Lectures on Liouville Theory and Matrix Models

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Lecture1. Introduction

1. The Liouville gravity

1. Theory of gravity. Since Einstein the term gravity means the dynamic theory of the space-time metric structure. This dynamics may be either classical (classical gravity) or quantum, in which case we talk about quantum gravity. The main dynamical variable is the components of the metric tensor $g_{ab}(x)$.

In general this theory of gravity is very complicated structure, both from mathematical point of view and conceptually. Even in classical gravity the equations of motion imposed on the metric are highly non-linear and lead to soluthions which typically develop singularities where the space-time becomes highly curved and the classical Einstein theory itself fails to describe the physics near such singularities. In quantum gravity the situation is much worse, especially from the point of view of interpretations. Having lost the calssical "rigid" space-time frame to settle his experimental equipment, the virtual observer feels himself somewhat "dissolved" and is forced to look for new interpretational possibilities. The simplest (and quite common) solutionis to forget about coordinates and consider only coordinate-independent observables. Such approach, which can be called the topological gravity in some extended sense, is reasonably consistent and suffers from the only problem: how to make contact with the semiclassical limit, where, as each of us know, the everyday life has apparently nothing to do with the topological gravity. Anyhow, the problem of interpretations, the problem of correct choice of observables, is still of primary importance in quantum gravity. In other words we still don't know what are correct questions to be asked.

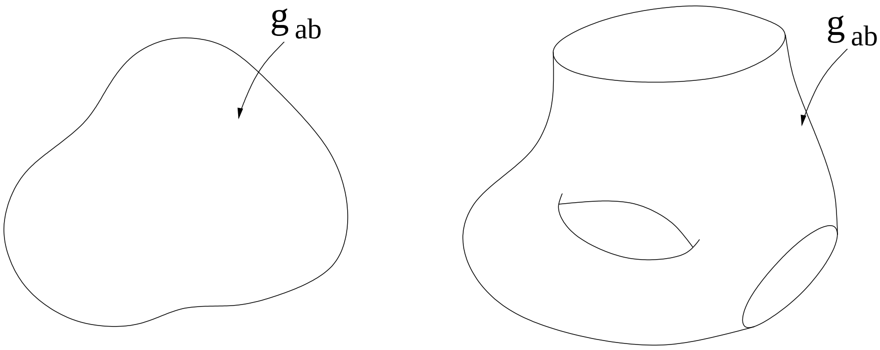
To this order any simplified model, which softens the severe mathematical problems of gravity but shares the same questions of interpretation, can be considered useful and worth studying. Below I'll concentrate on gravity in two dimensions (2D) where many technical simplifications are immediately come to play. Even there only few very particticular and most simple tasks are taken, mainly to illustrate the general pattern of problems coming even in this very simplified model.

2. Two-dimensional gravity. From now on we imply a two-dimensional manifold equipped with a metric g_{ab}. Morover, I restrict myself to the so called euclidean gravity, where the metric is positive definite $g > 0$.

I remind here the peculiarities and simplifications of two-dimensional metric geometry.

1. In 2D the Riemann curvature is completely described by the scalar curvature R.

2. Metric g_{ab} contains only three independent components. Therefore by an appropriate choice of the coordinate system (a two-parameter freedom) it can be described by only one field-like dynamical variable. As an example, we can take the so called isothermic (or conformal) coordinate system, which can always be chosen locally in two dimensions and



where the metric tensor has the form

$$g_{ab}(x) = \delta_{ab} e^{\sigma(x)} \tag{1.1}$$

and $\sigma(x)$ characterises completely the metric structure of the manifold. E.g. the scalar curvature reads

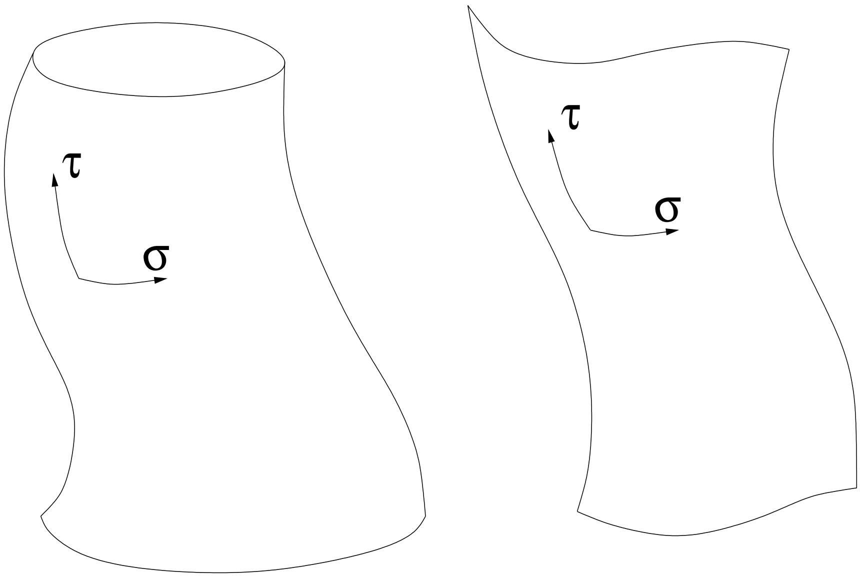
$$R(x) = -e^{-\sigma(x)} \partial_a^2 \sigma(x)$$

3. The action functional. To construct a classical covariant theory of gravity we have first of all to choose the action, which must a covariant (coordinate independent) functional of the metric $A[g_{ab}]$. At first sight it seems natural to to take a local action, i.e., with the density a local function of metric and its derivatives. General covariance prescribes this density to be constructed from coordinate tensors such as metric and Riemann curvature, like

$$A[g_{ab}] = \mu \int \sqrt{g} d^2x + k \int R\sqrt{g} d^2x + \left(\begin{array}{c} \text{terms of higer degree in } R \\ \text{and it's derivatives} \end{array} \right) \tag{1.2}$$

The first term here is simply the 2-volume of the surface. Coupling μ is called therefore the cosmological coupling constant. Second quoted term is nothing but the famous Eistein action. It is another peculiarity of the two-dimensional gravity, that the Einsten action in 2D does not lead to any essential local dynamics: the Gauss-Bonnet theorem permits to reduce the Einstein action to a number which depens on the topology only. In particular, it doesn't influence the local equations of motion. In principle we can consider next terms in (1.2) to create a non-trivial dynamics. I will not follow this line here. First, these terms play small role in the most interesting case of big surfaces (such terms are called irrelevant). The second and more important reason is that it seems more natural to construct the gravitational action as the effective one induced by certatin matter fields living on the surface. Such induced action is not nessesserily local, the series of local terms (1.2) being nothing but its long-wave expansion.

4.Induced action. If certain generally covariant matter lives over the surface, it generates an effective gravitational action, which although in general non-local, is authomatically covariant. Since long people argue (A.D.Sakharov) that the standard 4d Einstein action is simply a first term in the short wave expansion of the effective action generated by massive



material degrees of freedom. Another, more relevant in 2D example, is how 2D gravity appears in the string theory context. The trajectory of a string in the target space-time is a two-dimensional surface (the world sheet), either with a boundary in the case of open string, or compact for closed strings. Embedding coordinates $\vec{X}(\sigma, \tau)$ can be considered as fields on the world seet. In the simplest example of purely bosonic string the dynamics is prescribed by the standard string action

$$A_{\text{string}}\left[g_{ab}, \vec{X}\right] = \frac{1}{2}\int g^{ab}\partial_a\vec{X}\partial_b\vec{X}\sqrt{g}d^2x \tag{1.3}$$

Of course, once the "matter fields" $\vec{X}$ are integrated out, this results in certain effective action dependent on the metric only. This effective action is highly non-local, because, as one can see immediately from (1.3) the fields $\vec{X}$ are massless and thus have infinite correlation length.

More generally, one can plug-in any relativistic field theory, massless or massive. Let me write down explicilty an example of the generator of the effective gravity the familiar two-dimensional sin-Gordon model immersed to a general relativity background

$$A\left[g_{ab}, \varphi\right] = \frac{1}{2}\int \left(g^{ab}\partial_a\varphi\partial_b\varphi - \frac{m^2}{\beta^2}\cos(\beta\varphi)\right)\sqrt{g}d^2x$$

From the point of view of effective gravity we have to disinguish the "heavy" matter theories, which have correlation length (or inverse mass scale) much less then the characteristic scale L of the surface itself $R_c \ll L$, and the "light" matter theories, either massless or having the inverse mass scale compatible with the scale L. With "havy" theories the situation is simple. From the scale L the induced action is almost local and we're back to the long wave expansion. As a result, the "heavy" matter contributes only to the cosmological term and topological Einstein action. Further local terms are less relevant. For "light" matter the situation is much more complicated: the action is no more local and much less universal. Very important simplifications taking place in two dimensions with a special matter content are considered in the next subsections. To this order I remind few facts about conformal field theory (CFT).

5. Conformal matter. Among two-dimensional relativistic field theories there is a class of massless theories which are scale covariant, i.e. they don't have any distinguished mass scale and behave self-similarly as the scale changes. Typically such theories posess, in addition to ordinary relativistic and scale covariance, much higher conformal symmetry, which in 2D can be enlarged to infinite dimensional Virasoro symmetry. Such theories are called the conformal theories. Well familiar examples of conformal theories are the two dimensional free bosonic and fermionic fields

$$\begin{aligned} L &= \frac{1}{2}\left(\partial_a \varphi\right)^2 \qquad\qquad c = 1 \\ L &= i\bar{\psi}\gamma^a\partial_a\psi \qquad\qquad c = 1/2 \end{aligned}$$

These fields are free and it is not a big deal to treat them explicitly. There are however interacting non-trivial conformal theories. Due to their enlarged symmetry, conformal theories are studied much better then general relativistic field theories. Many conformal field theories are constructed explicitly.

All conformal theories are characterized by certain number c called the central charge and a set of local observables which are called the primary fields $\{\Phi_i, \Delta_i\}$ with their characteristic "dimensions" Δ_i. This dimensions describe variations of the fields Φ_i with respect to the scale transformations

$$\mathrm{CFT} = \left\{ \begin{array}{ll} c & \text{charge centrale} \\ \{\Phi_i, \Delta_i\} & \text{champs primaires} \end{array} \right.$$

One of the most important properties of conformal theories is the very explicit and simple way they are coupled to curved spacetime background and simplified reaction on the variation of the metric background. This in order is due to the following simple statements [1]

Stress tensor anomaly. Consider the stress tensor as a succeptibility of the system with respect to the variations of the background metric

$$\delta\mathcal{A}[g_{ab}] = -\frac{1}{4\pi}\int T_{ab}(x)\delta g^{ab}(x) g^{1/2}(x) d^2x$$

In 2D CFT the trace of the stress tensor $T_{ab}(x)$ is in fact a c-number (i.e., proportional to the identity operator) and reads explicitly

$$\theta(x) = g^{ab}T_{ab} = -\frac{c}{12}R + \mu \text{ (cosm. constant)} \tag{1.4}$$

where R is the scalar curvature of the background metric and c is precisely the central charge mentioned above.

Primary fields. The primary ϕ_i fields mentioned above vary very simply (here the primary fields are supposed to be scalars)

$$\delta\Phi_i(x) = -\Delta_i \Phi_i(x) \delta\sigma(x) \tag{1.5}$$

under the Weil variations of the metric

$$\delta g_{ab}(x) = g_{ab}(x)\delta\sigma(x) \tag{1.6}$$

All other local fields behave less simple but all of them can be constructed as the operator product expansions of primaries and the nontrivial stress tensor components. Especially simple are the so called rational CFT's which involve only finite number of primary fields (and therefore contain finite spectrum of dimensions Δ_i).

Conceptually these two properties (which were in fact abstracted from explicit calculations in certain simple examples like free field theories) are enough to develop the whole structure as rich as the conformal field theory. Moreover, the basic properties (21.39) and (1.5) can be taken as the very definition of the conformal field theory [2].

6. Liouville action. Simple and universal reaction of conformal theories to the variations of the Weil factor leads to very simple and universal form of the effective action of gravity generated by conformal matter, which is called the Liouville action.

$$\{g_{ab}\} = \left\{ \begin{array}{c} \text{finite dimensional} \\ \text{moduli space} \end{array} \right\} \otimes \{\text{Weil factor } \sigma\}$$

If we fix certain "background" metric $\widehat{g}_{ab}$ then

$$A_{\text{eff}}[g] = A_{\text{eff}}[\hat{g}] + A_{\text{L}}[\sigma, \hat{g}]$$

where

$$A_{\text{L}}[\sigma, \hat{g}] = -\frac{c}{96\pi} \int \left(\hat{g}^{ab} \nabla_a \sigma \nabla_b \sigma - 2\sigma \hat{R} \right) \hat{g}^{1/2} d^2x + \mu \int e^{\sigma} \hat{g}^{1/2} d^2x$$

There are two relevant remarks. Being an effective action of massless field theory the effective action is apparently non-local. It is a good chance that the Weil factor enters this action in a formally local way, and this opens a possibility to interprete the action as the one of local field theory. Second, the background metric for any given complex structure can be chosen at will, in particular in some cases possessing some symmetries simplifying the treatment. In the case of sphere for example it can be taken the maximal symmetric metric of sphere. Another convenient possibility is related to the fact that once except for one point the sphere can be globally mapped on the infinite plane, where the background metric can be taken flat $g_{ab} = \delta_{ab}$. Although this map is singular at one point, the flat background metric opens a possibility to use methods of field theory in flat space. In particular, with this choice the Liouville action reads

$$A_{\text{L}}[\sigma] = -\frac{c}{96\pi} \int (\partial_a \sigma)^2 d^2x + \mu \int e^{\sigma} d^2x$$

Here we added local cosmological term generated either by "heavy" modes or by local regularisation procedure. The extremum is achieved at the fields satisfying the well known Liouville equation

$$\frac{c}{96\pi}\Delta\sigma = \mu e^{\sigma} \quad \Longrightarrow \quad R = \text{const}$$

In this equation c is the sum of all central charges of all conformal components of the matter living over the surface.

7. Quantization and Liouville gauge. In the framework of the Feynmann path integral approach, the quantization of the gravity introduces the functional integrals of the form

$$Z_{\text{g}} = \int D[g_{ab}] \exp\left(-A_{\text{eff}}[g_{ab}]\right) \tag{1.7}$$

??where the integration is over all metrics over the surface modulo the diffeomorphism equivalent g_{ab}'s. To get rid of this redundancy, we need the gauge fixing. One of the possibilities particulary useful in 2D gravity is to choose the metric in the form (21.35) (the conformal gauge). As usual, the gauge fixing introduces the Faddeev-Popov determinant, which is needed to preserve the gauge invariance (general covariance in this case) of the measure. As it was first pointed out by Polyakov 1981 when fixing the conformal gauge in quantum theory the Faddeev-Popov determinant is in order expressed in terms of Liouville action with a specific negative value of central charge $c_{\text{gh}} = -26$. Together with the matter contribution above this gives

$$A_{\text{eff}} = \frac{26-c}{96\pi}\int (\partial_a\sigma)^2\, d^2x + \dots$$

The functional integral (**??**) aquires the form

$$Z_{\text{g}} = \int D[\sigma] \exp\left(-\frac{26-c}{96\pi}\int (\partial_a\sigma)^2\, d^2x + \dots\right) \tag{1.8}$$

There is certain problem with the integration measure $D[\sigma]$ over the Liouville field configurations. Complete definition of the path integral (1.8) requires ultraviolet cutoff, which, from the physical point of view, must depend itself on the the scale factor $\exp(\sigma)$. This means that the integration measure differs from the ordinary (linear) integration measure where the cutoff is defined with respect to certain fixed metric. The direct evaluation of (1.8) with this non-linear measure turns out quite difficult both technically and conceptually. However, in 1988 it was suggested by F.David and J.Distler&H.Kawai [3, 4] that the effect of this complicated non-linear measure can be reduced to certain finite renormalization of the parameters. This means that in (1.8) ordinary linear measure (with respect to fixed reference metric) can be consistently used once the parameters in $A_{\text{L}}[\sigma]$ are chosen properly. Then the renormalized parameters can be determined from the consistency conditions. This assumption is not in fact well justified theoretically. The only serious support might come from actual calculations in this framework and comparison of the results with other known facts in 2D quantum gravity. Among them are the results of the discrete, or matrix model approach (see e.g. the review [5] and references therein) and the field-theoretic calculations in the different gauge (called the light-cone or Polyakov gauge) [6].

This quantum theory with linear integration measure and renormalized parameters is again a CFT and is called the Liouville theory. To conform with the standard notations in quantum Liouville theory let's parameterize the Liouville central charge c_{L} in terms of another parameter b as

$$c_L = 1 + 6\left(b + b^{-1}\right)^2$$

and renormalize the Liouville field σ as $\sigma \to 2b\phi$ so that now the field $\exp(2b\phi)d^2x$ is interpreted as the volume form. The renormalized Liouville field theory action now reads

$$A_L[\phi] = \int \left[\frac{1}{4\pi}(\partial_a\phi)^2 + \mu e^{2b\phi}\right] d^2x \tag{1.9}$$

Parameter b is related to the matter central charge c through the "central charge balance" relation

$$c_L = 26 - c \tag{1.10}$$

The spectrum of primary fields in this CFT consists of exponential operators $\exp(2a\phi)$ with continous parameter a and corresponding dimensions

$$\Delta_a = a(Q - a)$$

Here

$$Q = b + b^{-1}$$

is often called the background charge.

8.Perturbed CFT's. A very important observation in 2D relativistic field theory is that relativistic non-conformal field theories can be generated as perturbations of confomal theories by certain primary operators with $\Delta \leq 1$ called relevant. Formally one writes

$$A\left(\lambda_i\right) = A_{\text{CFT}} + \lambda_1 \int \Phi_1 d^2x + \lambda_2 \int \Phi_2 d^2x + \ldots \tag{1.11}$$

where λ_i are in general dimensional coupling constants. It is not known if every relativistic field theory can be described in this way. There are some examples (e.g., sigma models) where such description cannot be taken literally and at minimum needs some important modifications. Nevertheless there are also many interesting and practically important theoris where this formalism has perfect sense and may be successfully applied.

9. Perturbed CFT coupled to Liouville gravity. Consider the perturbed CFT (1.11) in the curved background metric g_{ab} (21.35). The perturbed action now reads

$$A[g, \lambda] = A_{\text{CFT}}[g] + \lambda \int \Phi\sqrt{g}d^2x \tag{1.12}$$

where $A_{\text{CFT}}[g]$ stands for the CFT action in the background and for simplicity we restict to a single perturbation by a primary field Φ of dimension Δ. In the conformal gauge (21.35) the simple transformation properties (**??**) allow to "flatten" the metric, i.e., to rewrite (1.12) as

$$A[g,\lambda] = A_{\mathrm{CFT}} - \frac{c}{96\pi}\int (\partial_a\sigma)^2 d^2x + \mu\int e^{\sigma}d^2x + \lambda\int \Phi e^{(1-\Delta)\sigma}d^2x \tag{1.13}$$

where A_{CFT} is for the CFT action in "flat" $\sigma = 0$. If we want now to quantize the geometry as before, we have to include again the ghost contribution to the Liouville action and repeat the Distler-Kawai-David renormalization prescription. The (1.13) becomes

$$A[\phi,\mu,\lambda] = A_{\mathrm{CFT}} + A_{\mathrm{L}}[\phi] + \lambda\int \Phi e^{2a\phi}d^2x$$

where self-consistency condition requires the dimension Δ_a of the "dressing" Liouville exponential in the interation term to satisfy the "dimensional balance"

$$\Delta + \Delta_a = 1 \tag{1.14}$$

which can be regarded as an equation to determine the dressing parameter a.

Take for example the gravitational partition sum of the perturbed CFT

$$Z_{\mathrm{g}}(\mu,\lambda) = \int D[\phi]\exp\left(-A_{\mathrm{CFT}} - A_{\mathrm{L}}[\phi] - \lambda\int \Phi e^{2a\phi}d^2x\right)$$

Formal development in λ gives

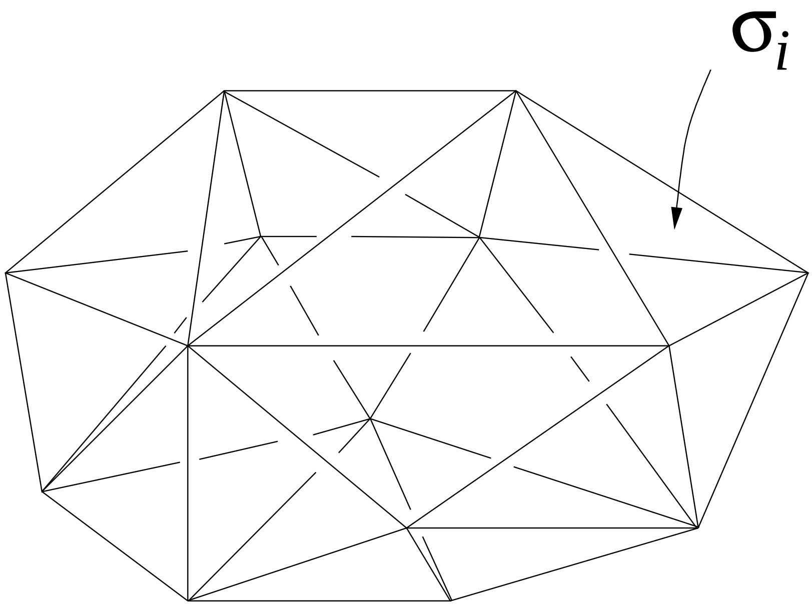
$$Z(\mu,\lambda) = Z(\mu,0)\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}\int_{x_1,\ldots,x_n}\langle\Phi(x_1)\ldots\Phi(x_n)\rangle_{\mathrm{CFT}}\left\langle e^{2a\phi(x_1)}\ldots e^{2a\phi(x_n)}\right\rangle_{\mathrm{L}} \tag{1.15}$$

where $\langle\ldots\rangle_{\mathrm{CFT}}$ and $\langle\ldots\rangle_{\mathrm{L}}$ stand for the correlation functions in ordinary flat CFT and Liouville respectively. In exactly solvable CFT the multipoint functions are known (at least in principle). If the Liouville multipoint functions are also known, this expression gives a constructive tool to evaluate the perturbative development of the partition function. There are very serious reasons to believe that the series in λ in certain cases is convergent and therefore completely determines $Z(\mu,\lambda)/Z(\mu,0)$.

However this expression is not very easy to employ. Even if the multipoint correlation functions are known explicitly, it remains to solve two additional problems

1. Compute the multiple integral over x_i (moduli)
2. Sum up the perturbative series.

In the above considerations we implicitly assumed that the conformal coordinate (21.35) are chosen globally over all the surface. This is only possible on the sphere (with one puncture) and torus. On the surfaces of more complicated topology this cannot be done and some (althoght rather simple) modifications should be made in the above calculations. This will be made in a more accurate manner in the subsequent lectures. In particular, even on the sphere, expansion (1.15) shouldn't be taken literally, since after fixing the conformal gauge, there is still a residual gauge ambiguity, which requires to introduce some ghost insertions



in the correlation functions in (1.15) and reduces the number of integrations in each term from n to $n-3$.

This above consideration also can be easily generalized to treat other observables like correlation functions, and also for the non-compact surfaces with boundaries.

At present there are no observable physical systems whose mathematical description has anything to do with the 2D gravity. I intentially do not mention the string theory, because this enormously popular branch of mathematical physics has precisely the same experimental status as 2D gravity: luck of experimentally observable predictions. Thus, a natural question arises: all these things, what are ther good for?

10. "Experimental" data. There is however a source of, conventionally speaking, "experimental" data for 2D gravity. I'm talking about the results coming from completely different approach to the problem, the approach which is known as the *discrete gravity* or *random lattices* [5]. This framework, conseptually completely different from the field theoretic one, allows to obtain in certain models very detailed exact information. In our more "theoretical" field theory approach, these results can be used as an "experimental" reference point to compare and test the validity of certain assumptions and approximations. In a sense one can view the relation beween the discrete approach to the continuous Liouville gravity (or other field theory aproaches to 2D gravity) in a similay way as that between the lattice statistical systems on the regular lattice (either solvable or not) and the field theoretic continuous descritption of the problem near criticality.

