2)}_n$, L^{Δ}_n, L^{Coset}_n are given by the formula $L^{\dagger}_n = L_{-n}$. Therefore we proved unitarity for any module obtained by coset construction (6.14).

It is easy to see that any module of the form $\mathrm{L}_{h,c}$ for $c = 1-6/m(m+1)$, $h = h_{r,s}$, $m \in \mathbb{Z}_{\ge 2}$, $1 \le r \le s < m$ appears on the right side of (6.14). $\square$

Problem 6.4. *Let $l \in \mathbb{Z}_{\ge 0}$ and k is generic. Find the decomposition of $\mathcal{L}_{i,1} \otimes \mathcal{L}_{l,k}$ as* $\mathrm{Vir} \oplus \widehat{\mathfrak{sl}}(2)$ *module.*

Problem 6.5 (*)**.** *Consider coset $(\hat{E}_8)_1 \oplus (\hat{E}_8)_1/(\hat{E}_8)_2$. Find its central charge and highest weights. Compare to Ising minimal model (3/4)*

7 Drinfeld-Sokolov reduction

7.1 Finite-dimensional case

Classical case. Consider finite dimensional simple Lie algebra $\mathfrak{g} = \mathfrak{sl}(2)$. Denote its standard generators by e, h, f, as before we will also denote them as $J_+.J_0, J_-$ sometimes.

The dual vector space $\mathfrak{g}^*$ has the structure of Poisson manifold. This means that the algebra of function on $\mathfrak{g}^*$ is a Poisson algebra. One can restrict to the algebraic functions, this algebra is an algebra of polynomials $S(\mathfrak{g}) = \mathbb{C}[e, h, f]$ and the bracket of generator is defined by $\{J_a, J_b\} = [J_a, J_b]$. This bracket is called Kostant-Kirillov bracket.

By N we denote nilpontent subgroups which consist of matrices $g(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$.

Proposition 7.1. *The coadjoint action of $\mathfrak{g}^*$ is hamiltonian with $H = e$.*

Proof. This proposition means that for any function $F \in \mathbb{C}[e, h, f]$ we have

$$\frac{d}{d\alpha} \left(g(\alpha) \cdot F\right)\Big|_{\alpha=0} = \{H, F\}.$$

It is sufficient to check this relation on the generators $F = J_a$ In this case the left side will be just adjoint action $[e, J]$. This finishes the proof. □

Remark 7.1. This proof does not use any special properties of our situation. Using the same words one can show that coadjoint action of any group G on $\mathfrak{g}^*$ is hamiltonian.

Definition 7.1. The *hamiltonian reduction* $\mathfrak{g}^*$ by the coadjoint action of N is $\{x \in \mathfrak{g}^* | e(x) = \epsilon\}/N$.

This construction depends on the ϵ, i.e. depends on the of hamiltonian. Action of the torus relates all nonzero values of ϵ, so actually there are two cases $\epsilon = 0$ and, say, $\epsilon = 1$.

In order to write this quotient explicitly it is convenient to identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using trace form:

$$J \in \mathfrak{g} \leftrightarrow \mathrm{Tr}(J\cdot) \in \mathfrak{g}^*.$$

Under this identification the set $\{x \in \mathfrak{g}^* | e(x) = \epsilon\}$ goes to the set of matrices $\begin{pmatrix} * & * \\ \epsilon & * \end{pmatrix}$.

Proposition 7.2. *For $\epsilon \neq 0$ any matrix of the form $\begin{pmatrix} * & * \\ \epsilon & * \end{pmatrix}$ can be uniquely reduced to the form $\begin{pmatrix} 0 & t \\ \epsilon & 0 \end{pmatrix}$ by conjugation of N.*

Proof. Direct computation. □

This proposition means that for $\epsilon \neq 0$ the reduction can be identified with affine line $\epsilon f + \langle f \rangle$. This is called a slice.

Now consider the algebra of functions on the reduction. By the definition it is $\mathbb{C}[e, h, f]/(e - \epsilon))^N$. Since the reduction is an affine line this algebra is an algebra of polynomials in one variable t. One can also argue that this algebra is generated by image of the Casimir element $t = C = ef + fe + h^2/2$.

In terms of the matrices $\begin{pmatrix} * & * \\ \epsilon & * \end{pmatrix}$ this invariant is proportional to the determinant, this gives the same formula.

Quantum case. Now we want to quantize this construction. It well known that quantization of Poisson algebra $S(\mathfrak{g})$ is the universal enveloping algebra $U(\mathfrak{g})$.

Definition 7.2. The *quantum hamiltonian reduction* $\mathfrak{g}^*$ by the coadjoint action of N is $(U(\mathfrak{g})/(e-\epsilon))^N$.

Since the group N is connected the condition of N invariance can be stated infinitesimally, as invariance under commutativity with e. In terms of the quotient it means that for any $X \in (U(\mathfrak{g})/(e-\epsilon))^N$ there exists $Y \in U(\mathfrak{g})$ such that $eX - Xe = Y(e-\epsilon)$. Therefore the quantum hamiltonian reduction is an algebra

$$\Big(X_1 + Y_1(e-\epsilon)\Big)\Big(X_2 + Y_2(e-\epsilon)\Big) = \Big(X_1X_2 + (X_1Y_2 + Y_1X_2 + Y_1Y)(e-\epsilon)\Big),$$

where $eX_2 - X_2e = Y(e-\epsilon)$.

Problem 7.1. *Find $(U(\mathfrak{sl}_2)/(e-\epsilon))^N$ depending on value of $\epsilon \in \mathbb{C}$.*

There is another way to define quantum hamiltonian reduction. Consider Clifford algebra Cl generated by ψ, ψ^* with relations

$$\psi^2 = (\psi^*)^2 = 0 \quad \psi^*\psi + \psi\psi^* = 1.$$

Let $A = U(\mathfrak{sl}(2)) \otimes \mathrm{Cl}$. The algebra A is actually a superalgebra, where $U(\mathfrak{sl}(2))$ are even and ψ, ψ^* are odd. Let $\mathrm{Q}_\epsilon = \psi(e-\epsilon)$. The operator $\mathrm{ad}_{\mathrm{Q}_\epsilon}$ acts on A as a commutator on even elements and as anticommutator on odd elements. Clearly $\mathrm{ad}^2_{\mathrm{Q}_\epsilon} = 0$.

Problem 7.2 (*)**.** *Find cohomology of $\mathrm{ad}_{\mathrm{Q}_\epsilon}$ on A.*

One can also construct representations using quantum hamiltonian reduction. This will be is more meaningful in other examples, but for completeness let us give the construction here. For any representation V of $U(\mathfrak{sl}_2)$ one can consider $V \otimes \Lambda$, where Λ is 2-dimensional representation of Cl. The algebra A acts on the $V \otimes \Lambda$ and one can define cohomology space $H(V \otimes \Lambda, \mathrm{Q}_\epsilon)$. The algebra $H(A, \mathrm{Q}_\epsilon)$ acts on this space

Problem 7.3 (*)**.** *a) Let $V = \mathcal{V}_l$ be a Verma module generated by the highest weight vector v_l such that $ev_l = 0$, $hv_l = 0$. Find $H(V \otimes \Lambda, \mathrm{Q}_\epsilon)$, compute the action of $H(A, \mathrm{Q}_\epsilon)$.*

b) Let $V = \mathcal{V}_l^*$ be a dual Verma module. This module can be also realized as a space of polynomials $\mathbb{C}[x]$ with action of

$$e = -x^2\partial_x - lx, \quad h = 2x\partial_x + l, \quad f = \partial_x$$

Find $H(V \otimes \Lambda, \mathrm{Q}_\epsilon)$, compute the action of $H(A, \mathrm{Q}_\epsilon)$.

c) Let V be a finite dimensional representation of $\mathfrak{sl}_2$. Find $H(V \otimes \Lambda, \mathrm{Q}_\epsilon)$, compute the action of $H(A, \mathrm{Q}_\epsilon)$.

7.2 Affine case

In this section we follow [Bel89], [Dic03], [FF90], [FBZ04]. This particular case of hamiltonian reduction is called Drinfeld-Sokolov reduction.