The essence of the discret approach can be seen on the following very simple model. Take, for example, an arbitrary irregular lattice (called more correctly a graph) constructed from N triangles and having the topology of a sphere, like in fig.**??**. Let $\{G_N\}$ be ensemble of such topologicaly different graphs. Now, let's consider a grand partition sum over these ensemble and over N, taking as the statistical weight simply some activiy M attached to each face of the lattice, so the total statistical weight be M^N. A simple but non-trivial generalization is to attach to each site or face of the graph a "spin" variable σ_i or some other degree of freedom to simulate the "matter". The partition sum reads, for example

$$Z_N(K) = N \sum_{\{G_N\}} \sum_{\{\sigma_i\}} e^{K \sum_{\text{nearest neighbours}} \sigma_i \sigma_j} \tag{1.16}$$

As it is, the problem (1.16) seems not easier then the continuos path integral over g_{ab}. To simulate a continuous surface one should take a kind of thermodynamic limit, where the size of the graph N goes to infinity. In this limit calculation of the partition sum seems a complicate problem. Fortunately, for certain choices of statistical weights, there is a powerful machinery which permits to calculate effectively the thermodynamic limit of the sums like (1.16). This is famous matrix models technique. Here I'm not going to go into any details of this interesting theory, see for example the review [5]. I'd only like to mention some important things.

Explicitly (W.Tutte 1962) for $N = 2k$ triangles

$$T(N) = \frac{(4k-3)!}{6k(3k-1)!k!} \sim \frac{1}{8\sqrt{3\pi}} \left(\frac{16}{\sqrt{3}}\right)^N N^{-7/2}$$

$$\frac{\sqrt{4}\,(4)^{4k-3}}{6k\sqrt{3}\,(3)^{3k-1}\,\sqrt{2\pi}\,(k)^{7/2}} \tag{1.17}$$

- **Random lattice Ising model** (RLIM) – spins $\sigma_i = \pm 1$, $i = 1, 2, \ldots N$ attached to faces. Spin energy

$$\mathcal{H}[\sigma_i] = K \sum_{\langle ij \rangle} I_{i,j} \sigma_i \sigma_j + \sum_i H \sigma_i$$

  $I_{i,j}$ – adjacency of faces (through common link), K – parameter (exchange integral), H – magnetic field.

- **Thermodynamics** described by the partition sum

$$Z_N(K,H) = N \sum_{\{G_N\}} \det{}^{-D/2} \Delta^{\text{lat}}_{i,j} \sum_{\{\sigma_i\}} \exp\left(-\mathcal{H}[\sigma_i]\right)$$

  with

$$\Delta^{\text{lat}}_{i,j} = (I_{i,j} - 3\delta_{i,j})'$$

- **Exactly solvable** "pure" model $D = 0$ (V.Kazakov and D.Boulatov, 1986) by *matrix model technique* (E.Brézin, C.Itsikson, G.Parisi and J.-B.Zuber, 1978; F.David, 1985; V.Kazakov 1985; V.Kazakov, I.Kostov and A.Migdal, 1985)

- **Large** $N \gg 1$ are interesting for statistical mechanics. In RLIM ($H = 0$) and $N \gg N_\text{c}$, N_c – "correlation volume" (N – interpreted as space volume). Ordinarily $N_\text{c} \sim 1$

$$Z_N(K) \sim \mathcal{Z}(K) N^{-5/2} e^{-E(K)N}$$

$E(K)$ – specific (per unit volume) free energy

$-5/2$ – "pure gravity" critical exponent

- **Criticality** – occurs at $K = K_\text{c}$ – critical temperature

$$Z_N(K_\text{c}) \sim \mathcal{Z}_\text{c}(K) N^{-7/3} e^{-E(K_\text{c})N}$$

$-7/3$ – "gravitational Ising" critical exponent

- **Crossover** scaling behavior at $|\tau| \ll 1$

$$\tau = \frac{K - K_\text{c}}{K_\text{c}}$$

let

$$N_\text{c} = L_0 \, |\tau|^{-3} \gg 1$$

L_0 – (non-universal) scale parameter.

At $1 \ll N \ll N_\text{c}$ – "Ising" behavior

At $N \gg N_\text{c}$ – "pure gravity" – spins correlate locally, contributing only to $E(K)$

$$Z_N(K) \sim \mathcal{Z}(K_\text{c}) N^{-5/2} e^{-E_\text{reg}(K)N - E_\text{sing}(\tau)N}$$

$E_\text{sing}(\tau)$ – universal contribution of long-range correlations of Ising spins

$$E_\text{sing}(\tau) = e_0 \, |\tau|^3$$

e_0– amplitude of critical singularity (depends on the choice of L_0)

- $N \sim N_\text{c}$ – *crossover scaling function* $F(y)$

$$Z_N(K) \sim F\left(\frac{N}{N_\text{c}}\right) N^{-7/3} e^{-E_\text{reg}(K)N}$$

Properties

$$F(y) \sim \mathcal{Z}_\text{c} \quad y \ll 1$$
$$F(y) \sim F_\infty y^{-1/6} \exp\left(-\pi f_0 y\right)$$

with

$$F_\infty = \mathcal{Z}(K_\text{c}) N_\text{c}^{-1/6}$$
$$\pi f_0 = e_0 L_0$$

- **Explicit scaling** functions in exactly solvable random lattice models. In RLIM

$$F(y) = 3^{2/3}\Gamma(2/3)\mathcal{Z}_{\rm c}\mathrm{Ai}\left(l_{\rm eg}^2(3y)^{2/3}/4\right)$$

Ai(t) – Airy function

$$l_{\rm eg} = 2\gamma(1/3)\gamma^{2/3}(3/4)$$

and

$$f_0 = -\frac{l_{\rm eg}^3}{4\pi} = -0.563...$$

- Loop gas on triangulation

$$Z_N(n, M) = N \sum_{\{G_N\}} \sum_{\rm loops} n^{\#\ \text{of components}} M^{\rm length}$$

Critical behavior near

$$\tau = \frac{M_{\rm c} - M}{M_{\rm c}}$$

determined by ν

$$n = 2\cos\pi\nu$$

$0 \leq \nu \leq 1$. Basic equation (transcendental)

$$x = u + tu^{1-\nu}$$

Scaling function

$$-\nu\frac{\partial^2 Z(x,t)}{\partial x^2} = u^\nu$$

(Ising model at $\nu = 1/3$) Fixed area scaling entire function

$$\frac{Z_a(t)}{Z_a(0)} = \sum_{n=0}^{\infty} \frac{\Gamma(1-\nu)(ta^\nu)^n}{n!\Gamma((\nu-1)(n-1))}$$

(Scaling in regular lattice)	$\leftrightarrow$	("flat" field theory)
(Scaling in random lattice)	$\leftrightarrow$	(2D gravity LFT)

First, using the matrix model approach, one is often able to compute exactly the partition function and other observables. As one could expect from general reasoning, these solutions show critical points wich are accompanied with certain critical behavior in their vicinity. In fact, a wide variety of classes of critical behavior has been observed in the matrix model framework. For many universality classes the critical exponents are computed exactly and, moreover, some scaling functions (which characterize scaling regions near critical points and depend on the scale independent combinations of the coupling parameters) are exactly calculable.

Finally, in most cases it is possible to interpret the critical behavior of the random lattice system in terms of 2D gravity with certain content of critical or perturbed matter. In this way one can verify that the critical exponents calculated in this way are the same as computed, on the other hand, in the field theoretic framework. This supports the conjecture that the continuous limit of the discrete gravity is described by the continuous 2D gravity and, in particular, by the Liouville gravity.

On of the main disadvantages of the matrix model approach (which is by itself is quite beautiful and involves lots of profound mathematics) is that it is not the matrix theorist who determines from the beginning the nature of the graviy and the corresponding matter content to study. Instead, he takes certain matrix model which he can solve and tries to interpret the results in terms of gravity with certain matter content. This is why I call this approach "experimental" and you shouldn't see any disdain in this point of view. The only thing I'd like to stress using this term is that in the matrix model it is not the matrix theorist who has the whole control over the gravity model. Instead it is the matrix model itself who enjoys this after all.

Another important disadvantage of the matrix model approach in its present formulation is, to my opinion, that the outcoming results are automatically coordinate independent (topological) observables like thermodynamic characteristics or integrated correlation functions (better to say the correlation numbers).

On the contrary, in the field theory "theoretic" framework the theorist is free to construct a model with every consistent matter content. This probably gives him a chance to achieve a deeper insight to the structure of two-dimensional gravity. Also, in any field theoretic framework one needs a gauge fixing at certain step and then considers coorinate (and gauge) dependent characteristics. This probably gives access to more detailed information about the structure of the theory.

Anyway, I believe that the field theoretic studies of 2D gravity might help to understand better the main question of quantum gravity: what are the right questions to ask?

2. Outline of the lectures to follow

1. General properties of quantum field theory

- Hamiltonian formalizm vs. Feynmann path integral approach.
- Relation between Minkowskian and euclidean field theory. The Wick rotation to imaginary time.
- Two dimensional free bosons and fermions.
- Relation to lattice models of statistical mechanics near criticality. Non-Lagrangian formalizm and operator product expansions.

- Field theory (euclidean) in the curved background metric.
- Stress tensor as a responce to the variations of the background metric vs. as a generator of coordinate transformations.
- Some peculiarities of the 2D geometry which simplify the differential calculus.

3. Examples of minimal conformal field theories

- Free Majorana fermion.
- Yang-Lee CFT.
- Generalized minimal models
- Conformal bootstrap and structure constants in minimal models.
- Mathematical appendix: Barnes double gamma-function and related special functions.

4. Liouville field theory

- Conformal invariance of Liouville field theory.
- Continuous spectrum of primary exponential fields.
- Degenerate exponentials.
- Conformal bootstrap. Two-point function. Liouville reflection amplitude.
- Conformal bootstrap and three-point function. Structure constants of the operator product algebra.
- General conformal block and four-point function.
- Tower of higher equations of motion.

5. Minimal Liouville gravity

- Liouville partition function.
- Two- and three-point functions in minimal gravity.
- Comparison with the "experimental" matrix model results.
- Higher equations of motion and four-point function.
- Example of non-minimal gravity; numerical results.

6. Dynamical lattice systems and matrix models.

- One matrix model and gravitational Yang-Lee model
- Two-matrix model and gravitational Ising
- Loop gas on a dynamical triangulation

7. "Applied" Liouville gravity

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Lecture 2. General Properties of Quantum Field Theory

3. Hamiltonian formalism vs. Feynman path integral

Traditionally there are two basic formalisms in quantum field theory: the Hamiltonian formalism and the Feynman path integral. In this brief introduction I would like to remind briefly both settlements and try to reveal how they are related to each other. I will always have in mind a field theory in two space-time dimensions, so that we have the time t, and one spatial coordinate x.

3.1. Elements of classical field theory

To define a field theory one chooses a collection of fields, ϕ_i, which are the degrees of freedom in the dynamical problem. The fields are functions of the spatial coordinate, and also depend on the time, so that they are the functions of two variables, $\phi_a = \phi_a(x,t)$. Then, the central object is the action S, which is a functional of the fields, $S[\phi_a(x,t)]$. The action incorporates everything that is to be said about the dynamics of the system, it specifies all interactions between the degrees of freedom. Classical trajectories are the local extremals (stationary points) of the action (here and below I omit the label a for the fields),

$$\phi^{\mathrm{cl}}(x,t) \; : \qquad \frac{\delta S}{\delta\phi(x,t)}\,[\phi^{\mathrm{cl}}] = 0\,. \tag{3.1}$$

Although the term "field theory" is used sometimes in wider sense, here I discuss *local* field theories. Intuitively, that means that the degrees of freedom associated with different, finitely separated, points of the space-time are not allowed to interact directly. The formal statement is that the action is the space-time integral of local lagrangian density ,

$$S = \int \mathcal{L}\left(\phi(x,t), \partial_\mu\phi(x,t)\right)\, dx\, dt \tag{3.2}$$

where $\mathcal{L}\left(\phi(x,t), \partial_\mu\phi(x,t)\right)$ is a function of the fields $\phi(x,t)$ and their first derivatives $\partial_\mu\phi(x,t) = (\partial_x\phi(x,t), \partial_t\phi(x,t))$, taken at the same space-time point (x,t). I will not allow the higher derivatives to enter the Lagrangian density, because in that case there is no universal way to develop canonical formalism needed below. Typical example is a scalar field theory

$$\mathcal{L}\left(\phi, \partial_\mu\phi\right) = -\frac{1}{4\pi}\left(-(\partial_t\phi)^2 + (\partial_x\phi)^2 + V(\phi)\right) \equiv -\frac{1}{4\pi}\left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right) \tag{3.3}$$

with some "potential" $V(\phi)$ (the potential is assumed to be bounded from below). The second form involves the Lorentz pseudometric $\eta^{\mu\nu} = diag(-1,1)$, and makes explicit the invariance of this Lagrangian with respect to the Lorentz transformations of the space-time. In what follows I always have in mind relativistic (i.e. Lorentz invariant), or, in more general context, generally covariant field theories.

3.2. Canonical quantization

The canonical quantization usually goes through the *Hamiltonian formalism.* In the most common setting, one fixes some moment of time, say $t = 0$, and deals with the phase space spanned by the functions

$$(\phi(x), \pi(x)) \tag{3.4}$$

where $\phi(x) = \phi(x, t = 0)$, and $\pi(x)$ represents the associated canonical momenta,

$$\pi(x) = \frac{\partial \mathcal{L}\left(\phi(x), \partial\phi(x)\right)}{\partial\left(\partial_t \phi(x)\right)}\,. \tag{3.5}$$

All elements of the Hamiltonian mechanics are introduced as usual - the phase space is endowed with the standard simplectic form $\Omega = \int \delta\pi(x) \wedge \delta\phi(x)\, dx$, one defines the Hamiltonian H by standard Legandre transform, etc. For instance, in the above example we have

$$H = \frac{1}{4\pi} \int \big(\alpha\, \pi(x)^2 + (\partial_x \phi)^2 + V(\phi)\big)\, dx \qquad \alpha = (2\pi)^2\,. \tag{3.6}$$

Canonical quantization consists of replacing the phase coordinates $\phi(x), \pi(x)$ by operators $\hat{\phi}(x), \hat{\pi}(x)$, acting in a suitable Hilbert space $\mathcal{H}$, and satisfying the canonical commutators

$$\Big[\hat{\pi}(x), \hat{\phi}(x')\Big] = -i\hbar\, \delta(x - x')\,. \tag{3.7}$$

Finding useful representation of these commutators in a field theory can be problematic if one sticks to mathematically rigorous approach (which I do not). For intuitive consideration, one can think of $\mathcal{H}$ as the space of functionals $\Psi[\phi(x)]$ of the functions $\phi(x)$, with the metric

$$||\Psi||^2 = \int \Psi^\star[\phi]\, \Psi[\phi]\;\, D[\phi]\,. \tag{3.8}$$

The integration here is over the space of functions $\phi(x)$. Generally, such integrals require careful definitions, but at the moment I am ignoring such subtleties, relegating more detailed discussion to the future. Anyway, the operators $\phi(x)$ act by the multiplication

$$\hat{\phi}(x) * \Psi[\phi(x)] = \phi(x)\Psi[\phi(x)] \tag{3.9}$$

while $\pi(x)$ are represented by the functional derivatives

$$\hat{\pi}(x) * \Psi[\phi(x)] = -i\hbar \frac{\delta}{\delta\phi(x)} \Psi[\phi(x)] \tag{3.10}$$

The time evolution is controlled by the canonical Hamiltonian operator $\hat{H}$, e.g. for the above example

$$\hat{H} = \frac{1}{4\pi} \int \left((2\pi\,\hbar)^2 \frac{\delta^2}{\delta\phi(x)^2} + (\phi_x)^2 + V(\phi) \right) dx \tag{3.11}$$

In the Schrödinger picture the wave-functions are time dependent and evolve through the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi[\phi(x), t] = \hat{H} \Psi[\phi(x), t] \,. \tag{3.12}$$

In terms of the unitary time evolution operator

$$\hat{U}(t) = \exp\left(-\frac{i}{\hbar}\, \hat{H}\, t \right) \tag{3.13}$$

we have

$$\Psi[\phi(x), t] = \hat{U}(t - t')\Psi[\phi(x), t'] \tag{3.14}$$

In the Heisenberg representation the states are time independent, while the quantum fields evolve in time according to

$$\hat{\phi}(x, t) = \hat{U}(t)\hat{\phi}(x)\hat{U}(-t) \tag{3.15}$$

i.e.,

$$i\hbar \frac{\partial \hat{\phi}}{\partial t} = \left[\hat{H}, \hat{\phi} \right] \tag{3.16}$$

The most common objects of analysis in the quantum field theory are the Green's functions (also called the Whiteman functions)

$$\mathcal{G}_n\left((x_1, t_1), (x_2, t_2), ..., (x_n, t_n)\right) = \frac{\langle 0 \mid \hat{\phi}(x_1, t_1)\, \hat{\phi}(x_2, t_2) \ldots \hat{\phi}(x_n, t_n) \mid 0 \rangle}{\langle 0 \mid 0 \rangle}\,, \tag{3.17}$$

where $\mid 0\rangle$ stands for the *vacuum state*, i.e. the ground state $\Psi_0[\phi(x)]$ of the Hamiltonian operator. From the definition of the Heisenberg operators we have for the above Green's function

$$\frac{\langle 0 \mid \hat{\phi}(x_1)\, \hat{U}(t_1 - t_2)\, \hat{\phi}(x_2) \ldots\, \hat{U}(t_{n-1} - t_n)\hat{\phi}(x_1)\ \mid 0 \rangle}{\langle 0 \mid\ \hat{U}(t_1 - t_n)\ \mid 0 \rangle} \tag{3.18}$$

In writing this equation I have used the fact that

$$\hat{U}(t) \mid 0 \rangle = e^{-\frac{i}{\hbar} E_0\, t}\ \mid 0 \rangle \tag{3.19}$$

where E_0 is the vacuum energy,

$$\hat{H} \mid 0 \rangle = E_0\ \mid 0 \rangle \,. \tag{3.20}$$

Generally, the Heisenberg operators $\hat{\phi}(x, t)$ generally do not commute, the order in which they appear inside the Green's function matters. However, the operators $\hat{\phi}(x, t)$ and $\hat{\phi}(x', t)$, taken at the same time commute, since they essentially coincide with the corresponding

Schrödinger operators. More generally, in relativistic field theory the operators $\hat{\phi}(x,t)$ (as well as all local field operators) commute if the separation between the associated space-time points is space-like

$$\left[\hat{\phi}(x,t),\hat{\phi}(x',t')\right]=0 \quad \text{if} \quad (x-x')^2-(t-t')^2>0\,. \tag{3.21}$$

It is intuitively clear why it should be so - the space-like separated events are simultaneous in appropriate Lorentz frame. Deeper meaning of this condition, known as the *local commutativity*, will be discussed later.

3.3. Feynman path integral

In this representation, the time evolution operator $\hat{U}(t)$ is represented through the functional $U_t[\phi(x),\tilde{\phi}(x)]$ of two functional variables, with the action defined as the convolution

$$\hat{U}(t)\Psi[\tilde{\phi}(x)]=\int U_t[\tilde{\phi}(x),\phi(x)]\,\Psi[\phi(x)]\;D[\phi(x)]$$

As in the ordinary quantum mechanics, one can argue (following Feynman, see R.P.Feynman, A.R.Hibbs, "Quantum theory, Path Integrals", McGraw-Hill, 1965) ...) that the functional kernel in the above integral itself admits representation in terms of the *path integral*

$$U_{t_0}[\phi_f(x),\phi_i(x)]=\int\exp\left(\frac{i}{\hbar}\,S[\phi(x,t)]\right)\;D_{\substack{\phi(x,0)=\phi_i(x)\\ \phi(x,t_0)=\phi_f(x)}}[\phi(x,t)] \tag{3.22}$$

where the action is defined as

$$S[\phi(x,t)]=\int_0^{t_0}dt\int dx\,\mathcal{L}\left(\phi(x,t),\partial_\mu\phi(x,t)\right)\,,$$

and the functional integration is over all the *path*, i.e. the functions of two variables $\phi(x,t)$, with the constraint on the initial (at $t=0$) and the final (at $t=t_0$) configurations,

$$\begin{cases}\phi(x,0)=\phi_i(x)\\ \phi(x,t_0)=\phi_f(x)\end{cases}$$

In quantum mechanics, usual definition of the path integral involves splitting the time into many small intervals and approximating the path by discrete sequence of configurations at the discrete moments of time, and then taking the continuous limit by shrinking the sizes of the time intervals while simultaneously increasing their number. This procedure is more or less straightforward in the case of finitely many degrees of freedom. In a field theory, building finite-dimensional approximations of the path integral requires also discretization of the space, thus, roughly speaking, replacing the space-time continuum by something like a lattice. As it turns, in this case the limiting procedure is far more subtle. I am going to come back to this point shortly.

At the moment, let me concentrate on another subtlety. The integrand in (3.22) is a phase factor, with the absolute value equal to one. As the result, the path integral, if converges, never converges absolutely. The integrals which do not converge absolutely are best understood as the limits (in some parameters) of absolutely convergent integrals. The most instructive way to make the path integral absolutely convergent is to make the time a complex variable. In the extreme case, when the time takes pure imaginary values,

$$t = -iy$$

the Minkowski space-time metric becomes the Euclidean metric on $\mathbb{R}^2$,

$$x^2 - t^2 = x^2 + y^2$$

and the relativistic field theory becomes the Euclidean field theory.

3.4. Euclidean field theory

Take again the time-evolution operator $\hat{U}(t)$, and consider it as the function of complex variable t. As usual, we assume that the Hamiltonian operator is bounded from below (otherwise, for instance, there is no vacuum state). It is immediately clear that the operator is well defined at all t in the lower half-plane of the complex t-plane. More precisely, for any two normalizable states $\mid \Psi_1\rangle$ and $\mid \Psi_2\rangle$ the matrix element

$$\langle\Psi_1 \mid \hat{U}(t) \mid \Psi_2\rangle$$

is analytic function of t if $\Im m\, t < 0$. At this point, let us take pure imaginary t, with the negative imaginary part

$$t = -iy\,, \qquad y > 0$$

and define the Hermitian (at real y) operator

$$\hat{T}(y) = \exp\left(-\hat{H}y\right)\,, \tag{3.23}$$

Starting from this moment I omit the factor $\hbar$; in other words, I choose the system of units in which

$$\hbar = 1 \tag{3.24}$$

(Note that we already ignore the speed of light c, i.e. in our system of units $c = 1$). Rephrasing the above statement, the operator $\hat{T}(y)$ is bounded in the right half-plane of the complex y-plane, and its matrix elements (between normalizable states) are analytic in this domain. The unitary operator $\hat{U}(t)$ can be obtained as the analytic continuation of $\hat{T}(y)$ to the imaginary axis $y = it$. It is usually not a good idea to deal with the operator (3.23) at negative y.

The operator $\hat{T}(y)$ is referred to as the "Euclidean time evolution operator", and sometimes as the "transfer-matrix". Analog of the path integral representation exists for this operator (see e.g. R.P.Feynman, "Statistical mechanics, a set of lectures", Benjamin, 1972). One makes the substitution $t = -iy$ everywhere, so that

$$\partial_t \phi \to i\,\partial_y \phi\,, \qquad \int (...)\; dxdt \to -i \int (...)\; dxdy\,.$$

The representation of the kernel $T[\tilde{\phi}(x), \phi(x)]$ has the form

$$T_y[\phi_f(x), \phi_i(x)] = \int e^{-\mathcal{A}[\phi(x,y)]}\, D_{\substack{\phi(x,0)=\phi_i(x)\\ \phi(x,y_0)=\phi_f(x)}} [\phi(x,y)]\,. \tag{3.25}$$

Here the integration now is over the space of functions $\phi(x,y)$ of the Euclidean coordinates (x,y), again, subject to the "boundary conditions" at the "equal-time" lines $y = 0$ and $y = y_0$,

$$\begin{cases} \phi(x,0) = \phi_i(x) \\ \phi(x,y_0) = \phi_f(x) \end{cases},$$

and the "Euclidean action" $\mathcal{A}[\phi]$ is the integral

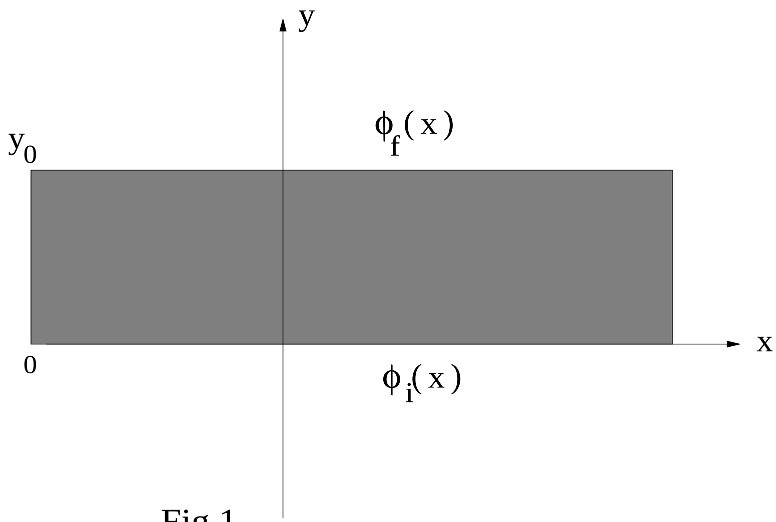

Fig.1

$$\mathcal{A}[\phi] = \int \mathcal{L}_E(\phi(x,y), \partial_\mu \phi(x,y))\, dxdy$$

where the action density is obtained from the original Lagrangian density by the above substitutions,

$$\mathcal{L}_E(\phi, \partial_x \phi, \partial_y \phi) = -\mathcal{L}(\phi, \partial_x \phi, i\,\partial_y \phi)\,.$$

For example, for the scalar field theory

$$\mathcal{L}_E\,(\phi, \partial_\mu \phi) = \frac{1}{4\pi}\left((\partial_x \phi)^2 + (\partial_y \phi)^2 + V(\phi)\right) = \frac{1}{4\pi}\left(\delta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi)\right) \tag{3.26}$$

This functional is explicitly positive if $V(\phi)$ is positive, and if also $V(\phi)$ grows at large ϕ the integral (3.25) appears to be absolutely convergent. This structure of the kernel $T_y[\phi_f, \phi_i]$ is schematically depicted in the Figure 1. The path integral is performed over the fields $\phi(x, y)$ living in the slab $y_0 > y > 0$, with the fixed values at the boundary.

The following observation is important. Consider the kernel $T_y[\phi_f(x), \phi_i(x)]$, and take the limit $y \to \infty$. Since by definition

$$\hat{T}(y) = \sum_n \mid n\rangle\, e^{-E_n\, y}\, \langle n \mid$$

where $\mid n\rangle$ are normalized eigenstates of $\mathcal{H}$ with the eigenvalues E_n, the limit $y \to \infty$ produces the projector onto the vacuum state,

$$\hat{T}(y) \to e^{-E_0\, y} \mid 0\rangle\langle 0 \mid \quad \text{as} \quad y \to \infty\,.$$

Equivalently,

$$T_y[\phi_f(x), \phi_i(x)] \to e^{-E_0\, y}\ \Psi_0[\phi_f(x)]\, \Psi_0^*[\phi_i(x)]\,,$$

i.e. when the slab grows wide, the dependence of the "boundary values" $\phi_i(x)$ and $\phi_f(x)$ factorizes in terms of the ground-state wave functionals $\Psi_0[\phi]$.

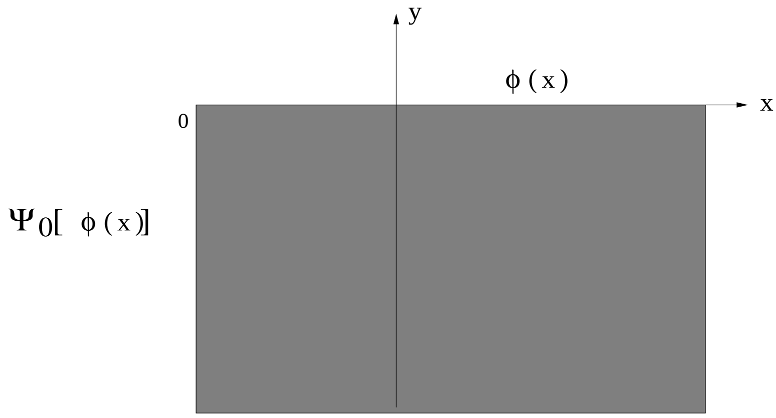

Fig.2

When y becomes very large, we can think of one of the boundaries as departing to infinity. This allows one to associate the vacuum wave functional $\Psi_0[\phi(x)]$ with the functional integral over the fields $\phi(x, y)$ living in the half-plane $y < 0$, with fixed boundary value $\phi(x, y = 0) = \phi(x)$; more precisely, (un-normalized) wave functional is generated by the functional integral

$$\int e^{-\mathcal{A}[\phi]}\ D_{\phi(x,0)=\phi(x)}[\phi(x, y), 0 > y > -Y] \sim e^{-E_0\, Y}\ \Psi_0[\psi(x)]\,, \qquad Y \to \infty \tag{3.27}$$

where Y is (large) y-size of the system. The structure is depicted in Fig.2.

It is also instructive to rewrite the relation

$$\hat{T}(y_1)\hat{T}(y_2) = \hat{T}(y_1 + y_2)\,, \tag{3.28}$$

obvious from the definition (3.23), in terms o the path integrals. It translates as follows:

$$T_{y_1+y_2}[\phi_f(x), \phi_i(x)] = \int T_{y_2}[\phi_f(x), \tilde{\phi}(x)]\, T_{y_1}[\tilde{\phi}(x), \phi_i(x)]\, D[\tilde{\phi}(x)] \tag{3.29}$$

The two slabs associated with T_{y_1} and T_{y_2} are attached one to another by the common line $y = y_1$, and the integration over the boundary values at the junction $y = y_1$ completes the path integral, "soldering" the two slabs to form the thicker slab $T_{y_1+y_2}$, as shown in Fig.3.

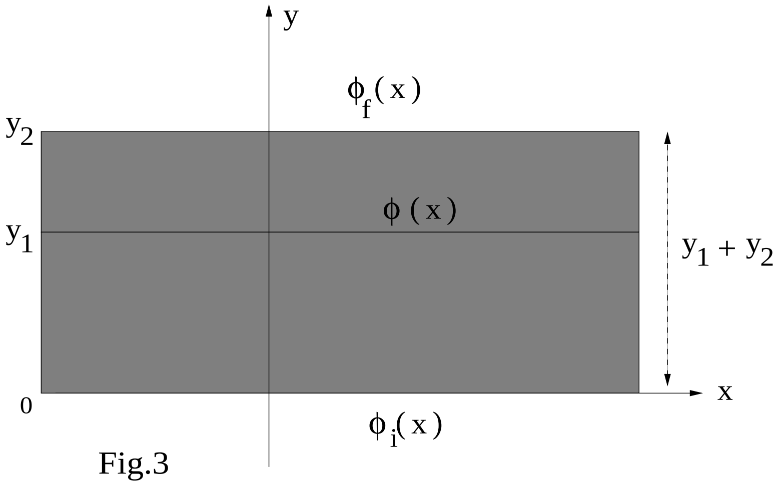

Fig.3

It is very instructive to reconsider the correlation function defined above, Eq.(3.17) from this point of view. One observes, using the representation (3.18), that the correlation function, taken as the function of complex time variables $t_1, t_2, ..., t_n$, is analytic in the domain where the imaginary parts of all separations $t_k - t_{k+1}$ are negative. Taking pure imaginary values of $t_k = -iy_k$,with

$$y_k > y_{k+1}\,, \tag{3.30}$$

we define the Euclidean version of the Green's functions (3.17) (for the reasons to become clear shortly, they are usually called the *correlation functions*)

$$G_n\left((x_1, y_1), ..., (x_n, y_n)\right) = \frac{\langle 0 \mid\ \hat{\phi}(x_1)\,\hat{T}(y_1 - y_2)\,\hat{\phi}(x_2)\,...\,\hat{T}(y_{n-1} - y_n)\,\hat{\phi}(x_n, t_n)\ \mid 0\rangle}{\langle 0 \mid \hat{T}(y_1 - y_n)\ \mid 0\,\rangle}\,. \tag{3.31}$$

This expression acquires even more suggestive form if one uses the Eq.(3.27) which expresses the ground-state wave function as the functional integral over the fields in the half-plane. we have

$$(3.31) = \lim_{\substack{Y_+ \to +\infty \\ Y_- \to -\infty}} \frac{\langle 0 \mid\ \hat{T}(Y_+ - y_1)\,\hat{\phi}(x_1)\,\hat{T}(y_1 - y_2)\,...\,\hat{T}(y_{n-1} - y_n)\,\hat{\phi}(x_n, t_n)\,\hat{T}(y_n - Y_-) \mid 0\rangle}{\langle 0 \mid \hat{T}(Y_+ - Y_-)\ \mid 0\,\rangle}\,. \tag{3.32}$$

The limit exists because the diverging factors $e^{-E_0 Y_+}$ and $e^{+E_0 Y_-}$ coming from (3.27) cancel between the numerator and the denominator.

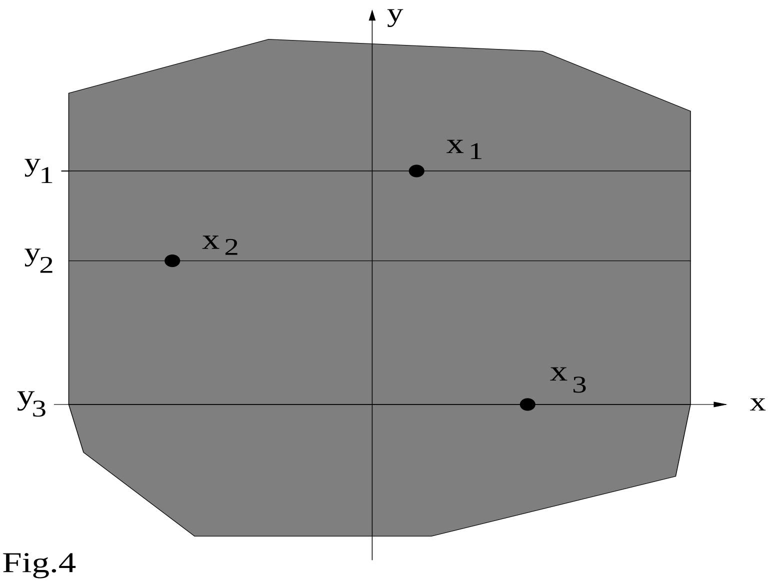

Fig.4

The structure of the last expression corresponds to the following picture (Fig.4). The 2D plane is sliced into slabs $y_k < y < y_{k+1}$, each representing the operator $\hat{T}(y_k - y_{k-1})$, and this stack of slabs is sandwiched between two half-planes associated with the vacuum states at y_1 and y_n. The slabs are joined at the lines $y = y_k$, where the "soldering" integration over the boundary values $\phi_k(x)$ is performed. The operators $\hat{\phi}(x_k)$ are inserted between the slabs. Since these operators act as multiplication by $\phi(x_k)$ (see Eq.(3.9)), the kernel associated with the product

$$\hat{T}(y_{k-1} - y_k)\,\hat{\phi}(x_k)\,\hat{T}(y_k - y_{k+1})$$

is given by the functional integral

$$\int \phi(x_k, y_k)\, e^{-\mathcal{A}[\phi]}\, D_{\substack{\phi(x,y_{k+1})=\phi_i(x)\\ \phi(x,y_{k-1})=\phi_f(x)}}[\phi(x,y), y_f > y > y_i] \tag{3.33}$$

over the functions $\phi(x, y)$ confined to the combined slab $y_f > y > y_i$. The integration over the boundary values $\tilde{\phi}(x)$ at $y = y_k$ "solders" the two slabs, the operator insertion $\hat{\phi}(x_k)$ leaving behind the factor $\phi(x_k, y_k)$ in the integral (3.33). This structure is illustrated in the Fig.5.

Combining these observations, we arrive at the following compact representation of the correlation functions

$$G_n\left((x_1, y_1), ..., (x_n, y_n)\right) = \frac{1}{Z} \int \phi(x_1, y_1)...\phi(x_n, y_n)\, e^{-\mathcal{A}[\phi(x,y)]}\, D[\phi(x, y)], \tag{3.34}$$

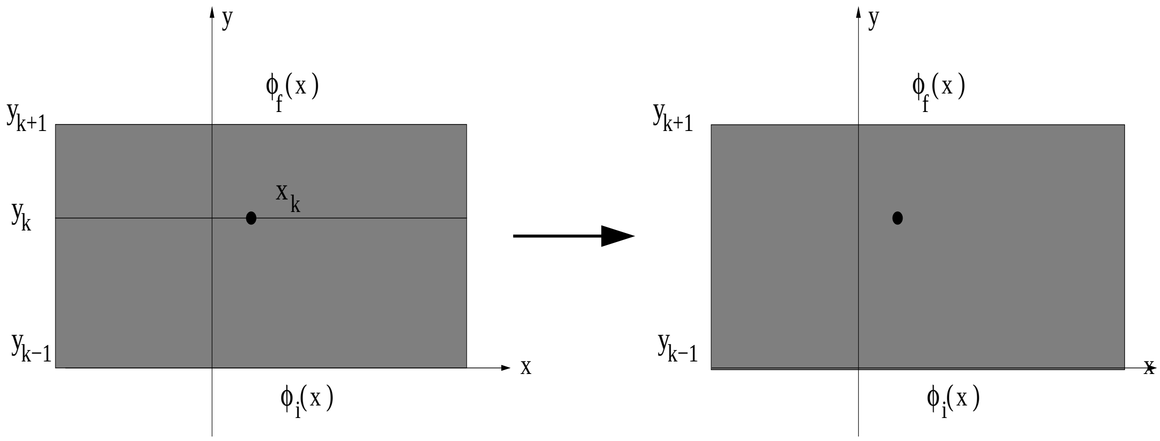

Fig.5

where Z is the "partition function"

$$Z = \int e^{-\mathcal{A}[\phi(x,y)]}\, D[\phi(x,y)]\,. \tag{3.35}$$

In both expressions (3.34) and (3.35) the integration is over the functions $\phi(x,y)$ on the whole plane $\mathbb{R}^2$.

We conclude that the real-time Green's functions of a quantum field theory, being analytically continued to pure imaginary values of time, $t = -iy$, become the correlation functions of the Euclidean field theory. Conversely, the Euclidean correlation functions, defined through the functional integrals (3.34), can be continued to pure imaginary values of one of the Euclidean coordinates, $y = it$; in this domain they reproduce the Green's functions of the real-time quantum field theory.

3.5. Relation to classical statistical mechanics

The representation (3.34), and especially the expression (3.35) for the "partition function", strongly suggest interpretation of the 2D quantum field theory in terms of *classical* statistical mechanics in two-dimensional space. Recall that for a dynamical system with the phase coordinates $\{q_a, p_a\}$ the thermal equilibrium state at a temperature T is described by the Gibbs probability distribution

$$P(q_a, p_a) = Z^{-1} \exp\left(-\frac{H(q_a,p_a)}{kT}\right) \tag{3.36}$$

where $H(q_a,p_a)$ is the (classical) Hamiltonian, k is the Boltzmann's constant, and the partition function Z is given by the statistical integral

$$Z = \int e^{-\frac{H(q,p)}{kT}} \prod_a \frac{dq_a dp_a}{2\pi} \tag{3.37}$$

Typical Hamiltonian has the form (imagine a gas, or a crystal)

$$H(q_a, p_a) = \sum_a \frac{1}{2} p_a^2 + U(q_a)\,, \tag{3.38}$$

with some potential energy $U(q_a)$. Then the momenta p_a in (3.37) are easily inegrated out,

$$Z = (kT/2\pi)^N \int e^{-\frac{U(q_a)}{kT}} \prod_a dq_a \tag{3.39}$$

here N is the number of the degrees of freedom. The integral (3.35) is analogous to the configuration-space integral (3.37), with the Euclidean action playing the role of the potential energy (more precisely, $\mathcal{A}$ is interpreted as U/kT). The analogy becomes more close if we consider a special case, where the degrees of freedom are associated with the "atoms" sitting at the nodes of a two-dimensional crystalline lattice (for instance, q_a might describe small displacements of the "atoms"). Taking for simplicity a square lattice with the nodes labelled by the double index $a = (i, j)$, one can consider a potential energy looking something like this

$$U(q_{i,j}) = \sum_{i,j} \left(K\,(q_{i,j} - q_{i+1,j})^2 + K\,(q_{i,j} - q_{i,j+1})^2 + v(q_{i,j}) \right)\,, \tag{3.40}$$

which combines contributions of the displacement potentials $v(q)$ and the nearest-neighbor interactions. To obtain direct analogy to the theory (3.26), let us renormalize the variables, $q_{i,j} = \sqrt{\frac{kT}{4\pi K}}\,\phi_{i,j}$, so that

$$\mathcal{A}(\phi_{i,j}) \equiv \frac{U(\phi i, j)}{kT} = \frac{\Delta^2}{4\pi} \sum_{i,j} \left(\frac{(\phi_{i,j} - \phi_{i+1,j})^2}{\Delta^2} + \frac{(\phi_{i,j} - \phi_{i,j+1})^2}{\Delta^2} + V_0(\phi_{i,j}) \right)\,. \tag{3.41}$$

where Δ is the lattice spacing. The associated statistical integral

$$\int \exp\left\{-\mathcal{A}(\phi_{i,j})\right\} \prod_{i,j} d\phi_{i,j} \tag{3.42}$$

furnishes the most straightforward discretization of the functional integral (3.35). Moreover, the correlation functions

$$\langle \phi_{i_1,j_1} ... \phi_{i_n,j_n} \rangle\,, \tag{3.43}$$

defined as the product of the variables averaged over the Gibbs ensemble, are of immediate interest in statistical mechanics; obviously, they provide discretized versions of the correlation functions (3.34). Note that the linear integration measure appearing in this example serves as the discrete approximation of the functional measure $D[\phi(x, y)]$ in (3.34), (3.35),

$$\prod_{i,j} d\phi_{i,j} \;\to\; D[\phi(x, y)]\,. \tag{3.44}$$

Some (but not all) important properties of the functional measure $D[\phi(x,y)]$ are abstracted from this approximation. For instance, it is assumed that the functional measure is invariant under shifts of the functional variable,

$$D[\phi(x,y) + C(x,y)] = D[\phi(x,y)]\,, \tag{3.45}$$

for arbitrary function $C(x,y)$.

Naively, we could just take a limit $\Delta \to 0$, expecting to recover the continuous functional integral as the result. This naive procedure (with minor adjustments) proves valid in one-dimensional (one "time", no space) systems, which appear in the path integral approach to ordinary quantum mechanics with finitely many degrees of freedom. However, in two dimensions the situation appears to be far more subtle (it is yet more complex in more then two dimensions). Without special precautions, the naive limit produces a theory in which all correlation functions vanish unless some of the points (x_i, y_i) coincide - one says that the theory has zero correlation length. Generating more interesting result requires taking a *scaling limit*, in which relevant parameters of the Lagrangian are sent to special "critical" values, simultaneously with taking Δ to zero. The situation is understood in terms of Renormalization Group and critical behavior. Interpretation in terms of the classical statistical mechanics is especially useful in this context. I am going to return to this problem later. For now, we just assume that appropriate continuous limit exists and taken, and proceed with formal manipulations with the functional integrals.

3.6. Different Hamiltonian pictures

Instead of starting with the canonical quantization, as we have done in the §1.2 , one can take the Euclidean-space functional integral (3.34) as the very definition of the quantum field theory. In fact, this is the most common contemporary point of view on the quantum field theory. From this perspective, the Hamiltonian formalism, involving the Hilbert space, operators, etc, is just a method of evaluation of the functional integral (3.34). Starting from (3.34), we can *define* the Euclidean-time evolution operator through the kernel $T_{y_0}[\phi_f(x), \phi_i(x)]$, given by the functional integral (3.25) over the fields $\phi(x,y)$ in the slab $0 < y < y$, with the prescribes values at the boundaries. Then, by standard formal manipulations with the functional integral (which are explained in many textbooks, including the above Feynman's monographs) one can show that the operator $\hat{T}(y_0)$ thus defined has the form (3.23), i.e.

$$\hat{T}(y_0) = e^{-\hat{H} y_0}\,, \tag{3.46}$$

with H being the Hamiltonian related to the Lagrangian density in a standard manner (it is important that the Hamiltonian appearing this way is local, i.e. it has the form of the integral over dx of a local energy density). At this point it is useful to recall the relation between the Lagrangian and the Hamiltonian formalism, this time using the language of the Euclidean time. The canonical momentum is defined as

$$\pi(x) = i\,\frac{\partial \mathcal{L}_E\,(\phi, \partial_x\phi, \partial_y\phi)}{\partial\,(\partial_y\phi)}\,, \tag{3.47}$$

and then

$$H[\pi(x), \phi(x)] = \int \left(i\,\pi\partial_y\phi + \mathcal{L}_E\left(\phi, \partial_x\phi, \partial_y\phi\right)\right)\, dx \tag{3.48}$$

where it is understood that the variable $\partial_y\phi(x)$ (the derivative of $\phi(x, y)$ taken at $y = 0$) is excluded from (3.48) in favor of $\pi(x)$, using the Eq.(3.47).

The Euclidean functional integral representation (3.34) has enormous advantage in that it makes it evident that the same field theory may have many different (often inequivalent) operator representations. The Hamiltonian formalism exposed above was based on the choice of one of the Cartesian coordinates (y) as the "Euclidean time"; correspondingly, the space of states $\mathcal{H}$ (the space of functionals $\Phi[\phi(x)]$) was associated with the "equal-time slice" - the infinite line parallel to the x-axis, and the related transfer-matrices $\hat{T}(y)$ "read" the space $\mathbb{R}^2$ slice by slice along the y-direction. This is just one of many possible choices. Trivial possibility is to choose another direction, the y-axis in somewhat rotated Cartesian coordinate system. The resulting operator formalism is identical to the original one, and hardly brings any new insight in the case of the infinite plane geometry (It is important to realize that the operator representations corresponding to different choices of the direction of y-axis are not unitary equivalent; this is in contrast with the real-time operator formalism, where the representations associated with different Lorentz frames are related by unitary transformations. There real time has its advantages, after all.) This possibility brings much new insight in the case of more complex geometry, notably in the important case when the Euclidean space is a cylinder $\mathbb{R}^2/\mathbb{Z}$, as I am intend to discuss in some details later on.

Much more interesting possibility, even in the case of $\mathbb{R}^2$, emerges if one allows non-Cartesian coordinates. In using generic coordinates ξ^μ, it is useful to introduce the metric tensor, so that $ds^2 = dx^2 + dy^2 = g_{\mu\nu}d\xi^\mu d\xi^\nu$. The form of the action depends on $g_{\mu\nu}$,

$$\mathcal{A}[\phi] \to \mathcal{A}[\phi, g]\,, \tag{3.49}$$

enters the Lagrangian density in a covariant way. For instance, the Lagrangian density (3.26) should look like

$$\mathcal{L}_E = \sqrt{g}\ \left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right)\,. \tag{3.50}$$

Instructive example is provided by taking the polar coordinates

$$x = r\,\cos\theta\,, \qquad y = r\,\sin\theta\,. \tag{3.51}$$

There are now two essentially different possibility in choosing the Hamiltonian picture. One is to choose the the angular variable θ to play the role of the Euclidean time - this leads to the so-called *angular quantization*. Another possibility consists of interpreting the radial coordinate r as the "time"; the corresponding approach is known as the *radial quantization*. In both cases, it is convenient to trade the radial coordinate r for its logarithm ρ,

$$r = e^\tau\,. \tag{3.52}$$

This makes the transformation $(x, y) \to (\tau, \theta)$ a conformal one,

$$ds^2 = dx^2 + dy^2 = e^{2\tau}\ \left(d\tau^2 + d\theta^2\right)\,. \tag{3.53}$$

Note that unlike r, the variable τ ranges from $-\infty$ to $+\infty$. The Euclidean Lagrangian density (3.26) the new coordinates becomes (remember, it transforms as a density!)

$$\mathcal{L}_E = \frac{1}{4\pi}\,\left((\partial_\tau\phi)^2 + (\partial_\theta\phi)^2 + e^{2\tau}\,V(\phi)\right)\,, \tag{3.54}$$

and

$$\mathcal{A} = \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} d\tau\ \mathcal{L}_E\left(\phi, \partial_\tau\phi, \partial_\theta\phi\right)\,. \tag{3.55}$$

Now, let me briefly discuss the angular and the radial pictures, using this scalar field theory as the example.