Classical case. Consider now affine Lie algebra $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}(2)$. The commutation relations were given above in (6.1). The dual space $\widehat{\mathfrak{g}}^*$ has Poisson structure with Kostant-Kirillov bracket.

The function K on $\widehat{\mathfrak{g}}^*$ is a Casimir function, in other words it belongs to the Poisson center. One can fix its value by number, we denote this number by k and the corresponding affine hyperplane in $\widehat{\mathfrak{g}}^*$ by $\widehat{\mathfrak{g}}^*_k$. The algebra of functions on $\widehat{\mathfrak{g}}^*_k$ is an algebra of polynomials with generators e_n, f_n, h_n. In terms of currents (6.2) the Poisson structure has the form

$$\begin{aligned} \{e(z), e(w)\} = \{f(z), f(w)\} = 0, \quad \{e(z), f(w)\} = h(w)\delta(z, w) + k\delta'(z, w), \\ \{h(z), e(w)\} = 2e(w)\delta(z, w), \quad \{h(z), f(w)\} = 2f(w)\delta(z, w), \\ \{h(z), h(w)\} = 2k\delta'(z, w), \end{aligned}$$

where $\delta(z, w) = \sum_{a+b=-1} z^a w^b$.

By $\widehat{N}$ we denote a group which consist of matrices $\begin{pmatrix} 1 & \alpha(z) \\ 0 & 1 \end{pmatrix}$, where $\alpha(z) = \sum \alpha_n z^{-n}$.

Proposition 7.3. *The action of $\widehat{N}$ on $\mathfrak{g}^*$ is hamiltonian. The Hamiltonian corresponding to one dimensional subgroup $\{g(t\alpha(z))\}$ is $H = \sum \alpha_n e_{-n} = \oint \alpha(z)e(z)dz$.*

This proposition means that for any function F

$$\frac{d}{dt}\left(g(t\alpha(z)) \cdot F\right)\Big|_{t=0} = \{H, F\}. \tag{7.1}$$

It was already mentioned in Remark 7.1 that the proof of proposition 7.1 works in this case. The space of all hamiltonians is generated by e_n, $n \in \mathbb{Z}$

It is convenient to consider functions e_n, f_n, h_n as a functions on the space of linear differential operators

$$L = k\frac{d}{dz} + A(z) = k\frac{d}{dz} + \begin{pmatrix} h(z)/2 & f(z) \\ e(z) & -h(z)/2 \end{pmatrix}. \tag{7.2}$$

Then the action of the group $\widehat{N}$ can be given by conjugation of differential operator or gauge transformations of $A(z)$, namely

$$k\frac{d}{dz} + A \mapsto g(k\frac{d}{dz} + A)g^{-1} = k\frac{d}{dz} + \left(gAg^{-1} - g'g^{-1}\right) \tag{7.3}$$

Indeed, one can see from this formula that

$$\frac{d}{dt}\left(g(t\alpha(z)) \cdot e_n\right)\Big|_{t=0} = 0, \quad \frac{d}{dt}\left(g(t\alpha(z)) \cdot h_n\right)\Big|_{t=0} = \sum_{a+b=n} \alpha_a h_b,$$

$$\frac{d}{dt}\left(g(t\alpha(z))\cdot f_n\right)\Bigg|_{t=0} = -\sum_{a+b=n}\alpha_a f_b + kn\alpha_n,$$

in agreement with the formula (7.1).

Definition 7.3. The hamiltonian reduction of $\widehat{\mathfrak{g}}_k^*$ by $\widehat{N}$ is $\left\{A(z) = \begin{pmatrix} h(z)/2 & f(z) \\ \epsilon & -h(z)/2 \end{pmatrix}\right\}/\widehat{N}$, where the action of $\widehat{N}$ is given by gauge transformations

In principle ϵ can depend on z but for simplicity below we take $\epsilon \in \mathbb{C}$. As in nonaffine case there are actually two cases $\epsilon = 0$ and $\epsilon \neq 0$. Similarly to nonaffine case one can show that