Angular quantization. In the angular Hamiltonian picture, we define the canonical momentum conjugated to $\phi(\tau)$ as (I uses the same notation π for different quantity!)

$$\pi(\tau) = i\,\frac{\partial\mathcal{L}_E}{\partial(\partial_\theta\phi)} = \frac{i}{2\pi}\,\partial_\theta\phi(\tau)\,, \tag{3.56}$$

and then the Hamiltonian

$$K = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\alpha\,\pi(\tau)^2 + (\partial_\tau\phi)^2 + e^{2\tau}\,V(\phi)\,,\right)\,d\tau\,, \qquad \alpha = (2\pi)^2\,. \tag{3.57}$$

Now, again, the canonical quantization consists of promoting the canonical variables $(\pi(\tau), \phi(\tau))$ to the operators, acting in the space of states $\mathcal{H}_{\text{angular}}$ (loosely speaking, the space of functionals $\Psi[\phi(\tau)]$), and obeying the standard commutators

$$\left[\hat\pi(\tau), \hat\phi(\tau')\right] = -i\,\delta(\tau - \tau')\,, \tag{3.58}$$

identical to the ones we had in the standard picture, Eq.(3.7). The angular Hamiltonian (3.57) becomes operator $\mathcal{K}$ acting in the space $\mathcal{H}_{\text{angular}}$, and then, as before, we can define the Euclidean time (angular) evolution operator

$$\hat T(\theta) = e^{-\hat K\theta}\,. \tag{3.59}$$

The associated kernel $T_{\theta_0}[\phi_f(\tau), \phi_i(\tau)]$ is interpreted as the functional integral over the fields $\phi(\tau, \theta)$ within the wedge $0 < \theta < \theta_0$, with the boundary conditions $\phi(\tau, 0) = \phi_i(\tau)\,,\ \phi(\tau, \theta_0) = \phi_f(\tau)$, see Fig.6.

There are two important differences with respect to the Cartesian coordinate picture. One is the dependence, trough the factor $e^{2\tau}$, of the potential term in (3.57) of the spatial coordinate τ. This renders entirely different nature to the two "spatial" infinities $\tau \to \pm\infty$. Massess (conformal) left, "Mass barrier" at the right.

Angular nature of the "time". Trace. For $\theta_1 > \theta_2 ... > \theta_n$

$$\langle\phi(\tau_1, \theta_1)...\phi(\tau_n, \theta_n)\rangle = \frac{\text{tr}\left(\hat\phi(\tau_1, \theta_1)...\hat\phi(\tau_n, \theta_n)\,e^{-2\pi\,\hat K}\right)}{\text{tr}\left(e^{-2\pi\,\hat K}\right)} \tag{3.60}$$

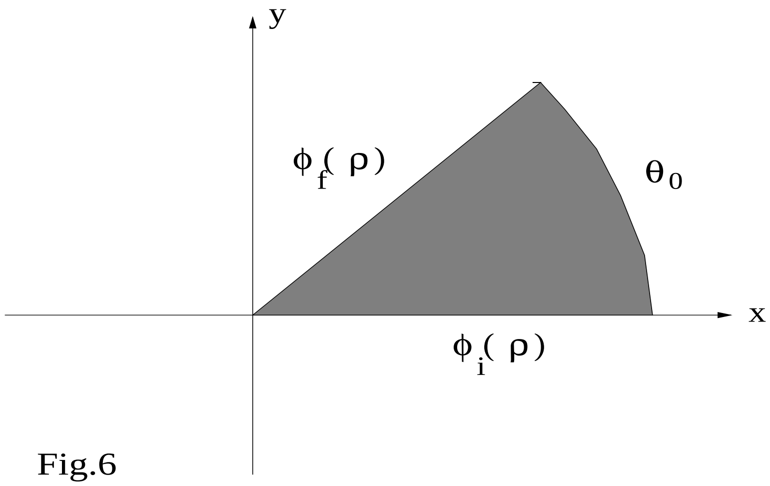

Fig.6

where

$$\hat{\phi}(\tau,\theta) = e^{-\hat{K}\theta}\,\hat{\phi}(\tau)\,e^{\hat{K}\theta} \tag{3.61}$$

Radial quantization. We start with the same action (3.54),(3.55), but this time interpret the radial variable τ as the Euclidean time. The angular variable θ then appears as the spatial coordinate, which now ranges in the finite interval $0 < \theta < \pi$, with the periodicity implied. As usual, we introduce the canonical variables $(\phi(\theta), \pi(\theta))$, which now are assumed to be periodic functions, $\phi(\theta + 2\pi) = \phi(\theta)$ and $\pi(\theta + 2\pi) = \pi(\theta)$, and the Hamiltonian

$$D(\rho) = \frac{1}{4\pi}\int_0^{2\pi} \left(\alpha\,\pi(\theta)^2 + (\partial_\theta\phi)^2 + e^{2\rho}\,V(\phi)\right)\,d\theta\,. \tag{3.62}$$

In the canonical quantization, we introduce the corresponding operators, with the canonical commutators,

$$\left[\hat{\pi}(\theta), \hat{\phi}(\theta')\right] = -i\,\delta(\theta-\theta')\,, \tag{3.63}$$

which act in the space $\mathcal{H}_{\rm radial}$ of the functionals $\Psi[\phi(\theta)]$ of *periodic* functions $\phi(\theta)$.

The most important feature (besides the compactness of the spatial coordinate θ) which distinguishes (3.62) from the Cartesian coordiante case (3.6) is the explicit "time" dependence which enters (3.62) through the factor $e^{-2\tau}$ in front of the potential term.

$$-\partial_\tau\Psi[\phi(\theta),\tau] = \hat{D}(\tau)\Psi[\phi(\theta),\tau]$$

$$\hat{T}(\tau_f,\tau_i) = P\exp\left\{-\int_{\tau_i}^{\tau_f} D(\tau)\,d\tau\right\}$$

Complete set of states $\Psi_j[\phi(\theta)]$. Associate with local operators.

$$\Psi_n[\phi,\tau] = t_{nm}(\rho)\Psi_m[\phi]\,,$$

Generic "time" evolution

4. Formal properties of the functional integral

From the Euclidean point of view, the main object of quantum field theory is a set of its correlation functions, defined through the Euclidean functional integral,

$$\langle\phi(x_1)...\phi(x_n)\rangle = Z^{-1}\int \phi(x_1)...\phi(x_n)\, e^{-\mathcal{A}[\phi(x)]}\, D[\phi(x)]\,. \tag{4.1}$$

From now on, I use the symbol x to denote the points of the two-dimensional space, so that, for instance, in the Cartesian coordinates $x = (\mathrm{x}, \mathrm{y})$. The integration in (4.1) is over the space of functions $\phi(x)$, with $x \in \mathbb{R}^2$. The action $\mathbb{A}[\phi]$ has a local form,

$$\mathcal{A}[\phi] = \int \mathcal{L}_E\left(\phi(x), \partial_\mu\phi(x)\right)\, d^2x\,, \tag{4.2}$$

where the (Euclidean) Lagrangian density $\mathcal{L}_E$ involves the fields $\phi(x)$ and their first derivatives $\partial_\mu\phi(x)$, taken at the same point x; the typical form is

$$\mathcal{L}_E = \frac{1}{4\pi}\,\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right)\sqrt{g}\,. \tag{4.3}$$

In the flat space, in the Cartesian coordinates, the the metric is just $g_{\mu\nu} = \delta_{\mu\nu} = \mathrm{diag}(1, 1)$. However, the above form accommodates for the possibility of more general coordinate systems, and explicitly suitable for addressing the problem of the field theory in general curved space.

The notion of the correlation functions admits natural, and indeed necessary, generalization. Along with the "fundamental" field $\phi(x)$, one can consider various "composite fields", which are local functions of $\phi(x)$, like $\phi^k(x)$. More generally, a composite field may involve derivatives $\partial_\mu\phi(x)$, for example $\phi^k(x)\partial_\mu\phi(x)\partial_\nu\phi(x)$. It is only important that no degrees of freedom located at finitely separated points are allowed. In quantum field theory the composite fields require renormalizations, but here I ignore this problem as well. The collection of such local composite fields can be regarded as a vector space, which I generally denote $\mathcal{F}$. Thus, loosely speaking

$$\mathcal{F} = \mathrm{span}\left\{\phi^k(x), \phi^k(x)\partial_\mu(x), \phi^k(x)\partial_\mu\phi(x)\partial_\nu\phi(x), ...\right\} \tag{4.4}$$

I will generally use the notation $O_j(x)$ for the basic vectors of this space. The expression (4.1) generalizes as

$$\langle O_{j_1}(x_1)...O_{j_n}(x_n)\rangle = Z^{-1}\int O_{j_1}(x_1)...O_{j_n}(x_n)\, e^{-\mathcal{A}[\phi(x)]}\, D[\phi(x)]\,. \tag{4.5}$$

Although defining the functional integral is generally a complex problem, at this point I would like to ignore it, at just proceed on a formal level, assuming some natural (but hard to prove) properties of the functional measure.

Equations of motion. These are what one has to solve in the classical field theory. In quantum theory the equations of motion also play important role - the have the meaning of certain identities for the correlation functions.

Assume that the functional measure is invariant with respect to shifts of the functional variables,

$$D[\phi(x)+c(x)] = D[\phi(x)]\,, \tag{4.6}$$

with arbitrary function $c(x)$. This shift is just a change of the integration variables, and the value of the integral does not change. Assume the function $c(x)$ to be infinitesimal. Then the action changes as

$$\delta\mathcal{A} = \int \delta\mathcal{L}_E(x)\, d^2x = \int \left(\frac{\partial \mathcal{L}_E}{\partial \phi}\, c(x) + \frac{\partial \mathcal{L}_E}{\partial(\partial_\mu \phi)}\, \partial_\mu c(x) \right) d^2x\,. \tag{4.7}$$

If we choose the function $c(x)$ that decays sufficiently fast at the infinity (say, has a finite support), the second term can be integrated by parts,

$$(4.7) = \int R(x)\, c(x)\, d^2x\,, \tag{4.8}$$

where $R(x) \in \mathcal{F}$ is the composite field

$$R(x) = \frac{\partial \mathcal{L}_E}{\partial \phi}(x) - \partial_\mu \frac{\partial \mathcal{L}_E}{\partial(\partial_\mu \phi)}(x)\,. \tag{4.9}$$

Therefore, we have the identity

$$\sum_{k=1}^{n} \langle\, O_1(x_1)...\delta_c O_k(x_k)...O_n(x_n)\rangle - \int d^2x\, c(x)\, \langle\, R(x)\, O_1(x_1)...O_n(x_n)\rangle = 0\,. \tag{4.10}$$

Here $\delta_c O(x)$ is the variation of the composite field under the infinitesimal shift $\phi(x) \to \phi(x) + c(x)$. It is important that in view of the local form of the composite field $O(x)$, the variation $\delta_c O(x)$ may only involve the function $c(x)$ and its derivatives, taken at the point x,

$$\delta_c O(x) = A(x)c(x) + B^\mu(x)\partial_\mu c(x) + ... \tag{4.11}$$

Since (4.10) must hold for any function $c(x)$, we conclude that

$$\langle\, R(x)\, O_1(x_1)...O_n(x_n)\,\rangle = 0 \qquad \text{if} \quad x \neq x_1, x_2, ..., x_n\,. \tag{4.12}$$

The form (4.9) is very familiar from the classical field theory - the Euler-Lagrange equations are just the statement that $R(x)$ must vanish on the field configurations $\phi^{\rm cl}(x)$ which extremize the action $\mathcal{A}$. For example, for the model (4.3)

$$R(x) = \frac{1}{4\pi}\, \left(V'(\phi(x)) - \Delta\phi(x)\right)\,, \tag{4.13}$$

where Δ is the Laplacian. We see that

A field having this property - that any correlation function involving this field vanishes unless the its position x coincides with one of the other insertion points - is called the "redundant field". In what follows I will write

$$R(x) \simeq 0 \tag{4.14}$$

to indicale this property.

The field (4.9) is not the only composite field with this property. Existence of many more redundant fields can be inferred from the following argument. Consider more general transformation of variables

$$\phi(x) \to \phi(x) + c(x) F(\phi(x)), \tag{4.15}$$

where again $c(x)$ is an infinitesimal function, and F depends on the value of ϕ at the point x. Generally, there is no reason to expect the measure $D[\phi]$ to be invariant with respect to such transformations. However, the product form of its discrete prototype

$$\prod_{i,j} d\phi_{i,j}, \tag{4.16}$$

which appeared in the context of the lattice model, strongly suggests the following general form of the measure transformation,

$$D[\phi(x) + c(x) F(\phi(x))] = \exp\left\{ -\int O_F(x)\, c(x)\, d^2x \right\} D[\phi(x)], \tag{4.17}$$

where $O_F \in \mathcal{F}$ is certain local field. It is not clear how to find O_F for a given transformation (4.15). Nonetheless, we can repeat the above analysis, and derive the new redundant field

$$R_F(x) = R(x) F(\phi(x)) + O_F(x). \tag{4.18}$$

It is likely that similar argument holds if the function F in (4.15) involves also derivatives of $\phi(x)$. There are reasons to believe that the the general form (4.17) holds for any local transformation of the field variables, but this is less obvious (systematic analysis should involve arguments based on the Renormalization Group). Existence of corresponding redundant fields can be confirmed in renormalized perturbation theory.

Ward identities. Important identities for the correlation functions come from the symmetries of the theory. Suppose there is a special local transformation of variables

$$\phi(x) \to \phi(x) + \delta_\varepsilon \phi(x), \tag{4.19}$$

(with an infinitesimal parameter ε) which leaves both the action and the measure invariant in virtue of explicit symmetry, i.e.

$$\mathcal{A}[\phi + \delta_\varepsilon \phi] = \mathcal{A}[\phi], \qquad D[\phi + \delta_\varepsilon \phi] = D[\phi]. \tag{4.20}$$

Simple example is the homogeneous translation of $\mathbb{R}^2$, which in terms of the cartesian coordinates x^μ is expressed as

$$x^\mu \to x^\mu + \varepsilon^\mu \,, \tag{4.21}$$

with constant vector ε. The homogeneous action is explicitly invariant. The same can be said about the measure - the transformation (4.21) corresponds to relabelling of the degrees of freedom $\psi(x)$. The corresponding transformation (4.15) is

$$\phi(x) \to \phi(x) + \varepsilon^\mu \, \partial_\mu \phi(x) \,. \tag{4.22}$$

More generally, a symmetry transformation can have the form

$$\delta_\varepsilon \phi(x) = \varepsilon \, E(\phi(x), \partial\phi(x), ...) \,. \tag{4.23}$$

Now, let us make the variable transformation (4.23), with the parameter ε replaced by a (infinitesimal) function $\varepsilon(x)$. Now, the action and the measure are expected to change, but the change must have the local form, i.e. it is described by an extra term $\delta_\varepsilon \mathcal{L}_E(x)$ added to the Lagrangian, which depends on $\varepsilon(x)$ and its derivatives taken at the point x. By the assumption (4.20) the variation must vanish at constant $\varepsilon(x) = \varepsilon$, therefore

$$\delta \mathcal{A}[\phi] = \int \partial_\mu \varepsilon(x) \, J^\mu(x) \, d^2x \,, \tag{4.24}$$

with some local field (the current) $J^\mu(x) = J^\mu(\phi(x), \partial\phi(x), ...)$ (I ignore the higher derivatives of $\varepsilon(x)$. They cannot come out from the variation of the canonical action, but in principle may appear as the result of the variation of the measure. Presence of such terms would not spoil the argument).

The Eq.(4.19) is still a transformation of the functional integration variable, hence the integral (4.5) does not change. By the same arguments that have led to (4.10), we find

$$\sum_{k=1}^{n} \langle \, O_1(x_1)...\delta_\varepsilon O_k(x_k)...O_n(x_n) \rangle - \int d^2x \, \partial_\mu \varepsilon(x) \, \langle \, J^\mu(x) \, O_1(x_1)...O_n(x_n) \rangle = 0 \,, \tag{4.25}$$

where now $\delta_\varepsilon O_k(x)$ are the variations of the inserted fields under this variable transformation.

$$\partial_\mu J^\mu(x) \simeq 0 \,. \tag{4.26}$$

Contact terms and variations of O. Charges as the contour integrals. Conventional integrals of motion.

Lecture 3. Role of the Energy-Momentum Tensor

5. Energy-momentum tensor

The energy-momentum tensor (or stress-energy tensor) plays central role in field theory. The most useful way to define it is in terms of the response of the system to infinitesimal variation of the background metric. We assume that the action $\mathcal{A}[\phi]$ in admits generally covariant extension

$$\mathcal{A}[\phi] \;\to\; \mathcal{A}[\phi, g] \;, \tag{5.1}$$

where g is arbitrary background metric

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu \;, \tag{5.2}$$

such that when g is set back to the metric of a flat space, $\mathcal{A}[\phi, g]$ returns to original form $\mathcal{A}[\phi]$. Since the action is assumed to be local, its variation with respect to g,

$$g_{\mu\nu}(x) \;\to\; g_{\mu\nu}(x) + \delta g_{\mu\nu}(x) \;, \tag{5.3}$$

must have the form

$$\delta\mathcal{A} = \frac{1}{4\pi} \int \sqrt{g}\; \delta g_{\mu\nu}(x)\, T^{\mu\nu}(x)\, d^2x = -\frac{1}{4\pi} \int \sqrt{g}\; \delta g^{\mu\nu}(x)\, T_{\mu\nu}(x)\, d^2x \;. \tag{5.4}$$

This defines the local field $T^{\mu\nu}(x)$- the energy-momentum tensor [1] . By the definition, the energy-momentum tensor is symmetric

$$T_{\mu\nu}(x) = T_{\nu\mu}(x) \;. \tag{5.5}$$

This equation simply reflects our assumption about the possibility of the generally covariant extension (5.1). For instance, for the scalar field $\phi(x)$ we may write

$$\mathcal{A}[\phi] = \frac{1}{4\pi} \int \sqrt{g}\; [g^{\mu\nu}\, \partial_\mu\phi\partial_\nu\phi + V(\phi)]\; d^2x \,. \tag{5.6}$$

In this case, by the above definition

$$T_{\mu\nu} = -\,\partial_\mu\phi\partial_\nu\phi + \frac{g_{\mu\nu}}{2}\, \big((\partial\phi)^2 + V(\phi)\big) \;. \tag{5.7}$$

[1] Note that our normalization of the energy-momentum tensor differs from the conventional one by a factor $1/2\pi$. Thus, in our normalization the energy density is $(1/2\pi)\, T_{yy}$, i.e. the Hamiltonian is

$$H = \frac{1}{2\pi} \int T_{yy}\, dx \;.$$

Of course, the extension (5.1) may not be unique. Correspondingly, there is intrinsic ambiguity in the definition of $T^{\mu\nu}(x)$, even in the flat space, which we will discuss in some details later in this course. For now, let me remind you some general properties which are independent of this ambiguity.

To simplify things, I start with the field theory in the flat background $\mathbb{R}^2$, with usual Euclidean metric

$$ds^2 = \delta_{\mu\nu}\, dx^\mu dx^\nu \ . \tag{5.8}$$

That means that after the variation (5.3) is made we will set

$$g_{\mu\nu}(x) \ \to \ \delta_{\mu\nu} \ . \tag{5.9}$$

5.1. Ward identities

There are special variations of the metric which leave the geometry unchanged, representing the changes of coordinate system. Thus, an infinitesimal coordinate transformation

$$x^\mu \ \to \ x^\mu + \varepsilon^\mu(x) \tag{5.10}$$

induces the variation of the metric

$$\delta g_{\mu\nu} = \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu \ , \tag{5.11}$$

Now let us look at the functional integral with the action (5.1),

$$\langle\, O\, \rangle = Z^{-1} \int O\, e^{-\mathcal{A}[\phi,g]}\, D[\phi]\, . \tag{5.12}$$

Here and often below I use the abbreviation

$$O = O_1(x_1) O_2(x_2) ... O_n(x_n) \tag{5.13}$$

for a product of local fields. For the integration variables $\phi(x)$ the change of coordinates is just a relabelling of the integration variables, which cannot affect the value of the integral [2]. Therefore we have the identity

$$0 = -\langle\, \delta_\varepsilon \mathcal{A}\ O\, \rangle + \langle\, \delta_\varepsilon O\, \rangle \ , \tag{5.14}$$

where $\delta_\varepsilon \mathcal{A}$ is the variation (5.4) associated with the metric variation (5.11). The term with $\delta_\varepsilon O$ describes the effect of the coordinate transform on the insertions $O_k(x_k)$ in (5.12),

$$\delta O = \sum_k O_1(x_1) \cdots \delta_\varepsilon O_k(x_k) \cdots O_n(x_n) \ . \tag{5.15}$$

[2] This implies that the functional measure is defined in a way consistent with this requirement. This property is not at all evident when $D[\phi]$ is understood as the limit of a lattice measures. In a rigorous approach, achieving this property would be highly nontrivial problem.

I will discuss the variations $\delta_\varepsilon O_i$ in detail a little later. Let me just mention here the obvious form of this variation in the case of the Euclidean transformations,

$$\varepsilon^\mu(x) = \delta a^\mu + \delta\theta\, \epsilon_{\mu\nu}\, (x - x_0)^\nu \tag{5.16}$$

i.e. rigid translations and rotations around some point x_0. Here and below $\epsilon_{\mu\nu}$ stands for the unit antisymmetric tensor; in Cartesian coordinates

$$\epsilon_{\rm xy} = -\epsilon_{\rm yx} = 1\ . \tag{5.17}$$

In this case we have

$$O_k(x_0)\ \rightarrow\ e^{i\, s_k\, \delta\theta}\, O_k(x_0 + \delta a)\ , \tag{5.18}$$

where s_k is the spin [3] of the field O_k (as usual, s_k is integer or half-integer, depending on whether O_k is a Bose or Fermi field). Therefore, under the Euclidean variation (5.16) we have

$$\delta_\varepsilon O_k(x_0) = \delta a^\mu \partial_\mu O_k(x_0) + i s_k\, \delta\theta\, O_k(x_0)\ . \tag{5.19}$$

For more general transformations $x^\mu \rightarrow x^\mu + \varepsilon^\mu(x)$ we only need to know at this point that the variations $\delta_\varepsilon O_k(x)$ are themselves local fields, i.e. they are linear combinations of the basic fields $\{O_k\}$ with coefficients which depend on $\varepsilon(x)$. The dependence on $\varepsilon(x)$ must be (a) linear and (b) local. The second statement means that for the variation $\delta_\varepsilon O_k(x)$ the coefficients may involve only ε and perhaps its derivatives taken at the same point x,

$$\varepsilon^\mu(x)\ , \qquad \partial_\mu\varepsilon^\nu(x)\ , \qquad \text{etc}\ . \tag{5.20}$$

Thus we have [4]

$$\delta_\varepsilon O_j(x) = \varepsilon^\mu(x)\, \partial_\mu O_j(x) + \text{terms with derivatives of }\ \varepsilon(x)\ . \tag{5.21}$$

where the explicit form of the first term follows from the fact that for constant ε^μ we have to recover (5.19).

We can now write down the identity (5.14) in more explicit form

$$\frac{1}{2\pi}\int \partial_\mu\varepsilon_\nu(x)\, \langle\, T^{\mu\nu}(x)\ O_1(x_1)\cdots O_N(x_N)\,\rangle\, d^2x = \sum_k \langle\, O_1(x_1)\cdots \delta_\varepsilon O_k(x_k)\cdots O_N(x_N)\,\rangle\ . \tag{5.22}$$

[3] Irreducible tensors and spin; spinors

[4] A bit more generally, we can write

$$\delta_\varepsilon O_k(x) = \varepsilon^\mu(x)\, \partial_\mu O_j(x) + i s_j\, \epsilon_{\mu\nu}\, \partial_\mu \varepsilon^\nu(x)\, O_j(x) + \big(\partial_\mu\varepsilon_\nu(x) + \partial_\mu\varepsilon_\mu(x)\big)\, O_j^{\mu\nu}(x) + \text{ second derivatives of }\ \varepsilon(x)\ .$$

The integration in the first term is over the whole space $\mathbb{R}^2$. Let us split it into two parts

$$\int_{\mathbb{R}^2} = \sum_k \int_{\mathbb{D}_k} + \int_{\bar{\mathbb{D}}} , \tag{5.23}$$

where $\mathbb{D}_i$ are small domains such that $\mathbb{D}_k$ contains x_k but no other of the insertion points, i.e.

$$x_k \in \mathbb{D}_k \ , \qquad \text{but} \qquad x_i \notin \mathbb{D}_k \quad \text{unless} \quad i = k \ . \tag{5.24}$$

and $\bar{\mathbb{D}}$ is the remaining part of $\mathbb{R}^2$ (**Fig.1**),

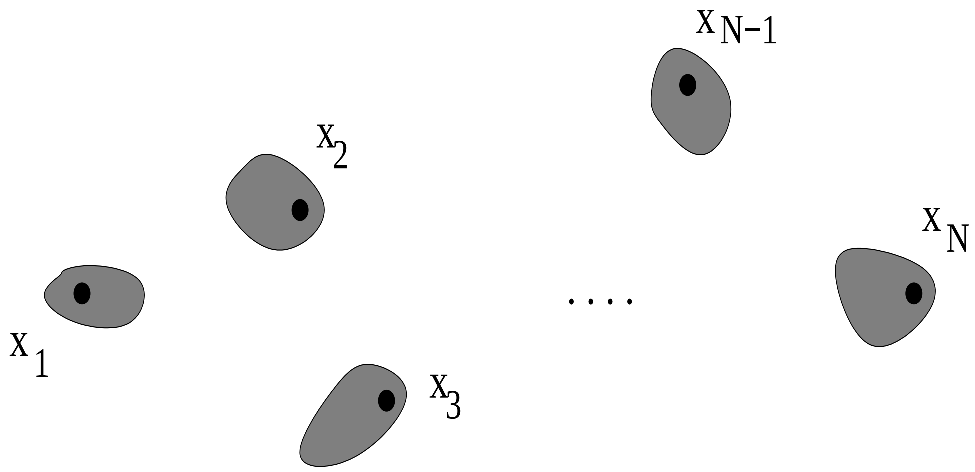

Figure 1:

$$\bar{\mathbb{D}} = \mathbb{R}^2 - \cup_i \mathbb{D}_i \ . \tag{5.25}$$

Let us transform the second integral in (5.23) by parts

$$-\frac{1}{2\pi} \int_{\bar{\mathbb{D}}} \varepsilon_\nu(x) \, \langle \, \partial_\mu T^{\mu\nu}(x) \, O_1(x_1) \cdots O_n(x_n) \, \rangle + \text{boundary terms} \tag{5.26}$$

Note that this is the only term in (5.22) which has dependence on $\varepsilon(x)$ outside the domains $\mathbb{D}_i$. It has to vanish separately from the rest of the terms. Therefore the correlation function in the integrand of (5.26) must vanish when $x \in \bar{\mathbb{D}}$, and since the domains $\mathbb{D}_i$ can be made arbitrary small around x_i this implies that it vanishes everywhere except when x coincides with one of the insertion points $x_1, \cdots, x_n$,

$$\langle \, \partial_\mu T^{\mu\nu}(x) \, O_1(x_1) \cdots O_n(x_n) \, \rangle = 0 \qquad \text{if} \quad x \neq x_1, \ldots, x_n \ . \tag{5.27}$$

In other words, the correlation function vanishes up to delta-function terms supported at the insertion points $x_1, \ldots, x_n$, or, as it is often expressed, it vanishes up to contact terms. We will express this kind of equations which hold up to contact terms as

$$\partial_\mu T^{\mu\nu}(x) \simeq 0 \ . \tag{5.28}$$

With (5.28) established, the bulk term in (5.26) can be dropped, while the boundary terms are calculated using the Stokes formula [5]. As the result, the left hand side of the identity (5.22) takes the form

$$\text{l.h.s. of the Eq.(5.22)} \quad = \quad \frac{1}{2\pi}\sum_{i=1}^{n}\int_{\mathbb{D}_i}\left(\,\cdots\,\right)d^2x \; - \\ \sum_{i=1}^{n}\oint_{C_i}\frac{dx^\mu}{2\pi}\,\varepsilon_\nu(x)\,\langle\,\tilde{T}^\nu_\mu(x)\,O_1(x_1)\cdots O_N(x_n)\,\rangle\;, \qquad (5.29)$$

where the integrations in the second part are over the contours $C_i = \partial\mathbb{D}_i$ - the boundaries of the domains $\mathbb{D}_i$, the direction being such that the interiors of $\mathbb{D}_i$ are always at the left (counterclockwise). In (5.29) and below I use the notation

$$\tilde{T}^\nu_\mu(x) = \epsilon_{\mu\lambda}\,T^{\lambda\nu}(x)\,. \qquad (5.30)$$

Since $\varepsilon^\mu(x)$ can be varied independently in each domain D_i, the equation (5.22) with the l.h.s in the form (5.29) must hold term by term, i.e. the variation of any field $O(x)$ can be expressed as

$$\delta_\varepsilon O(x) = \frac{1}{2\pi}\int_{D_x} d^2y\;\partial_\mu\varepsilon_\nu(y)\;T^{\mu\nu}(y)\,O(x) - \oint_{C_x}\frac{dy^\mu}{2\pi}\,\varepsilon_\nu(y)\,\tilde{T}^\nu_\mu(y)\,O(x) \qquad (5.31)$$

Here again the integration in the second term goes along the $C_i = \partial D_i$ which encircles x in the counterclockwise direction. The r.h.s. here is understood as the relation which applies for the insertions in the correlation functions (or, in terms of the operator product expansions, as I explain later). Note that the first term in (5.31) includes contributions of the contact terms. Let me stress that in view of (5.28) the r.h.s. does not depend on particular choice of D_x; in particular D_x can be made arbitrarily small, in full accord with the postulated local form of $\delta_\varepsilon O(x)$ described above.

Special role is played by *isometries* - the special coordinate transformations which leave the form of the metric tensor unchanged, i.e. the ones with $\partial_\mu\varepsilon_\nu(x) + \partial_\nu\varepsilon_\mu(x) = 0$. For such transformations the first term in (5.31) vanishes, and the variation can be expressed through the contour integral alone. For the Euclidean metric $g_{\mu\nu}(x) = \delta_{\mu\nu}$ those are the Euclidean transformations (5.16), i.e. homogeneous translations and rigid rotations. Comparing (5.31) with (5.19) we have

$$\oint_{C_x}\frac{dy^\nu}{2\pi}\,\tilde{T}^\mu_\nu(x)\,O_i(x) = \partial^\mu O(x)\;, \qquad (5.32)$$

[5] The Stokes theorem:

$$\int_D \partial_\mu A^\mu\,d^2x = \oint_{\partial D}\epsilon_{\mu\nu}\,A^\mu\,dx^\nu\;,$$

where integration in the r.h.s. is over the boundary, in such direction that D is at the right (clockwise, if D is a Disk, and counterclockwise if $D = \mathbb{R}^2 - \text{Disk}$).

and

$$\oint_{C_x} \frac{dy^\lambda}{2\pi\, i} \left[(y-x)^\nu\, T_{\nu\lambda}(y) - (y-x)_\lambda\, T^\nu_\nu(y)\right] O(x) = s\, O(x)\ , \tag{5.33}$$

where s is the spin of the field $O(x)$.

5.2. Energy-momentum and time evolution

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Evolution operator:

Hamiltonian picture - choosing an "equal time slice", a curve Γ (often compact). Let $\Gamma = \partial D$, and let $X^\mu(s)$ be parametric representation of that curve. Space of states is the

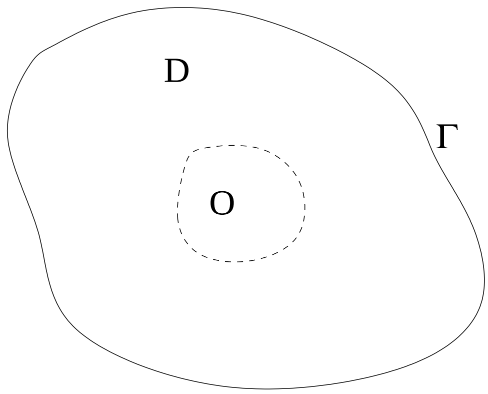


Figure 2:

space of functionals of the boundary values of the field $\phi(x)$ at Γ:

$$\mathcal{H}_\Gamma\ : \qquad \{\Psi[\phi(x)]\} \tag{5.34}$$

Here $\phi(s) = \phi\left(x^\mu = X^\mu(s)\right)$. Functional integral

$$\Psi_\Gamma[\phi(s)] = \int O\, e^{-\mathcal{A}[\phi(x)]}\, D_{\phi(X^\mu(s))=\phi(s)}[\phi(x)] \tag{5.35}$$

taken over the fields $\phi(x)$ with $x \in D$, constrained to having the fixed values $\phi(s)$ at Γ. O is some insertion which defines the state $\Psi \in \mathcal{H}_\Gamma$. The expression (5.35) has the form of (un-normalized) expectation value of O in the presence of boundary, I will write

$$\Psi_\Gamma[\phi(s)]\langle\, O\,\rangle_\Gamma \tag{5.36}$$

to remind about this structure.

Evolution is a change of Γ (with fixed O):

$$\tilde\Gamma : \qquad \tilde X^\mu(s) = X^\mu(s) + \delta X^\mu(s) \tag{5.37}$$

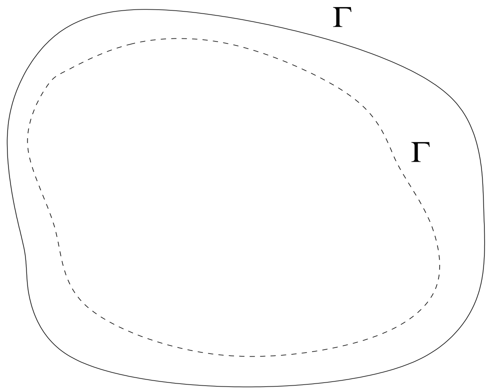


Figure 3:

Consider coordinate transformation (5.10), then

$$\Gamma \to \tilde{\Gamma} \; : \qquad \tilde{X}^\mu(s) = X^\mu(s) + \varepsilon^\mu\left(X(s)\right) \,, \tag{5.38}$$

Ward identity

$$0 = \langle\, \delta_\varepsilon O \,\rangle_\Gamma - \langle\, O \, \delta_\varepsilon \mathcal{A} \,\rangle_\Gamma + \delta_\Gamma \langle\, O \,\rangle_\Gamma \,, \tag{5.39}$$

where

$$\delta_\Gamma \langle\, O \,\rangle_\Gamma = \Gamma \langle\, O \,\rangle_{\tilde{\Gamma}} - \Gamma \langle\, O \,\rangle_\Gamma \tag{5.40}$$

We have, as before

$$\delta_\varepsilon \mathcal{A} = \frac{1}{2\pi} \int_D \left(\partial_\mu \varepsilon_\nu\right) T^{\mu\nu} \, d^2x = \frac{1}{2\pi} \int_D \partial_\mu \left(\varepsilon_\nu \, T^{\mu\nu}\right) \, d^2x \,. \tag{5.41}$$

By the Stokes theorem, the integral reduces to the line integral over the boundary $\Gamma = \partial D$

$$\delta_\varepsilon \mathcal{A} = - \int \mathcal{E}(s) \, ds \,, \tag{5.42}$$

where the "energy density" $\mathcal{E}$ is

$$\mathcal{E}(s) = \frac{1}{2\pi} \, T^{\mu\nu}(s) \, \varepsilon_\mu(s) \, n_\nu(s) \tag{5.43}$$

with $T^{\mu\nu}(s) = T^{\mu\nu}(X(s))$, $\varepsilon_\mu(s) = \varepsilon_\nu(X(s))$, and

$$n_\mu(s) = -\epsilon_{\mu\nu} \, \frac{dX^\nu(s)}{ds} \,. \tag{5.44}$$

Therefore the identity (5.39) reads

$$-\delta_\Gamma \, \langle\, O \,\rangle = \langle \int_\Gamma \mathcal{E}(s) \, ds \; O \,\rangle \,, \tag{5.45}$$

or, more explicitly

$$\delta_\Gamma \, \Psi_\gamma[\phi(s)] = ... \tag{5.46}$$

(From (5c): time evolution)

5.3. Callan-Symanzik equation

.

Dilations and CS equation. RG flow. C-theorem?

5.4. Fixed points and conformal invariance

At fixed point we have $T^\mu_\mu \simeq 0$ (problem of total derivative, as in Polchinski). Signals conformal invariance. Conformal transformations

$$\partial_\mu \varepsilon_\nu(x) + \partial_\nu \varepsilon_\mu(x) = \rho(x)\, \delta_{\mu\nu} = (\partial_\lambda \varepsilon_\lambda(x))\ \delta_{\mu\nu} \tag{5.47}$$

In 2D - analytic transformations

$$z \ \to \ w(z)\,, \qquad \bar{z} \ \to \ \bar{w}(\bar{z}) \tag{5.48}$$

of complex variables

$$z = \mathrm{x} + i\mathrm{y}\,, \qquad \bar{z} = \mathrm{x} - i\mathrm{y} \tag{5.49}$$

Infinitesimal form of (5.48)

$$w(z) = z + \varepsilon(z)\,, \qquad \bar{w}(\bar{z}) = \bar{z} + \bar{\varepsilon}(\bar{z})\,. \tag{5.50}$$

What happens:

• $T = T_{zz}$ and $\bar{T} = T_{\bar{z}\bar{z}}$ are holomorphic fields,

$$\begin{aligned} \partial_{\bar{z}} T(z,\bar{z}) = 0 \quad &\Rightarrow \quad T = T(z) \\ \partial_{\bar{z}} T(z,\bar{z}) = 0 \quad &\Rightarrow \quad \bar{T} = \bar{T}(\bar{z}) \end{aligned} \tag{5.51}$$

(From (5d))

5.5. Operator product expansions in CFT

Radial quantization: Γ-circle, radial coordinates

$$\mathcal{H}_{\text{radial}}: \qquad \{\Psi\,[\phi(\theta)]\} \tag{5.52}$$

Evolution

$$-\,\partial_\tau \Psi[\phi,\tau] = \hat{D}(\tau)\, \Psi[\phi,\tau]\,. \tag{5.53}$$

Logarithmic map

$$z = e^w = e^{\tau + i\theta}\,, \qquad \bar{z} = e^{\bar{w}} = e^{\tau - i\theta} \tag{5.54}$$

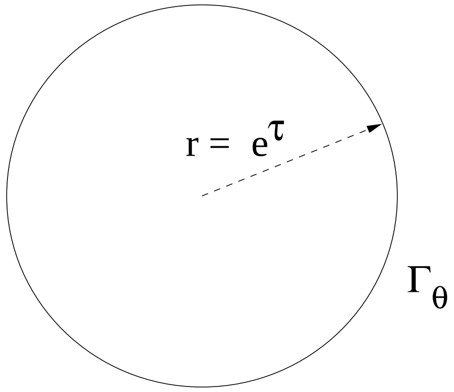


Figure 4:

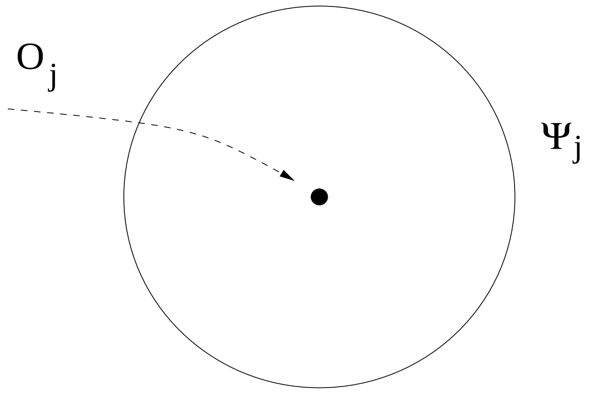


Figure 5:

6. Field theory in a curved space

So far, I have always assumed that the 2D field theory has the Euclidean plane $\mathbb{R}^2$ as the playing stage. In what follows we'll be often dealing with the field theory living on a manifold with some non-trivial metric (curved space). Our final goal will be even to quantize this background metric, i.e., to couple the field theory to two-dimensional (Euclidean) quantum gravity.

Let's consider a 2D manifold $\mathbb{M}^2$ with some Riemann metric described by the metric tensor $g_{\mu\nu}(x)$ which is supposed to be non-degenerate and positive definite (we're dealing with the Euclidean version) so that in any coordinate system $x = x^\mu$ the length interval

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{6.1}$$

is positive. In the curved background the path integral approach allows most straightforward generalization, so we'll take it as the starting point. The field theory is required to be generally covariant, i.e., the action functional, which involves the metric g as the functional

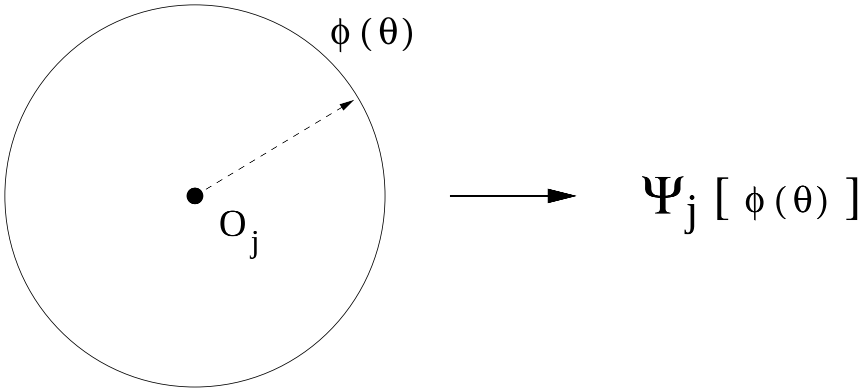


Figure 6:

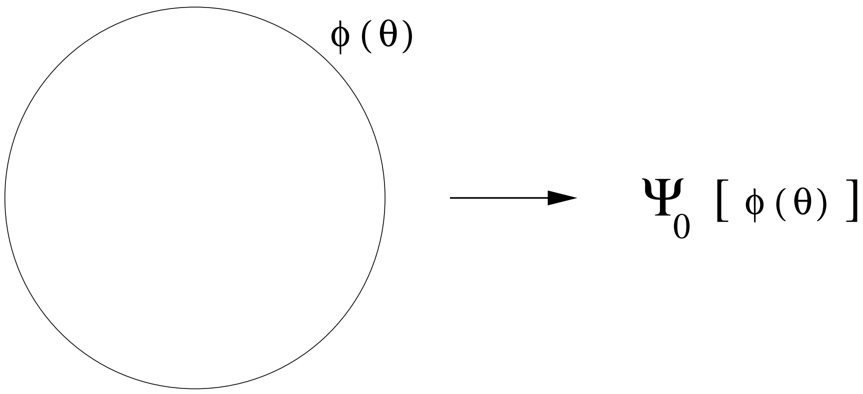


Figure 7:

parameter, $A[\phi(x), g(x)]$, must be invariant under any changes of the coordinates. Again, to be specific, let me write the scalar-field action in the background $g_{\mu\nu}(x)$ as

$$\mathcal{A}[\phi, g] = \frac{1}{4\pi} \int (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi)) \sqrt{g}\ d^2x \tag{6.2}$$

where now $g = \det g_{\mu\nu}$ is the metric tensor on the curved space $\mathbb{M}^2$. This form is manifestly covariant.

6.1. Energy-momentum tensor

. As before, it is defined as the response

$$\delta\mathcal{A} = \frac{1}{4\pi} \int \sqrt{g}\ \delta g_{\mu\nu}(x)\, T^{\mu\nu}(x)\, d^2x \tag{6.3}$$

to the variation of the background metric,

$$g_{\mu\nu}(x) \;\to\; g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\ . \tag{6.4}$$

For coordinate transformations $x^\mu \to x^\mu + \varepsilon^\mu(x)$ we have now (instead of (5.11))

$$\delta g_{\mu\nu} = \nabla_\mu \varepsilon_\nu + \nabla_\nu \varepsilon_\mu\ , \tag{6.5}$$

where ∇_μ is the covariant derivative

$$\nabla_\mu \varepsilon_\nu = \partial_\mu \varepsilon_\nu - \Gamma^\lambda_{\mu\nu}\varepsilon_\lambda\,, \tag{6.6}$$

with $\Gamma^\lambda_{\mu\nu}$ being the usual Riemann-Christoffel connection. Therefore under such variations

$$\delta_\varepsilon \mathcal{A} = \frac{1}{2\pi}\int \sqrt{g}\ \nabla_\mu \varepsilon_\nu(x)\, T^{\mu\nu}(x)\, d^2x\ . \tag{6.7}$$

Repeating the arguments in the previous section, we find that the energy-momentum tensor satisfies the covariant continuity equation

$$\nabla_\mu T^{\mu\nu} \simeq 0\ . \tag{6.8}$$

The equation (5.31) expressing the variation of a local field $O(x)$ generalizes in a straightforward way

$$\delta_\varepsilon O(x) = \frac{1}{2\pi}\int_{D_x} d^2y\, \sqrt{g(y)}\ \nabla_\mu \varepsilon_\nu(y)\ T^{\mu\nu}(y)\, O(x) + \oint_{C_x} \frac{dy^\mu}{2\pi}\ \varepsilon_\nu(y)\, \tilde{T}^\nu_\mu(y)\, O(x)\,, \tag{6.9}$$

where now

$$\tilde{T}^\nu_\mu(x) = e_{\mu\lambda}(x)\, T^{\lambda\nu}(x) = \sqrt{g(x)}\ \epsilon_{\mu\lambda}\, T^{\lambda\nu}(x)\,. \tag{6.10}$$

If the domain is very small, the geometry inside D_x has a approximate killing vector $\varepsilon^\mu(x)$ (constant vector in locally geodesic coordinates), therefore we can write

$$\partial_\mu O(x) = \frac{1}{2\pi}\oint_{C_x} \tilde{T}^\nu_\mu(y) O(x)\, dx^\mu\,, \qquad C_x \quad\to\quad x\,.$$

Note that contrary to the flat case, the limit of the small contour has to be taken (unless the geometry allows for a local Killing vector in a finite domain around x). This equation means that the integral of the appropriate components of the energy-momentum tensor generates shifts of the operator position.