Proposition 7.4. *For $\epsilon \neq 0$ any matrix $A(z) = \begin{pmatrix} h(z)/2 & f(z) \\ \epsilon & -h(z)/2 \end{pmatrix}$ can be uniquely reduced to the form $\begin{pmatrix} 0 & \epsilon t(z) \\ \epsilon & 0 \end{pmatrix}$ by the action of $\widehat{N}$.*

The modes of the function $t(z) = \sum t_n z^{-n-2}$ are coordinates on the hamiltonian reduction. Direct computation shows that $t(z) = \epsilon f(z) + h^2(z)/4 + kh'(z)/2$. One can check the $\{t(z), e(w)\} = 0$ as expected. One can see that

$$\{t(z), t(w)\} = -\frac{k^3}{2}\delta'''(z,w) + 2kt(w)\delta'(z,w) + kt'(w)\delta(z,w). \tag{7.4}$$

In other words t_n are functions on the space Vir^* with Kostant-Kirillov bracket. So we get (classically) Virasoro algebra from affine Lie algebra $\widehat{\mathfrak{sl}}_2$.

Remark 7.2. If $A(z) = \begin{pmatrix} h(z)/2 & f(z) \\ \epsilon & -h(z)/2 \end{pmatrix}$ then the differential equation

$$\left(k\frac{d}{dz} + A(z)\right)\begin{pmatrix}\varphi_1 \\ \varphi_0\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix} \tag{7.5}$$

is equivalent to the second order differential equation $-k^2\varphi_0''(z) + t(z)\varphi_0(z) = 0$. Gauge transformations by the group $\widehat{N}$ preserves φ_0 and therefore preserves this equation.

This second order differential equation is a first example of so called oper. Technically this equation is a row determinant of L.

Remark 7.3. It is worth to mention generalization of these construction to the other affine Lie algebra. Consider for simplicity the case of the algebra $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}(3)$.

- The group $\widehat{N} = \begin{pmatrix} 1 & \alpha_{12}(z) & \alpha_{13}(z) \\ 0 & 1 & \alpha_{23}(z) \\ 0 & 0 & 1 \end{pmatrix}$. The space of hamiltonians is generated by $e_{ij}[n]$, where $1 \leq i < j \leq 3$, $n \in \mathbb{Z}$.

- It is convenient to consider functions $e_{ij}[n]$ as a functions on the space of operators $L = \frac{d}{dz} + A(z)$. Hamiltonian reduction is defined as

$$\left\{ A(z) = \begin{pmatrix} * & * & * \\ 1 & * & * \\ 0 & 1 & * \end{pmatrix} \right\} / \widehat{N}$$

  One can prove that any $\widehat{N}$ orbit uniquely intersects each of the following sets

$$\left\{ A(z) = \begin{pmatrix} 0 & t_1(z) & t_2(z) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}, \qquad \left\{ A(z) = \begin{pmatrix} 0 & \tilde{t}_1(z) & \tilde{t}_2(z) \\ 1 & 0 & \tilde{t}_1(z) \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

  The coordinates $t_1(z)$, $t_2(z)$ are coefficients of the third order differential operator $\operatorname{rdet} L$. The coordinates $\tilde{t}_1(z)$, $\tilde{t}_2(z)$ correspond to Slodowy slice

- Any $\widehat{N}$ orbit intersects (but non uniquely) set

$$\left\{ A(z) = \begin{pmatrix} a_1(z) & 0 & 0 \\ 1 & a_2(z) & 0 \\ 0 & 1 & a_3(z) \end{pmatrix} \right\}$$

  Therefore we get

$$(-k\partial)^n + t_1(z)(-k\partial)^{n-2} + \ldots + t_n(z) = (-k\partial + a_1(z)) \cdot \ldots \cdot (-k\partial + a_n(z)).$$

  This is called Miura transform.

- Poisson bracket of $t_1(z)$, $t_2(z)$ is Gelfand-Dickey bracket for classical affine W algebra. This can be proven either in terms of t_k directly [Dic03, Sec 9.4] or in terms of Miura transformation.