Killing vectors and global symmetries

Peculiarities of 2D geometry. There are three independent components of $g_{\mu\nu}(x)$. Any domain $\mathbb{D} \in \mathbb{M}$, topologically a disk, admits conformal coordinates, such that

$$g_{\mu\nu}(x) = \delta_{\mu\nu}\, e^{\sigma(x)}\,, \tag{6.11}$$

in terms of a function $\sigma(x)$, related to the scalar curvature $R(x)$ as

$$R = -\Delta\sigma = -e^{-\sigma}\,\partial_\mu^2\sigma\,. \tag{6.12}$$

It is then convenient to change to *complex coordinates*

$$z = x^1 + i\,x^2\,, \quad \bar z = x^1 - i\,x^2\,, \tag{6.13}$$

so that

$$ds^2 = e^{\sigma(z,\bar z)}\,dz d\bar z\,. \tag{6.14}$$

Thus, the components of the metric tensor become

$$g_{zz} = g_{\bar z\bar z} = 0\;, \qquad g_{z\bar z} = g_{\bar z z} = \frac{1}{2}\,e^{\sigma(z,\bar z)}\;, \tag{6.15}$$

and, for the inverse tensor

$$g^{z\bar z} = 2\,e^{-\sigma(z,\bar z)}\;. \tag{6.16}$$

In this coordinates, we have from (6.12), we have

$$R = -4\,e^{-\sigma}\,\partial_z\partial_{\bar z}\sigma\;. \tag{6.17}$$

Furthermore, the covariant derivatives

$$\begin{aligned}\nabla_\mu A^\nu &= \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda}A^\lambda\\ \nabla_\mu A_\nu &= \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu}A_\lambda\end{aligned} \tag{6.18}$$

where $\Gamma^\lambda_{\mu\nu}$ are the conventional Riemann-Christoffel connection coefficients

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\,g^{\lambda\rho}\,\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)\,, \tag{6.19}$$

admit very simple form. The only non-zero coefficients are

$$\Gamma^z_{zz} = \partial_z\sigma\;, \qquad \Gamma^{\bar z}_{\bar z\bar z} = \partial_{\bar z}\sigma\;, \tag{6.20}$$

Consider an irreducible symmetric tensor $A^{\mu_1\mu_2..\mu_n}$, such that $g_{\mu\mu_1}A^{\mu_1\mu_2..\mu_n} = 0$. It has only two nonzero components

$$A^{(n)} = A^{\bar z\bar z\ldots\bar z}\,, \qquad \bar A^{(n)} = A^{zz\ldots z}\,, \tag{6.21}$$

and its contravariant components are related to these as

$$A_{(n)} \equiv A_{zz\ldots z} = 2^{-n}\,e^{n\,\sigma}\,\bar A^{(n)}\,, \qquad \bar A_{(n)} \equiv A_{\bar z\bar z\ldots\bar z} = 2^{-n}\,e^{n\,\sigma}\,A^{(n)}\,. \tag{6.22}$$

Using (6.20) we have

$$\begin{aligned}\nabla_z\bar A^{(n)} &= \left(\partial_z + n\,\partial_z\sigma\right)\,\bar A^{(n)} = e^{-n\,\sigma}\,\partial_z\left(e^{n\,\sigma}\,\bar A^{(n)}\right)\,,\\ \nabla_{\bar z}A^{(n)} &= \left(\partial_{\bar z} + n\,\partial_{\bar z}\sigma\right)\,A^{(n)} = e^{-n\,\sigma}\,\partial_{\bar z}\left(e^{n\,\sigma}\,A^{(n)}\right)\end{aligned} \tag{6.23}$$

Conformal transformations are the coordinate transformations which preserve the form (6.11) of the metric, may be changing the function $\sigma(x)$. It is not difficult to check that the conformal transformations are the holomorphic maps of the complex coordinates

$$(z, \bar{z}) \to (w(z), \bar{w}(\bar{z}) \,, \tag{6.24}$$

where $w = w(z)$ and $\bar{w} = \bar{w}(\bar{z})$ are respectively holomorphic and antiholomorphic functions $\bar{\partial} w = \partial \bar{w} = 0$. The scaling factor in the metric transforms as follows

$$e^{\sigma}(w, \bar{w}) = \frac{dz}{dw}\frac{d\bar{z}}{d\bar{w}} e^{\sigma}(z, \bar{z}) \tag{6.25}$$

$$\sigma(w, \bar{w}) = \sigma(z, \bar{z}) - \log\left(\frac{dw}{dz}\,\frac{d\bar{w}}{d\bar{z}}\right) . \tag{6.26}$$

In the Euclidean space time the complex coordinates $(z, \bar{z})$ are complex conjugate, so that to stay with real space the functions w and $\bar{w}$ are conjugate too. From the purely algebraic point of view, as we'll see before long, it doesn't make any difference to consider w and $\bar{w}$ as two independent holomorphic and antiholomorphic ones. Of course, the new coordinates are no more real. Such procedure can be sometimes justified even physically, due to the general analyticity properties of quantum field theory. In particular, if we make back the Wick rotation to Minkowskian space time $t = -iy$ the coordinates x^+ and x^-

$$\begin{aligned} x^+ &= x - t \\ x^- &= x + t \end{aligned} \tag{6.27}$$

are both real light-cone coordinates of the Minkowskian space time. The conformal transformations w and $\bar{w}$ are completely independent (but real analytic) functions of x^+ and x^- respectively.

Energy-momentum tensor. In the conformal complex coordinates, the covariant continuity equation (6.8) for the energy-momentum tensor takes the form

$$\nabla_{\bar{z}} T^{\bar{z}\bar{z}} + \nabla_z T^{z\bar{z}} = 0 \ , \tag{6.28}$$

together with the similar equation with the roles of z and $\bar{z}$ interchanged. Using the above relations, this equation reduces to

$$e^{-2\sigma}\,\partial_{\bar{z}}\big(e^{2\sigma}\,T^{\bar{z}\bar{z}}\big) + e^{-\sigma}\,\partial_z\big(e^{\sigma}\,T^{z\bar{z}}\big) = 0 \ , \tag{6.29}$$

or, in terms of the contravariant components

$$\partial_{\bar{z}} T_{zz} + e^{\sigma}\,\partial_z\big(e^{-\sigma}\,T_{z\bar{z}}\big) = 0 \ . \tag{6.30}$$

7. Conformal field theory

Conformal anomaly. In the flat space, conformal invariance of the theory is manifested by vanishing of the trace of the energy-momentum tensor. This condition generalizes to the curved background through the conformal anomaly equation,

$$T^{\mu}_{\mu}(x) = \alpha\, R(x) \ , \tag{7.1}$$

where α is a dimensionless parameter, which will be connected to the Virasoro central charge as

$$\alpha = -\frac{c}{12} \ . \tag{7.2}$$

Holomorphic energy-momentum pseudotensor. In view of (6.17) we have

$$T_{z\bar{z}} = -\alpha\ \partial_z \partial_{\bar{z}} \sigma \ . \tag{7.3}$$

Therefore in the conformal case (7.1) the continuity equation takes the form

$$\partial_{\bar{z}} T_{zz} - \alpha \left[-\partial_z \sigma \partial_z \partial_{\bar{z}} \sigma + \partial_z^2 \partial_{\bar{z}} \sigma \right] = 0 \ . \tag{7.4}$$

But the second term itself is a $\partial_{\bar{z}}$ derivative,

$$-\frac{\alpha}{2}\, \partial_{\bar{z}} \big(-(\partial_z \sigma)^2 + 2\, \partial_z^2 \sigma \big) \ , \tag{7.5}$$

and hence we have

$$\partial_{\bar{z}} T = 0 \ , \tag{7.6}$$

where

$$T = T_{zz} - \frac{\alpha}{2} \big\{ -(\partial_z \sigma)^2 + 2\, \partial_z^2 \sigma \big\} \, . \tag{7.7}$$

Similar way, one defines the antiholomorphic field

$$\bar{T} = T_{\bar{z}\bar{z}} - \frac{\alpha}{2} \big\{ -(\partial_{\bar{z}} \sigma)^2 + 2\, \partial_{\bar{z}}^2 \sigma \big\} \, , \tag{7.8}$$

which satisfies

$$\partial_z \bar{T} = 0 \ . \tag{7.9}$$

In view of the equation (7.6), this object is usually referred to as the holomorphic energy-momentum tensor. It is not difficult to see however that it does not transform as a tensor, but obeys certain anomalous transformation law. In view of the Eq.(13.9), the additional term

$$t_{zz} = -(\partial_z \sigma)^2 + 2\, \partial_z^2 \sigma \tag{7.10}$$

transforms as

$$t_{zz} = (\partial_z w)^2\, t_{ww} - 2 \big\{ w, z \big\} \ . \tag{7.11}$$

under the conformal transformations

$$z \;\rightarrow\; w(z)\,. \tag{7.12}$$

The last term in (7.11) involves the so called Schwartzian derivative (or simply the Schwartzian)

$$\{w,z\} = \frac{w_{zzz}}{w_z} - \frac{3}{2}\left(\frac{w_{zz}}{w_z}\right)^2 . \tag{7.13}$$

Since T_{zz} is the component of a true tensor,

$$T_{zz} = (\partial_z w)^2\; T_{ww}\,, \tag{7.14}$$

the holomorphic pseudotensor (7.7) must transform as

$$T(z) \;=\; (\partial_z w)^2\, T(w) + \alpha\,\big\{w,z\big\}\ . \tag{7.15}$$

We note that the infinitesimal form of this transformation law reads

$$\delta_\varepsilon T = \varepsilon\,\partial_z T + 2\;(\partial_z \varepsilon)\;T - \alpha\,\partial_z^3 \varepsilon \tag{7.16}$$

Lecture 4. Ising Field Theory

8. Phase transitions and criticality

9. Ising model

2D Ising model is a lattice model of classical statistical mechanics. In its simplest version, it is formulated as follows. Consider (infinite) square lattice with the lattice cites labelled by $\mathbf{x} = (n_1, n_2)$. The degrees of freedom are "spins" $\sigma_{\mathbf{x}}$, associated with the cites, and taking two values,

$$\sigma_{\mathbf{x}} = \pm 1\,. \tag{9.1}$$

The configuration space of the system thus is the collection $\{\sigma_{\mathbf{x}},\ \mathbf{x} \in \text{lattice}\}$ of all these spins (**Fig.1**).

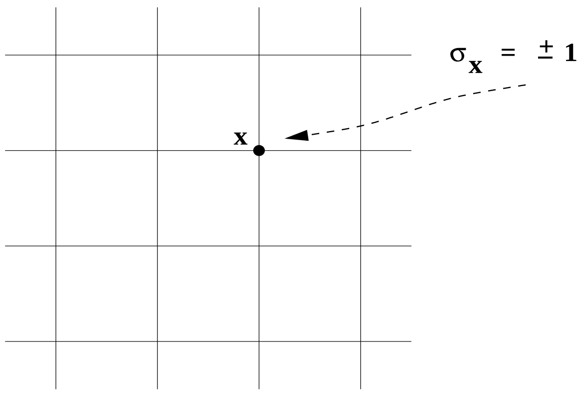


Figure 8:

Statistical mechanics is defined through the Gibbs distribution

$$P\{\sigma_{\mathbf{x}}\} = Z^{-1} \exp\left(-\frac{E\{\sigma_{\mathbf{x}}\}}{kT}\right) = Z^{-1} \exp\left(-\mathcal{A}\{\sigma_{\mathbf{x}}\}\right)\,, \tag{9.2}$$

where $E\{\sigma_{\mathbf{x}}\}$ is the energy functional, and Z is the partition function; I will use the notation $\mathcal{A}$ for the ratio E/kT to comply with the field-theoretic notations. The lattice action $\mathcal{A}$ is chosen to account for the *nearest neighbor* interactions only,

$$E\{\sigma_{\mathbf{x}}\} = -K \sum_{\mathbf{xy}=\text{nn}} \sigma_{\mathbf{x}}\sigma_{\mathbf{y}} - H \sum_{\mathbf{x}} \sigma_{\mathbf{x}}\,, \tag{9.3}$$

where K and H are parameters. I will assume $K > 0$; then the system can be regarded as a ferromagnet. The first term in $\mathcal{A}$ makes it energetically favorable for the neighboring "spins" to aline (i.e. take the same values), wile the last term describes interaction with external field H.

Qualitatively, thermodynamic properties of the Ising model (9.3) are well understood. In the (K, H) plane the system has a line of first-order phase transitions which is located at $H = 0$ and extends from some finite K_c to $+\infty$ (see **Fig.2**). The magnetization

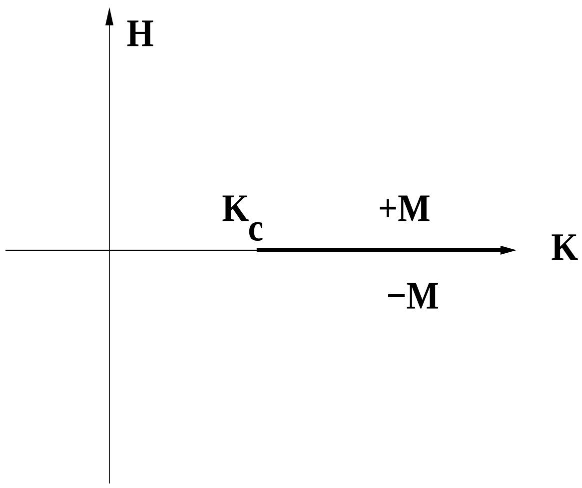


Figure 9:

$$M = \langle\, \sigma_{\mathbf{x}} \,\rangle \tag{9.4}$$

is discontinuous across this line, i.e.

$$M(K > K_c, H = +0) = -M(K > K_c, H = -0) \neq 0\,. \tag{9.5}$$

(but $M(K < K_c, H = 0) = 0$). The transition line ends at the critical point $(K = K_c, H = 0)$. If we restrict attention to the case $H = 0$ (zero external field) the point $K = K_c$ corresponds to the Curie point of the ferromagnet, the point where spontaneous magnetization first appears at sufficiently low temperatures. In terms of the variable K (or T) it is the second order phase transition.

To make quantitative analysis, one would like to find the partition function

$$Z(K, H) = \sum_{\{\sigma_{\mathbf{x}}\}} \exp\left(K \sum_{\mathbf{xy}=nn} \sigma_{\mathbf{x}}\sigma_{\mathbf{y}} + H \sum_{\mathbf{x}} \sigma_{\mathbf{x}} \right), \tag{9.6}$$

and the correlation functions

$$\langle\, \sigma_{\mathbf{x}_1} \cdots \sigma_{\mathbf{x}_n} \,\rangle = \sum_{\{\sigma_{\mathbf{x}}\}} \sigma_{\mathbf{x}_1} \cdots \sigma_{\mathbf{x}_n}\, P\{\sigma_{\mathbf{x}}\}\,. \tag{9.7}$$

The problem (for the partition function) was solved exactly by Onsager in the case $H = 0$. The theory with $H = 0$ reduces to the problem of free fermions on the lattice. I am not going to reproduce full solution of the lattice model here; some version of the solution can be found in virtually any textbook on statistical mechanics. But I will to show how the free-fermion structure emerges, introducing on the way important concept of the "disorder parameter" (which belongs to Kadanoff and Cheva).

So, let us restrict attention to the case of $H = 0$. It is instructive to take first a quick look at the high- and low- temperature expansions of the partition function.

9.1. Order-disorder duality

High-temperature expansion. At high temperatures T we have $K << 1$ and it is meaningful to expand Z in the powers of K. It is convenient to use the identity

$$e^{K\,\sigma_{\mathbf{x}}\sigma_{\mathbf{y}}} = \cosh K\,\Big(1 + \sigma_{\mathbf{x}}\sigma_{\mathbf{y}}\,\tanh K\Big) \tag{9.8}$$

to write

$$Z(K, H = 0) = \cosh^{2N} K\,\sum_{\sigma_{\mathbf{x}}}\,\prod_{\mathbf{xy}=nn}\,\Big(1 + \sigma_{\mathbf{x}}\sigma_{\mathbf{y}}\,\tanh K\Big)\,, \tag{9.9}$$

where N is the total number of the lattice sites ($N \to \infty$ for infinite lattice). Each factor in (1.8) corresponds to certain link of the lattice. If $K \to 0$ the first terms in these factors dominate and Z reduces to the trivial factor $2^N\,\cosh^{2N} K$. The high-temperature expansion in powers of

$$t = \tanh K$$

is obtained by taking the second term $t\,\sigma_{\mathbf{x}}\sigma_{\mathbf{y}}$ from some of the factors (i.e. for some of the lattice links) in (9.9). When the second term is taken, let us mark the associated link by a bold line. The t-expansion then is expressed in terms of graphs which are built from such bold links on the lattice. Since $\sigma_{\mathbf{x}}^2 = 1$ and $\sum_{\sigma=\pm 1}\sigma_{\mathbf{x}} = 0$, the summation over $\sigma_{\mathbf{x}}$ exterminates all odd powers of $\sigma_{\mathbf{x}}$ at the same cite $\mathbf{x}$. Hence only even numbers, i.e. 0, 2 or 4, of the bold links can meet at any lattice cite. The result is the sum of graphs which consist of continuous bold lines on the lattice which are allowed to cross at the "four-vertices", the cites where four of the bold links meet. The graphs are not necessarily connected, but each graph brings contribution

$$t^L \tag{9.10}$$

to the "renormalized" partition function $Z/2^N\,\cosh^{2N} K$, where L is the total length of the bold lines in the graph. Examples of such loops are shown in **Fig.3**.

The above analysis can be repeated for the case of the correlation functions (9.7); the result is that each spin insertion $\sigma_{\mathbf{x}}$ generates the end-point for the bold lines at the cite $\mathbf{x}$, as is depicted in the **Fig.4** in the case of the two-point function.

The bold lines in the graphs can be thought of the the Euclidean-space trajectories of particles. At the first glance, these particles appear to be interacting ones. Indeed, the

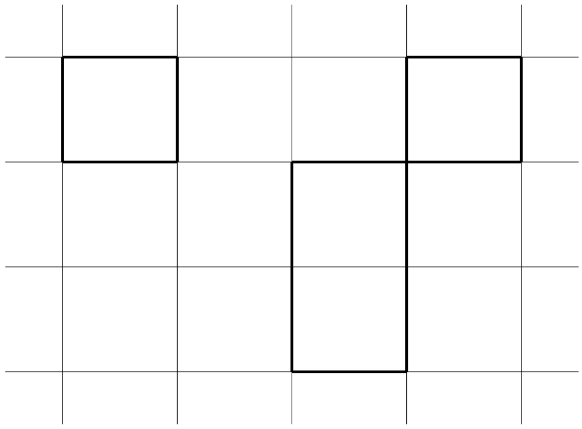


Figure 10:

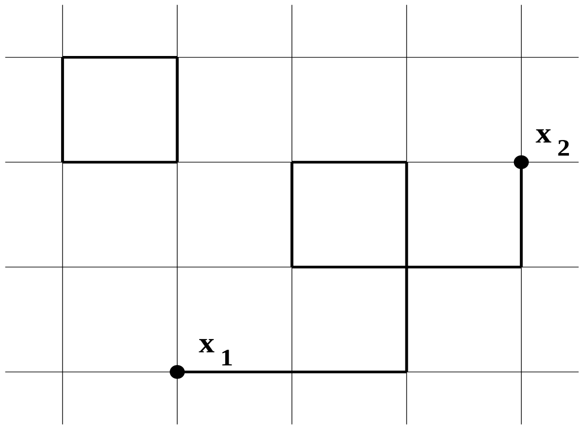


Figure 11:

vacuum trajectories (i.e. the trajectories with no endpoints) of free particles are combinations of closed loops, with the important property that the statistical weight of any combination of the loops is the product of the weights of the individual loops, no matter if they intersect (or self-intersect) or not. This seems not to be the case for the Ising graphs - the "four vertices" seem to represent nontrivial interactions between the particles. Fortunately, the sum of the Ising graphs can be transformed to the sum of non-interacting loops, at the price of giving some of these loops negative statistical weights. As the result the theory reduces to the free *fermionic* particles. I am not going to describe here combinatorics which leads to this result (see e.g. Landau&Lifshitz book on Statistical mechanics). Instead, later on I will derive equivalent result by different approach.

Low-temperature expansion. Now consider the case of low temperature, i.e. $K >> 1$. When T is strictly zero, there are two degenerate ground states of the energy functional (9.2) (with $H = 0$), the one with all spins equal +1 and the one with all spins equal −1. These two states have the same statistical weight $\left(e^K\right)^N$, but they have opposite spontaneous magnetization $M = \pm 1$. Let us concentrate attention at the state with all spins +1. If the temperature is small but not exactly zero (i.e. K is large but not infinite) some contributions to the partition function come from configurations with the majority of the spins equal to +1 but with some small fraction of the spins being equal to −1, as illustrated in Fig.4. These configurations can be given representations in terms of graphs if one introduces the "dual" lattice. The dual lattice is the lattice whose cites are the centers of the faces of the original lattice. For the case of the square lattice (which we stick to) the dual lattice is also a square lattice, see **Fig.5**. The cites of the original lattice are the faces of the dual lattice and vice

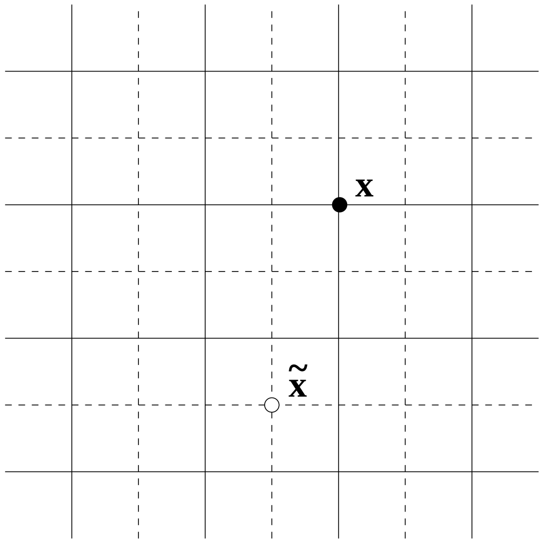


Figure 12:

versa. I will denote $\tilde{\mathbf{x}} = (k_1, k_2)$ (with half-integer k) the cites of the dual lattice. For any spin configuration $\{\sigma_{\mathbf{x}}\}$ on the original lattice, one can draw a graph *on the dual lattice* by applying the following rule. Take any link of the dual lattice. There are two neighboring cites $\mathbf{x}$ and $\mathbf{y}$ of the original lattice immediately at the sides of this link. If the spins $\sigma_{\mathbf{x}}$ and $\sigma_{\mathbf{y}}$ have opposite signs, i.e. if $\sigma_{\mathbf{x}}\sigma_{\mathbf{y}} = -1$, then mark this link bold. On the other hand, if $\sigma_{\mathbf{x}}\sigma_{\mathbf{y}} = +1$, then leave the link blank. Thus all possible spin configurations of the original lattice generate graphs on the dual lattice (**Fig.6**), and it is easy to see that these graphs are exactly of the same type as the graphs we encountered in the high-temperature expansion. Namely, only even number of bold links can meet at any cite $\tilde{\mathbf{x}}$ of the dual lattice, and hence the graphs consist of continuous bold lines with "four-vertices". Moreover, given spin configuration brings contribution

$$\tilde{t}^L \tag{9.11}$$

to the modified partition Z/e^{NK}, where L is the length (i.e. the number of bold links) of

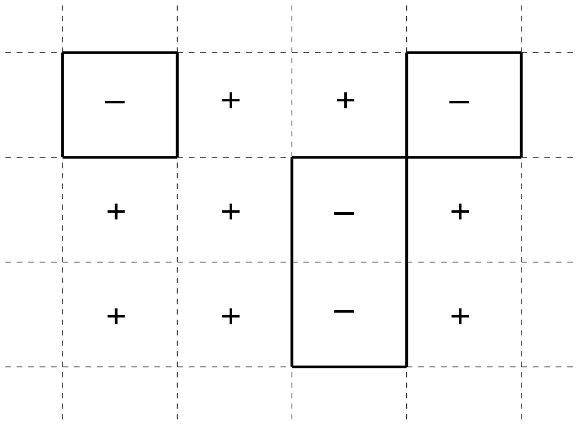


Figure 13:

the associated graph, and

$$\tilde{t} = e^{-2K}\,. \tag{9.12}$$

Indeed, each bold link separates opposite spins, so its statistical weight differs by the factor (1.11) from the statistical weight of the empty link (representing parallel neighboring spins.

Duality. We observe remarkable property of the Ising model: Its thermodynamic properties at low T are related to those at high T, namely

$$\frac{Z(K)}{2^N\,\cosh^{2N}K} = \frac{Z(\tilde{K})}{e^{N\tilde{K}}}\,, \tag{9.13}$$

where $\tilde{K}$ relates to K as

$$e^{-2K} = \tanh\tilde{K}\,. \tag{9.14}$$

Note that from (9.14) follows $e^{-2\tilde{K}} = \tanh K$, so that the relation (9.14) is inversion. Note also that (9.14) relates values of K in the low-T regime to its values in the high-T regime. Assuming that the critical point K_c is unique, it must satisfy $K_c = \tilde{K}_c$, i.e.

$$K_c = \frac{1}{2}\,\log\left(\sqrt{2}+1\right). \tag{9.15}$$

The graphs appearing in the low-T expansion also can be viewed as the Euclidean space-time histories of some particles. Although the graphs have the same structures and the same weights (in terms of the "dual" parameters, that is) as the high-T graphs, the interpretations of the particles are quite different. While the particles appearing in the high-T graphs can be called the spin particles (since the particle can be emitted by a single $\sigma_{\mathbf{x}}$ insertion, the lines in the low-T graphs have different relation to the spin configurations. The low-T graph lines separate domains of the lattice which are "filled" with the spins of the same sign - the "drops". Therefore the particles represented by the lines in the low-T graphs are rather the "kinks" separating degenerate vacua of the opposite magnetization. It is remarkable that despite this very different interpretations the low-T and the high-T particles have the same dynamical properties (they are free fermions, as we will see little later).

Duality as transformation. Let me present formal derivation of this duality relation, since it naturally introduces important notion of the *disorder parameter*. Consider again the partition function (with $H = 0$)

$$Z(K) = \sum_{\{\sigma_{\mathbf{x}}\}} e^{K \sum_{nn} \sigma_{\mathbf{x}}\sigma_{\mathbf{y}}} \,. \tag{9.16}$$

The expression in the sum factorizes in terms of variables

$$g_{\mathbf{xy}} = \sigma_{\mathbf{x}}\sigma_{\mathbf{y}} \,. \tag{9.17}$$

These variables are associated with the links $(\mathbf{xy})$ of the lattice, so that there are $2N$ of them. The complication is that these variables are not all independent, they must satisfy the constraints ("zero curvature conditions")

$$\prod_{\text{polygon}} g_{\mathbf{xy}} = +1 \,, \tag{9.18}$$

where the product involves $g_{\mathbf{xy}}$ associated with all links of any polygon on the lattice - this follows from the definition (9.17). Of course, all these constraints follow from the elementary constraints, associated with elementary polygons - the lattice faces. The latter are labelled by the cites of the dual lattice, so for every $\tilde{\mathbf{x}}$ we have

$$\prod_{\text{around } \tilde{\mathbf{x}}} g_{\mathbf{xy}} = 1 \qquad \text{for} \quad \text{every} \quad \tilde{\mathbf{x}} \,. \tag{9.19}$$

Then the summation in (9.16) can be performed over independent $g_{\mathbf{xy}}$ if we also insert the delta-symbol to enforce all the constraints (9.19),

$$\prod_{\tilde{\mathbf{x}}} \delta\Big(\prod_{\text{around } \tilde{\mathbf{x}}} g_{\mathbf{xy}} = 1 \Big) \,. \tag{9.20}$$

Now, each delta-function in (9.20) can be written as the sum

$$\delta\Big(\prod_{\text{around } \tilde{\mathbf{x}}} g_{\mathbf{xy}} = 1 \Big) = \frac{1}{2} \sum_{n=0,1} \Big[\prod_{\text{around } \tilde{\mathbf{x}}} g_{\mathbf{xy}} \Big]^n \,. \tag{9.21}$$

Indeed, if the product is equal to $+1$, the two terms in the sum add up to 1, but if the product equals -1 these two terms cancel each other. Since there are N delta-functions in (9.20), we will need N additional summation variables $n_{\tilde{\mathbf{x}}}$, one for each cite of the dual lattice.

With this we can write

$$Z(K) = \frac{1}{2^N} \sum_{\{g_{\mathbf{xy}}\}} e^{K \sum_{nn} g_{\mathbf{xy}}} \sum_{n_{\tilde{\mathbf{x}}}} \Big[\prod_{\tilde{\mathbf{x}}} g_{\mathbf{xy}} \Big]^{n_{\tilde{\mathbf{x}}}} \,. \tag{9.22}$$

For fixed configuration of $\{n_{\tilde{\mathbf{x}}}\}$ the expression (9.22) factorizes in terms of the link variables g, i.e. the summation over $\{g_{\mathbf{xy}}\}$ reduces to $2N$ identical sums

$$\sum_{g_{\mathbf{xy}}=\pm 1} e^{K\, g_{\mathbf{xy}}} \left[g_{\mathbf{xy}}\right]^{n_{\tilde{\mathbf{x}}}+n_{\tilde{\mathbf{y}}}} = e^{K} + \big(-1\big)^{n_{\tilde{\mathbf{x}}}+n_{\tilde{\mathbf{y}}}}\, e^{-K}\,, \tag{9.23}$$

where $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ are two cites of the dual lattice adjacent to the link $(\mathbf{xy})$. this expression can be brought to a nicer form by introducing new variables $\mu_{\tilde{\mathbf{x}}} = \pm 1$ instead of $n_{\tilde{\mathbf{x}}} = 0, 1$,

$$n_{\tilde{\mathbf{x}}} = \frac{1}{2}\big(1 - \mu_{\tilde{\mathbf{x}}}\big)\,. \tag{9.24}$$

It is easy to check that (9.23) can be written as

$$e^{K} + \big(-1\big)^{\frac{1-\mu_{\tilde{\mathbf{x}}}}{2}+\frac{1-\mu_{\tilde{\mathbf{y}}}}{2}}\, e^{-K} = e^{K} + \mu_{\tilde{\mathbf{x}}}\mu_{\tilde{\mathbf{y}}}\; e^{-K}\,. \tag{9.25}$$

And it further transforms with the use of the dual parameter $\tilde{K}$ from (9.14), $e^{-2K} = \tanh\tilde{K}$,

$$\frac{e^{K}}{\cosh\tilde{K}}\; e^{\tilde{K}\,\mu_{\tilde{\mathbf{x}}}\mu_{\tilde{\mathbf{y}}}}\,. \tag{9.26}$$

We finally obtain for (9.22)

$$Z(K) = \frac{e^{2NK}}{2^{N}\,\cosh^{2N}\tilde{K}} \sum_{\{\mu_{\tilde{\mathbf{x}}}\}} e^{\tilde{K}\,\sum_{nn}\,\mu_{\tilde{\mathbf{x}}}\mu_{\tilde{\mathbf{y}}}}\,. \tag{9.27}$$

Thus the duality transformation can be understood as certain non-local change of variables in the partition sum, which brings it to the sum over the dual variables $\mu_{\tilde{\mathbf{x}}}$, while the energy functional takes the original Ising form with K replaced by $\tilde{K}$. The variables $\mu_{\tilde{\mathbf{x}}}$ are usually called the "disorder parameter".

I would like to stress that the possibility to make this duality transformation strongly depends on the global Z_2 (spin reversal) symmetry of the theory with $H = 0$. If this symmetry is broken, as in the case of non-zero H, the transformation leads to theory involving special Z_2 gauge fields, and the dual theory becomes much more complicated. I will discuss some of related topics later on.

Properties of the disorder parameter. To understand the nature of the disorder parameter, consider again the partition sum (9.16). Take arbitrary simple closed contour Γ on the *dual* lattice (**Fig.7a**). It splits the original lattice into two parts, the part Λ which consists of the sites inside Γ, and the part $\bar{\Lambda}$ composed of the cites outside Γ. Let us make in (9.16) the following change of variables

$$\begin{aligned} \sigma_{\mathbf{x}} &\to \;\;\sigma_{\mathbf{x}} \qquad \text{if} \quad \mathbf{x}\in\bar{\Lambda}\,, \\ \sigma_{\mathbf{x}} &\to -\sigma_{\mathbf{x}} \qquad \text{if} \quad \mathbf{x}\in\Lambda\,, \end{aligned} \tag{9.28}$$

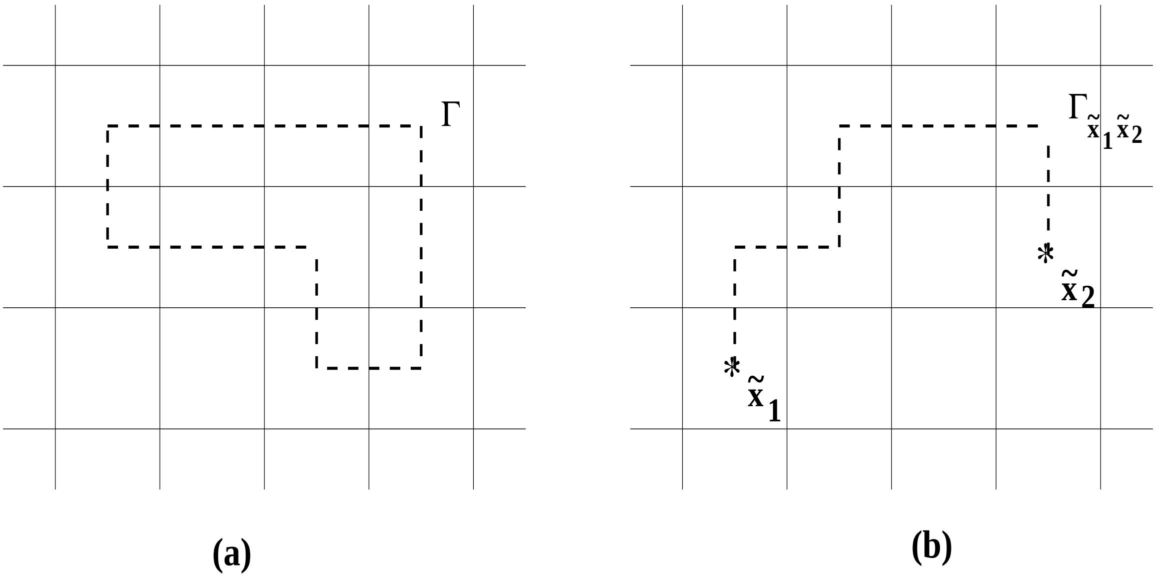


Figure 14:

The only terms in the energy functional affected by this transformation are those corresponding to the links $(\mathbf{xy})$ which cross the contour Γ; I will denote such links by perhaps clumsy symbol

$$(\mathbf{xy}) \in \Gamma\,.$$

These terms change sign in the exponentials in (9.16). Thus we have

$$Z(K) \equiv \sum_{\{\sigma_\mathbf{x}\}} P\{\sigma_\mathbf{x}\} = \sum_{\{\sigma_\mathbf{x}\}} P\{\sigma_\mathbf{x}\}\; T\{\sigma_\mathbf{x}\sigma_\mathbf{y}; (\mathbf{xy}) \in \Gamma\}\,, \tag{9.29}$$

where the insertion T takes into account this change of the signs,

$$T\{\sigma_\mathbf{x}\sigma_\mathbf{y}; (\mathbf{xy}) \times \Gamma\} = e^{-2K\, \sum_{\mathbf{xy}\in\Gamma}\, \sigma_\mathbf{x}\sigma_\mathbf{y}}\,. \tag{9.30}$$

I will often abbreviate (9.30) as $T\{\Gamma\}$. The Eq. (9.29) shows that, as the consequence of the global Z_2 symmetry, insertion of $T\{\Gamma\}$ with any closed Γ does not change the partition sum. One can insert $T\{\Gamma\}$ into more complicated correlation function which involves also some σ insertions. One finds

$$\langle\, \sigma_{\mathbf{x}_1} \cdots \sigma_{\mathbf{x}_n}\, T\{\Gamma\}\,\rangle = \big(-1\big)^{n_\Gamma}\, \langle\, \sigma_{\mathbf{x}_1} \cdots \sigma_{\mathbf{x}_n}\, T\{\Gamma\}\,\rangle\,, \tag{9.31}$$

where n_γ is the number of the σ insertions surrounded by Γ.

Now, consider some *open* contour $\Gamma_{\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2}$ on the dual lattice, with the end points at $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ (**Fig.7b**). Define $T\{\Gamma_{\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2}\}$ as in (9.30) with Γ replaced by $\Gamma_{\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2}$. The above analysis shows that the expectation value

$$\langle\, T\{\Gamma_{\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2}\}\,\rangle = Z^{-1} \sum_{\{\sigma_\mathbf{x}\}} P\{\sigma_\mathbf{x}\}\; T\{\Gamma_{\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2}\} \tag{9.32}$$

does not depend on the exact form of the contour $\Gamma_{\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2}$, but only on the positions of its end-points $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_1$. In fact, as the result of the Exercise 1 shows, this expectation value coincides with the two-point correlation function of the disorder variables $\mu_{\tilde{\mathbf{x}}}$,

$$(9.32) = \langle\, \mu_{\tilde{\mathbf{x}}_1}\, \mu_{\tilde{\mathbf{x}}_2} \,\rangle \equiv Z^{-1} \sum_{\{\mu_{\tilde{\mathbf{x}}}\}} \mu_{\tilde{\mathbf{x}}_1}\, \mu_{\tilde{\mathbf{x}}_2}\, e^{\tilde{K}\, \sum_{nn} \mu_{\tilde{\mathbf{x}}}\mu_{\tilde{\mathbf{y}}}}\,. \tag{9.33}$$

For further insight, consider expectation value of T together with some σ insertions,

$$\langle\, \sigma_{\mathbf{x}_1} \cdots \sigma_{\mathbf{x}_n}\; T\{\Gamma_{\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2}\}\,\rangle = Z^{-1} \sum_{\{\sigma_{\mathbf{x}}\}} P\{\sigma_{\mathbf{x}}\}\; \sigma_{\mathbf{x}_1} \cdots \sigma_{\mathbf{x}_n}\; T\{\Gamma_{\tilde{\mathbf{x}}_1,\tilde{\mathbf{x}}_2}\}\,. \tag{9.34}$$

This quantity does depend on the form of the contour Γ, but the dependence is "weak". The expectation value (9.34) does not change under deformations of Γ as long as the contour does not cross any of the points $\mathbf{x}_1, \cdots, \mathbf{x}_n$, and it changes sign when such crossing occurs. Another way of stating the same is to say that (9.34) is not a single-valued but a double-valued function of the points $\tilde{\mathbf{x}}_1$, $\tilde{\mathbf{x}}_2$ and $\mathbf{x}_1, \cdots, \mathbf{x}_n$, which change sign every time $\tilde{\mathbf{x}}_1$ or $\tilde{\mathbf{x}}_2$ goes around any of the points $\mathbf{x}_1, \cdots, \mathbf{x}_n$. The expectation value (9.34) defines the mixed correlation function involving both order and disorder parameters

$$(9.34) \;\equiv\; \langle\, \sigma_{\mathbf{x}_1} \cdots \sigma_{\mathbf{x}_n}\; \mu_{\tilde{\mathbf{x}}_1}\, \mu_{\tilde{\mathbf{x}}_2} \,\rangle\,. \tag{9.35}$$

This correlation function is double-valued function, with the monodromy properties described above.

The above construction admits straightforward generalization to include more insertions of the disorder field. In fact, it is convenient to introduce contours (on the dual lattice) $\Gamma_{\tilde{\mathbf{x}}}$ ending at the cite $\tilde{\mathbf{x}}$ of the dual lattice, with the other end brought away to infinity. Although it usually does not much matter, we will assume that the contour extends to the left horizontal infinity (**Fig.8**). We define

$$\mu_{\tilde{\mathbf{x}}} = T\{\Gamma_{\tilde{\mathbf{x}}}\}\,. \tag{9.36}$$

This definition is understood in terms of the insertion in the sum over the lattice spin configurations. This allows to define arbitrary mixed correlation functions

$$\langle\, \sigma_{\mathbf{x}_1}\, \cdots\, \sigma_{\mathbf{x}_n}\; \mu_{\tilde{\mathbf{x}}_1}\, \cdots\, \mu_{\tilde{\mathbf{x}}_m} \,\rangle\,. \tag{9.37}$$

In fact, this explicit construction is not very important it is seldom used in practice. What is important is our observation about the monodromy properties of the correlation functions (9.37). To recapitulate, the mixed correlation functions are double-valued functions of the points $\mathbf{x}_i$ and $\tilde{\mathbf{x}}_k$ involved. They change sign when any of the $\tilde{\mathbf{x}}_k$ is brought around any of the points $\mathbf{x}_i$. It does not change when any of $\mathbf{x}_i$ is brought around any other $\mathbf{x}_j$, and the same is true for the $\tilde{\mathbf{x}}$'s. This property will be brought out to the continuous field theory arising in the scaling limit of the Ising model.

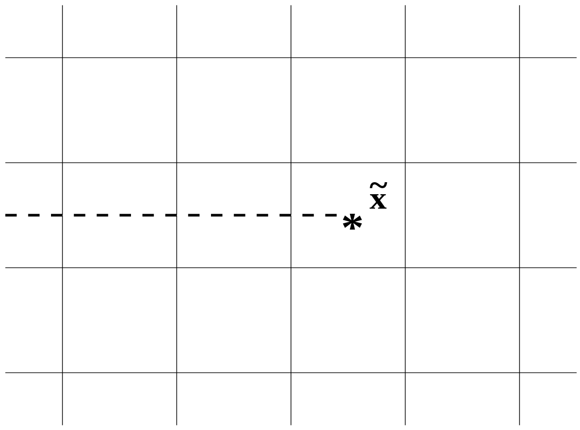


Figure 15:

9.2. Lattice fermions

Let us now show that the Ising model with $H = 0$ is the theory of free fermions, which makes it completely solvable. The fermions $\psi(\mathbf{x})$ appear as the products of the spin variable $\sigma_{\mathbf{x}}$ and the disorder variable $\mu_{\tilde{\mathbf{x}}}$ sitting at the nearby cite $\tilde{\mathbf{x}}$ of the dual lattice. There are four closest dual cites $\tilde{\mathbf{x}}$ to every cite $\mathbf{x}$, and to label them I introduce four vectors

$$\mathbf{e}_a\,, \quad a = 1,2,3,4\,, \tag{9.38}$$

each of the length $1/\sqrt{2}$, and each pointing at 45^o to the original lattice axes, in four possible directions NE, NW, SW, and SE

$$\mathbf{e}_1 \to \text{NE}\,, \quad \mathbf{e}_2 \to \text{NW}\,, \quad \mathbf{e}_3 \to \text{SW}\,, \quad \mathbf{e}_4 \to \text{SE}\,. \tag{9.39}$$

(of course this set has redundancy, $\mathbf{e}_3 = -\mathbf{e}_1$, $\mathbf{e}_4 = -\mathbf{e}_2$, but it is convenient to keep separate notations for all four, as in (9.39)). The four dual cites closest to $\mathbf{x}$ are

$$\mathbf{x} + \mathbf{e}_a\,, \quad a = 1,2,3,4\,. \tag{9.40}$$

as is shown in **Fig.9**.

The fermion variables $\psi_{a\,,\mathbf{x}}$ are defined as

$$\psi_{a\,,\mathbf{x}} = \sigma_{\mathbf{x}}\,\mu_{\mathbf{x}+\mathbf{e}_a}\,. \tag{9.41}$$

As these objects involve both σ and μ, there is the sign ambiguity which I have mentioned already. Precise way how this ambiguity is fixed is not important, but to make things as simple as possible I will always assume that the contour $\Gamma_{\tilde{\mathbf{x}}}$ associated with $\mu_{\tilde{\mathbf{x}}}$ is a horizontal straight line from minus infinity to $\tilde{\mathbf{x}}$, or it is deformable to such straight line. It is convenient to use pictorial representations of the objects (9.41) shown in **Fig.10**, where bold dots and crosses represent insertions of σ and μ, respectively.

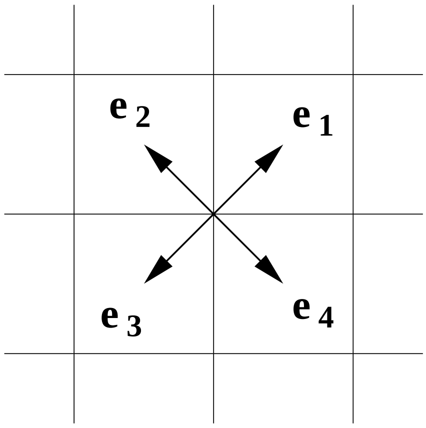


Figure 16:

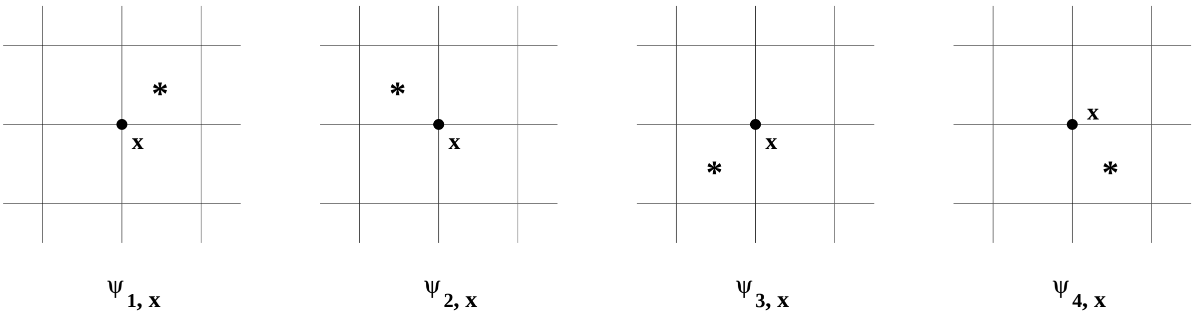


Figure 17:

let us show that the variables (9.41), being inserted into any correlation function

$$\langle \cdots \, \psi_{a\,,\mathbf{x}} \, \cdots \rangle \tag{9.42}$$

satisfy closed linear difference equation, which is the lattice version of the Dirac equation.

Consider for instance

$$\psi_{1\,,\mathbf{x}} = \sigma_{\mathbf{x}} \, \mu_{\mathbf{x}+\mathbf{e}_1} \,. \tag{9.43}$$

The disorder part $\mu_{\mathbf{x}+\mathbf{e}_1}$ is by definition a product of the factors

$$e^{-2K \, \sigma_{\mathbf{x}} \sigma_{\mathbf{x}'}} \tag{9.44}$$

along associated (horizontal) contour. One can split this product into the product representing insertion μ at the next dual cite to the left, times the factor associated with the last link $(\mathbf{x}+\mathbf{e}_2, \mathbf{x}+\mathbf{e}_1)$

$$\mu_{\mathbf{x}+\mathbf{e}_1} = \mu_{\mathbf{x}+\mathbf{e}_2} \, e^{-2K \, \sigma_{\mathbf{x}} \sigma_{\mathbf{x}+\mathbf{\Delta}_2}} \,, \tag{9.45}$$

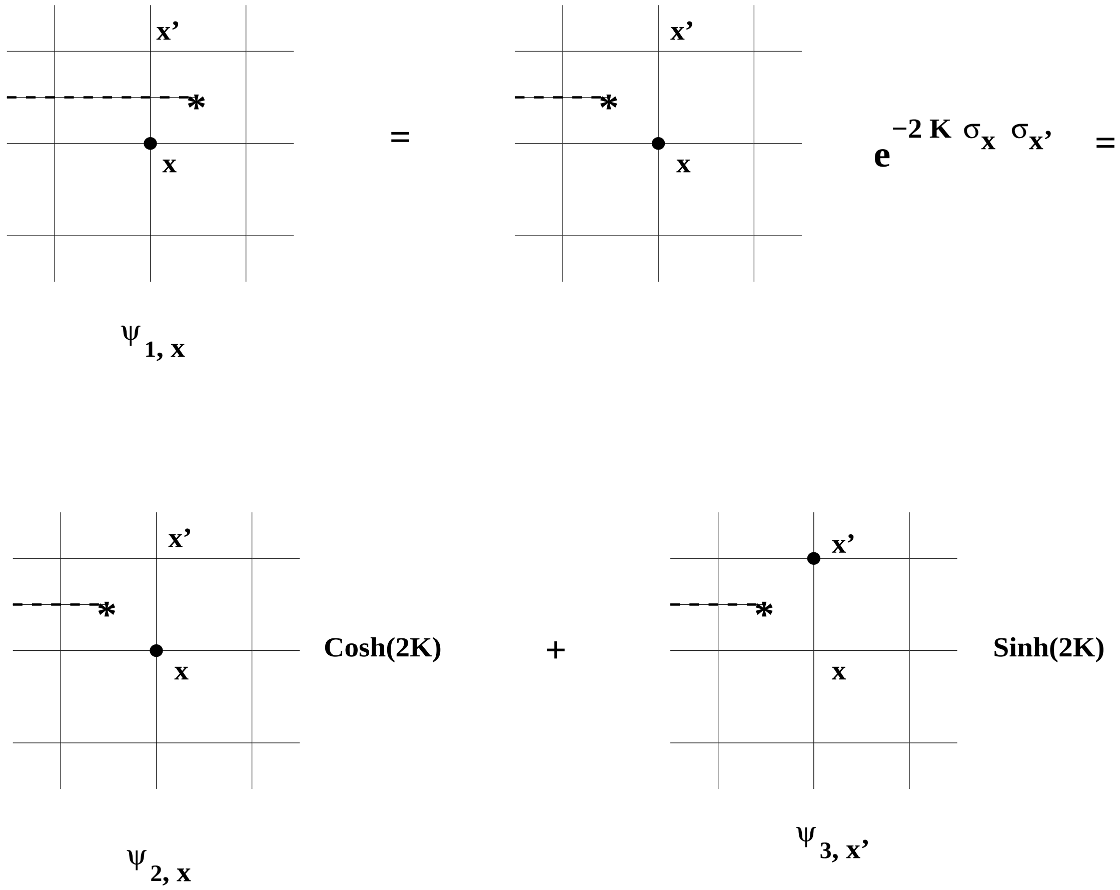


Figure 18:

where $\mathbf{\Delta}_2$ is one of the basic vectors of the lattice

$$\mathbf{\Delta}_1 = \mathbf{e}_1 + \mathbf{e}_4 = (1, 0)\,, \qquad \mathbf{\Delta}_2 = \mathbf{e}_1 + \mathbf{e}_2 = (0, 1)\,. \tag{9.46}$$

Now, as usual

$$e^{-2K\,\sigma_{\mathbf{x}}\sigma_{\mathbf{x}'}} = \cosh 2K - \sigma_{\mathbf{x}}\sigma_{\mathbf{x}'}\ \sinh 2K\,. \tag{9.47}$$

Substituting (9.45) into (9.43), and using (9.47) as well as the fact that $\sigma_{\mathbf{x}}^2 = 1$, one finds

$$\psi_{1,\mathbf{x}} = \big(\cosh 2K\big)\ \psi_{2,\mathbf{x}} - \big(\sinh 2K\big)\ \psi_{3,\mathbf{x}+\mathbf{\Delta}_2}\,. \tag{9.48}$$

This calculation is illustrated in **Fig.11**.