Example 7.4. In case of $\mathfrak{sl}_2$ we have $L = \frac{d}{dz} + \begin{pmatrix} 0 & t(z) \\ q & 0 \end{pmatrix}$. The form is given by $\omega(A, B) = \oint \operatorname{Tr}[L, A] B dz$. Cotangent vector corresponding to dt_n has the form $\delta_n = \begin{pmatrix} \lambda(z)/2 & \mu(z) \\ z^{n+1} & -\lambda(z)/2 \end{pmatrix}$. It should be $\widehat{N}$ invariant, namely $[L, \delta_n] = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. This gives $\lambda(z) = -k(n+1)z^n$. The form do not depend on the choice of μ, one can take $\mu = 0$ or fix it by the condition $[L, \delta_n] = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. This gives $\mu(z) = (n^2+n)/2z^{n-1} + t(z)z^{n+1}$ and

$$\{t_n, t_m\} = \omega(\delta_n, \delta_m) = \oint \operatorname{Tr}[L, \delta_n]\delta_m dz \sim -k^3 \frac{n^3}{2} \delta_{n+m} + k(n-m)t_{n+m}.$$

Quantum case. We do quantum reduction using fermions Introduce the anti-commuting generators ψ_n, ψ^*_m, $m, n \in \mathbb{Z}$ with the relations

$$\{\psi_n, \psi_m\} = \{\psi^*_n, \psi^*_m\} = 0, \quad \{\psi_n, \psi^*_m\} = \delta_{n+m,0}$$

By ClL we denote algebra generated by ψ_n, ψ^*_m, $m, n \in \mathbb{Z}$. It is also convenient to introduce currents $\psi(z) = \sum_n \psi_n z^{-n}$, $\psi^*(z) = \sum_n \psi^*_n z^{-n-1}$

Let A be a completion of the product $U(\widehat{\mathfrak{sl}}_2)_k \otimes \mathrm{Cl}$. Here completion allows us to consider infinite sums of generators which acts on the highest weight modules. Subindex k means taking the quotient by the ideal generated by central element $K - k$.

Introduce an operator

$$\mathrm{Q}_\epsilon = \oint_{|z|=1} (e(z) - \epsilon)\psi(z) dz.$$

The cohomology of $\mathrm{Ad}_{\mathrm{Q}_\epsilon}$ on A are called quantum hamiltonian reduction. It can be proven that for $\epsilon \neq 0$ the cohomology form Virasoro algebra.The Stress-energy-momentum tensor which generates the Virasoro symmetry reads

$$T_{\mathrm{DS}}(z) = T_{\mathrm{Sug}}(z) + \frac{1}{2}\partial_z h(z) - \psi(z)\partial\psi^*(z), \tag{7.6}$$

where T_{Sug} was defined in (6.3). The central charge of this Virasoro algebra equals

$$c_{\mathrm{DS}} = 1 - \frac{6(k+1)^2}{k+2}. \tag{7.7}$$

Remark 7.5. One can compute this cohomology using spectral sequence. Introduce two gradings on A

$$\deg_0 e_n = 1, \quad \deg_0 h_n = \deg_0 \psi_n = \deg_0 \psi^*_n = 0, \quad \deg_0 f_n = -1;$$
$$\deg_\infty f_n = \deg_\infty \psi_n = 1, \quad \deg_\infty h_n = 0, \quad \deg_\infty \psi^*_n = \deg_\infty e_n = -1.$$

We can decompose operator Q_ϵ as

$$\mathrm{Q}_\epsilon = \mathrm{Q}_0 - \epsilon\mathrm{Q}_\infty = \oint_{|z|=1} e(z)\psi(z)dz - \epsilon \oint_{|z|=1} \psi(z)dz.$$

It is easy to see that

$$\mathrm{Q}_0^2 = \mathrm{Q}_\infty^2 = \mathrm{Q}_0\mathrm{Q}_\infty + \mathrm{Q}_\infty\mathrm{Q}_0 = 0$$

The operator Q_0 preserves $\deg_\infty$ and shifts $\deg_0$ by 1, the operator Q_∞ preserves $\deg_0$ and shifts $\deg_\infty$ by 1.

It is easy to see that

$$[\mathrm{Q}_0, \psi(z)] = 0, \quad [\mathrm{Q}_0, h(z)] = -2e(z)\psi(z), \quad [\mathrm{Q}_0, :\psi^*(z)\psi(z):] = e(z)\psi(z),$$

here $[\cdot,\cdot]$ denotes super commutator. One can show that cohomology of Q_0 are generated by $\psi(z)$ and $J(z) = h(z) + 2 : \psi^*(z)\psi(z) :$. The $J(z)$ is a bosonic field with relations on generators $J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}$, $[J_n, J_m] = 2\kappa\delta_{n+m}$, where $\kappa = k + 2$.

Now we want to compute cohomology of Q_ϵ, one can view them in expansion on ϵ we start from cohomology of Q_0 and then add ϵ corrections. We have $[\mathrm{Q}_\infty, J(z)] = 2\epsilon\psi(z)$, and similarly for any field of the form $P(J(z), J'(z), \dots)$. Moreover, $[\mathrm{Q}_0, [\mathrm{Q}_\infty, P]] = 0$, therefore $[\mathrm{Q}_\infty, P]$ represents class in cohomology of Q_0. In order to have correction $P + \epsilon\tilde{P}$ such that $[\mathrm{Q}_\epsilon, P + \epsilon P_1] = o(\epsilon^2)$ we need to have $[\mathrm{Q}_\infty, P] = [\mathrm{Q}_0, P_1]$, in other words $[\mathrm{Q}_\infty, P]$ should represent Q_0 zero cohomology class.

Note that

$$[\mathrm{Q}_0, f(z)] = h(z)\psi(z) + k\partial_z\psi(z) =: J(z)\psi(z) : +\kappa\partial_z\psi(z)$$

The left side vanishes in Q_0 cohomology. Therefore in Q_0 cohomology we have $\partial_z\psi(z) = -\frac{1}{\kappa} : J(z)\psi(z) :$, in other words $\psi(z)$ is a vertex operator $\exp(-\int J(z)/\kappa)$. This operator is a screening operator and it well known that the algebra which commutes with this operator is the Virasoro algebra with central charge (7.7). The formula for the Virasoro algebra has the form

$$T_{\mathrm{DS}}(z) = \frac{1}{4(k+2)} : J(z)J(z) : + \left(\frac{1}{2} + \frac{1}{k+2}\right)\partial_z J(z) \tag{7.8}$$

Now we briefly discuss representations how to get representation of the Virasoro algebra using Drinfeld-Sokolov reduction.

By Λ we denote the Fock representation generated by the vector v

$$\psi_n v = 0 \text{ for } n \geq 0, \quad \psi_n^* v = 0 \text{ for } n > 0.$$

We introduce the grading on Λ by $\deg(v) = 0, \deg(\psi_n) = 1, \deg(\psi_n^*) = -1$. The operator $\mathrm{Q}_\epsilon = \oint_{|z|=1}(e(z)+1)\psi(z)dz$ acts on the space $V \otimes \Lambda$.

It is easy to see that $\mathrm{Q}^2 = 0$. We denote by $\mathrm{H}^i_{\mathrm{DS}}(V)$ the cohomology of the complex $(V \otimes \Lambda, \mathrm{Q})$, where i stands for the grading on Λ namely the um $\deg_0 + \deg_\infty$. These cohomology are called quantum Hamiltonian (or the Drinfeld-Sokolov) reduction of the V, sometimes terms BRST construction, and semi-infinite Lie algebra homology construction are also used. .

The current $T_{\mathrm{DS}}(z)$ commutes with Q. Therefore, this Virasoro algebra acts on $\mathrm{H}_{\mathrm{DS}}(V)$ if V is an any level k representation of $\widehat{\mathfrak{sl}}(2)$. It can be proven that for $\epsilon \neq 0$ and $V = \mathcal{L}_{l,k}$ we have $\mathrm{H}^i_{\mathrm{DS}}(\mathcal{L}_{l,k}) = 0$, for $i \neq 0$ and $\mathrm{H}^0_{\mathrm{DS}}(\mathcal{L}_{l,k})$ is irreducible representation of the Virasoro algebra or zero.

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Problem set