Similar equations can be derived for the other components ψ_a. This can be done exactly as above provided one first makes appropriate deformation of the contour associated with the disorder variable. To illustrate this last point, consider

$$\psi_{2,\mathbf{x}} = \sigma_{\mathbf{x}}\,\mu_{\mathbf{x}+\mathbf{e}_2}\,. \tag{9.49}$$

Let us first deform the contour associated with the μ insertion here as is shown in the **Fig.12**, and then apply the same transformation as in (9.45) to the last link of this contour.

$$\psi_{2,\mathbf{x}} = \big(\cosh 2K\big)\ \psi_{3,\mathbf{x}} - \big(\sinh 2K\big)\ \psi_{4,\mathbf{x}-\boldsymbol{\Delta}_1}\,. \tag{9.50}$$

The equations (9.48), (9.50), and similar equations for the other two components of ψ can be written in the following symmetric form

$$\psi_{a,\mathbf{x}} = \big(\cosh 2K\big)\ \psi_{a+1,\mathbf{x}} - \big(\sinh 2K\big)\ \psi_{a+2,\mathbf{x}+\boldsymbol{\Delta}_{a+1}}\,, \tag{9.51}$$

where ψ_a with $a \neq 1, 2, 3, 4$ are defined by the equations

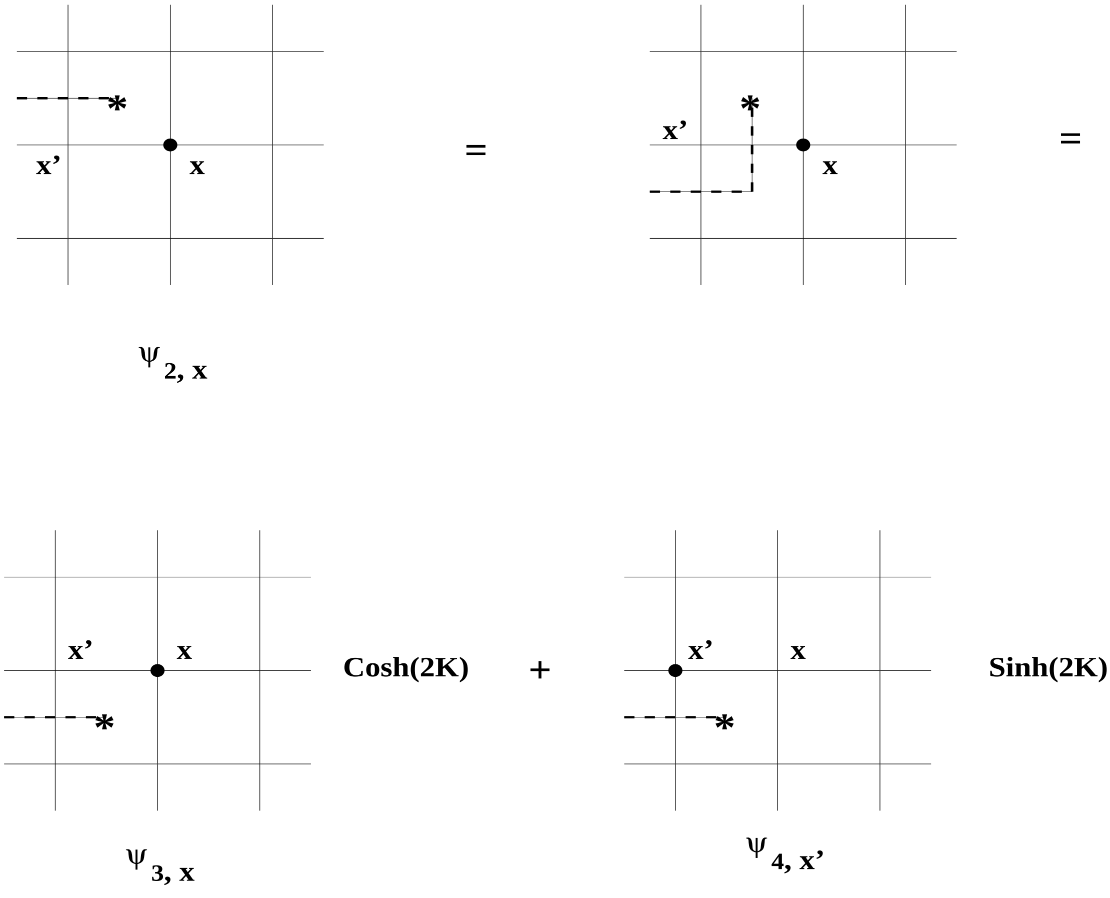


Figure 19:

$$\psi_{a+4,\mathbf{x}} = -\psi_{a,\mathbf{x}}\,. \tag{9.52}$$

Also, by definition,

$$\mathbf{e}_{a+4} = \mathbf{e}_a\,, \qquad \boldsymbol{\Delta}_{s+2} = -\boldsymbol{\Delta}_a\,. \tag{9.53}$$

The equation (9.52) is natural if one thinks of ψ_{a+4} as the result of four successive 90^o rotations of the object

$$\psi_{a\,,\mathbf{x}} = \sigma_{\mathbf{x}}\,\mu_{\mathbf{x}+\mathbf{e}_a}\,.$$

Although the full 360^o rotation returns σ and μ to the original positions, it is important to remember that after such rotation the contour associated with μ winds once around the point $\mathbf{x}$ (**Fig.13**); the minus sign in (9.52) appears as the result of "unwinding" of this contour.

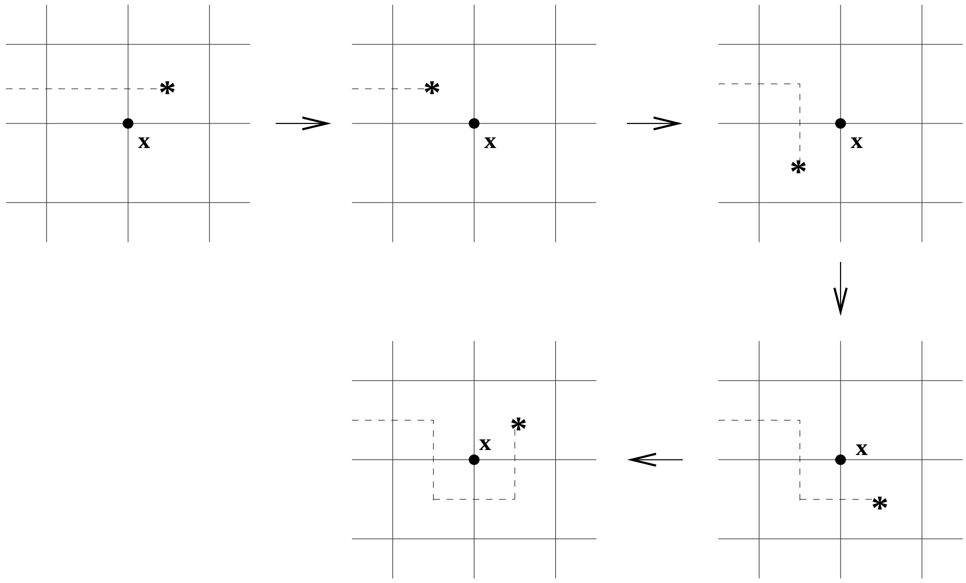


Figure 20:

We observe that the composite objects $\psi_{a\,,\mathbf{x}}$ obey linear "equations of motion" (9.51). This means that ψ_a is a free field. It is easy to see that ψ_a is fermi field. In the language of the lattice correlation functions the signature of a fermi field is the following property. Consider arbitrary correlation function of the form

$$\langle\,\cdots\psi_{a\,,\mathbf{x}_1}\,\psi_{a\,,\mathbf{x}_2}\cdots\,\rangle\,. \tag{9.54}$$

Let us move the points $\mathbf{x}_1$ and $\mathbf{x}_2$ in such a way that as the result of the move they interchange their positions, i.e.$\mathbf{x}_1 \to \mathbf{x}_1$ and $\mathbf{x}_2 \to \mathbf{x}_1$ (see **Fig.14a**). For a fermi field such move results in the change of the sign of the correlation function (9.54),

$$(9.54)\ \ \to\ -\langle\,\cdots\psi_{a\,,\mathbf{x}_2}\,\psi_{a\,,\mathbf{x}_1}\cdots\,\rangle\,. \tag{9.55}$$

In simple words, (9.54) is antisymmetric in $\mathbf{x}_1$ and $\mathbf{x}_2$. It is easy to see that the construction (9.49) guarantees this fermion exchange property - when one moves the points to interchange their positions, either the contour associated with $\psi_{\mathbf{x}_1}$ crosses $\mathbf{x}_2$, or the other way round, the contour attached to $\psi_{\mathbf{x}_2}$ crosses $\mathbf{x}_1$ (see **Fig.14b**), leading to the minus sign. And it is

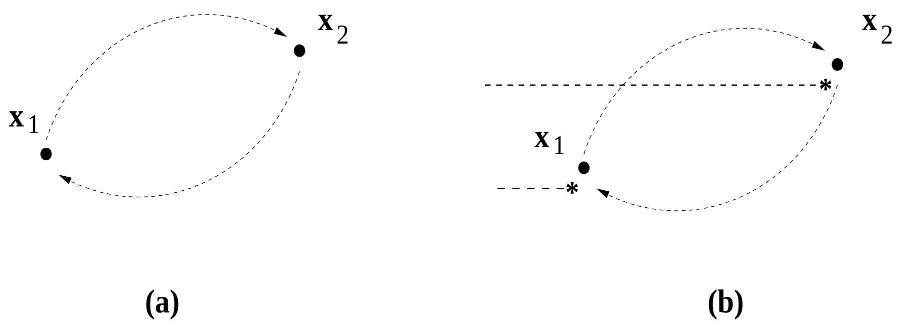


Figure 21:

easy to check that if in (9.54) one just brings $\mathbf{x}_1$ around $\mathbf{x}_2$ all extra signs cancel and (9.54) returns to its original value - the correlation function (9.54) is single valued, which means that $\psi_{a,\mathbf{x}}$ is *local* fermi field.

On the other hand, consider correlation function involving, besides ψ_a, any number of σ, or any number of μ, or both, for instance

$$\langle\, \psi_{a\,,\mathbf{x}}\, \sigma_{\mathbf{x}_1}\, \mu_{\tilde{\mathbf{x}}_2}\ \cdots \rangle\,. \tag{9.56}$$

It follows from the properties of σ and μ, and from our construction of ψ that such correlation function changes sign every time $\mathbf{x}$ is brought around either $\mathbf{x}_1$ or $\tilde{\mathbf{x}}_2$. One says that the fermi field $\psi_{a\,,\mathbf{x}}$ is not local with respect to σ and μ. Sometimes it is said that ψ is *semi-local* with respect to σ and to ψ. The term means that the product is a multi-valued function, which acquires a phase under the monodromy transformation (more general usage involves general finite-dimensional monodromy).

Let me mention here few simple identities involving these fermi fields. As follows directly from the definition (9.41), we have for instance

$$\sigma_{\mathbf{x}}\, \sigma_{\mathbf{x}+\mathbf{\Delta}_1} = \psi_{1\,,\mathbf{x}}\, \psi_{2\,,\mathbf{x}+\mathbf{\Delta}_1} = \psi_{4\,,\mathbf{x}}\, \psi_{3\,,\mathbf{x}+\mathbf{\Delta}_1}\,, \tag{9.57}$$

and similarly for $\sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{\Delta}_2}$, i.e. the energy density of the model

$$\varepsilon_{\mathbf{x}} = -\frac{J}{2}\sum_{a=1}^{4} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{\Delta}_a} \tag{9.58}$$

is expressed as the fermion bilinear. Similar expressions exist for the nearest-neighbor products

$$\mu_{\tilde{\mathbf{x}}}\, \mu_{\tilde{\mathbf{x}}+\mathbf{\Delta}_a}\,. \tag{9.59}$$

10. Scaling limit

In principle, the linear equations (9.51) (with suitable boundary conditions) can be used to find exact solution of the Ising model directly on the lattice. The solution shows critical point at

$$K = K_c \,, \qquad K_c = \frac{1}{2} \log\left(\sqrt{2}+1\right), \tag{10.1}$$

exactly as the duality predicts. When $K \to K_c$ the correlation length becomes large as compared to the lattice spacing, and one can obtain continuous field theory by taking the scaling limit, i.e. the limit $K \to K_c$ accompanied by an appropriate change of the length scale in order to keep the correlation length finite. This procedure is straightforward but somewhat cumbersome. We can get to the same result by taking the continuous limit directly in the linear equation (9.51).

Note that at the critical point $K = K_c$ we have

$$\cosh 2K_c = \sqrt{2}\,, \qquad \sinh 2K_c = 1\,. \tag{10.2}$$

One can check that with this coefficients the linear equations admit constant (i.e. $\mathbf{x}$-independent) solutions of the form

$$\psi_a = \omega^a\ C + \bar{\omega}^a\ \bar{C}\,, \tag{10.3}$$

where

$$\omega = e^{\frac{i\pi}{4}}\,, \qquad \bar{\omega} = e^{-\frac{i\pi}{4}}\,, \tag{10.4}$$

and C and $\bar{C}$ are arbitrary constants. This signals appearance of gapless modes with infinite correlation radius. Indeed, if one writes

$$\psi_a = \frac{\omega^a}{\sqrt{\pi}}\,\psi(\mathbf{x}) + \frac{\bar{\omega}^a}{\sqrt{\pi}}\,\bar{\psi}(\mathbf{x}) \tag{10.5}$$

and assumes that $\psi(\mathbf{x})$ and $\bar{\psi}(\mathbf{x})$ have very slow rate of change at the lattice scales, so that the lattice shift in the r.h.s. of (9.51) can be replaced by the derivative,

$$\psi(\mathbf{x}+\boldsymbol{\Delta}_a) \to \psi(\mathbf{x}) + \boldsymbol{\Delta}_a\,\partial_a\psi(\mathbf{x}) \tag{10.6}$$

the equation (9.51) leads to

$$\left(\partial_1 + i\partial_2\right)\psi(\mathbf{x}) = 0\,, \qquad \left(\partial_1 - i\partial_2\right)\bar{\psi}(\mathbf{x}) = 0\,. \tag{10.7}$$

This has the form of massless Dirac equation in 2D (more precisely, (10.7) involves self-conjugated spinor fields, i.e. it is the 2D Majorana equation).

If we slightly shift away from the critical point, i.e. set

$$K = K_c + k \tag{10.8}$$

with $k << 1$, the above gapless modes remain soft. Using

$$\cosh 2K = \sqrt{2} + 2k + O(k^2)\,, \qquad \sinh 2K = 1 + 2\sqrt{2}\,k + O(k^2)\,, \tag{10.9}$$

and neglecting all terms beyond the linear one in k, one finds instead of (10.7)

$$\begin{aligned}\big(\partial_1 + i\partial_2\big)\,\psi(\mathbf{x}) &= \quad im\,\bar\psi(\mathbf{x})\,,\\ \big(\partial_1 - i\partial_2\big)\,\bar\psi(\mathbf{x}) &= -im\,\psi(\mathbf{x})\,,\end{aligned} \tag{10.10}$$

which is massive Majorana equation, with the mass related to k

$$\varepsilon\, m = 4\, k\,, \tag{10.11}$$

where I have restored the notation ε for the lattice spacing to make dimensional counting straightforward. Taking the scaling limit amounts to sending k to zero, while looking at the theory at the length scales of the order of the correlation length

$$R \sim R_c = m^{-1}\,, \qquad R >> \varepsilon\,. \tag{10.12}$$

We see that in this limit the Ising model reduces to the free Majorana theory described by the equations of motion (10.10).

Strictly speaking, establishing the equations of motion (10.10) is not sufficient to prove equivalence. One has to find the boundary conditions which would determine the correlation functions, most important of which concerns with the singularities of the correlation functions at the coincident points. In deriving the equations (9.51) we have ignored possibility of other insertions in the correlation function, assuming that such extra insertions are located at finite lattice distance from the point $\mathbf{x}$. More careful analysis shows that if other fermion insertions are present at some points $\mathbf{x}_1, \;\cdots\; \mathbf{x}_n$, the equations (9.51) are violated by some constant (i.e. field independent) terms when $\mathbf{x}$ hits one of the points $\mathbf{x}_k$. In the scaling limit these terms modify (rather complete) the equations (10.10) as follows

$$\big(\partial_{x_1} + i\partial_{x_2}\big)\langle\, \psi(x)\,\psi(y)\;X\;\rangle = im\,\langle\, \bar\psi(x)\,\psi(y)\;O\;\rangle + \pi\,\delta(x-y)\,\langle\;O\;\rangle + \cdots\,, \tag{10.13}$$

where O stands for any combination of the fermion insertions, and the r.h.s. can have other delta-function terms if X contains ψ at other points. The second of the equations (10.10) is completed by similar delta-function terms. These equations now determine the correlation functions uniquely - for the correlation functions of the fermion fields the solution is given in terms of the sum of all Wick pairings.

In short, the scaling limit of the Ising model at $H = 0$ is the free fermion field theory whose properties can be encoded in the action

$$\mathcal{A}_{\rm FF} = \frac{1}{2\pi}\int\;\big[\psi\bar\partial\psi + \bar\psi\partial\bar\psi + im\,\bar\psi\psi\big]\,d^2x \tag{10.14}$$

In what follows I will use the complex coordinates

$$z = x_1 + ix_2\,, \qquad \bar z = x_1 - ix_2 \tag{10.15}$$

on the Euclidean plane. The derivatives in (10.14) stand for the complex derivatives

$$\partial = \partial_z = \frac{1}{2}\,(\partial_1 - i\partial_2)\,, \qquad \bar\partial = \partial_{\bar z} = \frac{1}{2}\,(\partial_1 + i\partial_2)\,. \tag{10.16}$$

The correlation functions in this free theory can be understood in terms of the gaussian functional integral

$$\langle \cdots \rangle = Z^{-1} \int \left[D\psi, D\bar{\psi}\right] \left(\cdots \right) e^{-\mathcal{A}_{\rm FF}[\psi,\bar{\psi}]} \tag{10.17}$$

over the Grassmanian (anticommuting) field variables $\psi(x)$, $\bar{\psi}(x)$.

As the field theory, (10.17) does not look terribly interesting. It contains a single sort of neutral particles with fermion statistics, which otherwise do not interact. The particle's mass is $|m|$ (remember, the parameter $m \sim K - K_c$, it can be positive or negative depending on whether we are in the low or in the high T phase). From the point of view of the functional integral (10.17) itself the sign of m is irrelevant. It can be changed by a simple change of variables in (10.17)

$$\psi \to \psi\,, \quad \bar{\psi} \to -\bar{\psi}\,, \qquad m \to -m\,. \tag{10.18}$$

The symmetry (10.18) is what the duality transformation of the Ising model does to the fermion field of (10.17). According to my convention $m\,\varepsilon = 4(K - K_c)$ the parameter m is positive in the low-T phase and it is negative in the high-T phase.

The special case $K = K_c$, i.e. $m = 0$, corresponds to the critical point. In the high-T domain the free particles are the "spin-particles" - we will see that the spin insertion $\sigma(x)$ can emit a single particle. In the low-T phase the particles are rather interpreted as the "kinks" separating domains with opposite orientations of the spins.

Basic thermodynamic properties of the Ising theory near criticality are readily derived from the gaussian functional integral (10.17). I'll skip explicit calculation. The specific free energy

$$F = -\frac{\log Z}{V} \tag{10.19}$$

(V is the 2-volume of the space) develops the famous Onsager's singularity

$$F_{\rm sing} = \frac{m^2}{8\pi}\,\log m^2\,, \tag{10.20}$$

which leads to logarithmic divergence of the heat capacity near the critical point. Magnetization and other related thermodynamic quantities will be discussed later on.

The free-fermion theory (10.14) represents the scaling limit of the Ising model with $H = 0$. Nonzero magnetic field H couples to the lattice spin variables $\sigma_{\mathbf{x}}$. In continuous limit, it couples to the local magnetization $\sigma(x)$, the local field associated with the lattice spins. This field is not local with respect to the fermi fields, and therefore no local expression for $\sigma(x)$ in terms of ψ and its derivatives can be expected. We will have to find out how this "spin" field is constructed in terms of the theory (10.14). But first let us discuss the conformal theory which appears at the critical point $m = 0$.

11. Conformal theory of the critical point

At $K = K_c$ we have $m = 0$, and the theory (10.14) acquires conformal invariance. It is straightforward to check that the massless action

$$\mathcal{A}_{\rm FF} = \frac{1}{2\pi}\int \left[\psi\partial_{\bar z}\psi + \bar\psi\partial_z\bar\psi\right]\, d^2x \qquad (11.1)$$

is invariant under the analytic conformal transformations

$$z \to w(z)\,, \qquad \bar z \to \bar w(\bar z)\,, \qquad (11.2)$$

provided the fields $\psi,\ \bar\psi$ are transformed as follows

$$\psi(z,\bar z) = (\partial_z w)^{\frac{1}{2}}\ \psi(w,\bar w)\,, \qquad \bar\psi(z,\bar z) = (\partial_{\bar z}\bar w)^{\frac{1}{2}}\ \bar\psi(w,\bar w)\,. \qquad (11.3)$$

The equations of motion (10.10) take the form of the Cauchy-Riemann equations

$$\partial_{\bar z}\psi = 0\,, \qquad \partial_z\bar\psi = 0\,, \qquad (11.4)$$

stating that ψ and $\bar\psi$ are the holomorphic and the anti-holomorphic fields, respectively

$$\psi = \psi(z)\,, \qquad \bar\psi = \bar\psi(\bar z)\,. \qquad (11.5)$$

As usual, this statement is understood in terms of the correlation functions. Consider

$$\langle\, \psi(z)\, O_1(x_1)\cdots O_n(x_n)\,\rangle\,. \qquad (11.6)$$

where O_i are some fields, local with respect to the fermions. It is single-valued holomorphic function of z, with possible singularities at the insertion points $x_1, x_2, \cdots, x_n$, where the equations of motion (11.4) are generally violated by contact terms. Important case is another insertion of ψ. The correlation function

$$\langle\, \psi(z)\, \psi(z_1)\, O\,\rangle \qquad (11.7)$$

has the first order pole at $z = z_1$, with the residue equal to the correlation function $\langle\, O\,\rangle$,

$$\langle\, \psi(z)\, \psi(z_1)\, O\,\rangle = \frac{1}{z - z_1}\,\langle\, O\,\rangle + \text{regular terms}\,. \qquad (11.8)$$

This relation follows directly from the Eq.(10.13) [6]. This, and similar relation for the field $\bar\psi$, can be compactly written in terms of the operator product expansions

$$\psi(z)\,\psi(z') = \frac{1}{z - z'} + \ \text{regular terms}\,, \qquad \bar\psi(\bar z)\,\bar\psi(\bar z') = \frac{1}{\bar z - \bar z'} + \ \text{regular terms}\,. \qquad (11.9)$$

[6] We recall that

$$\partial_{\bar z}\frac{1}{z} = \pi\,\delta^{(2)}(z)\,.$$

The recurrent relation

$$\langle\,\psi(z)\,\psi(z_1)\,...\,\psi(z_n)\,\rangle = \sum_{k=1}^{n}(-)^{k+1}\,\frac{1}{z-z_k}\,\langle\,\psi(z_1)\,...\,\psi(z_{k-1})\,\psi(z_{k+1})\,...\psi(z_n)\,\rangle \tag{11.10}$$

which follows from (17.45) is equivalent to the (fermionic) Wick's rule.

In flat space, the energy-momentum tensor of associated with the theory (11.1) is traceless, and its components $T = T_{zz}$ and $\bar{T} = T_{\bar{z}\bar{z}}$

$$T(z) = -\frac{1}{2}\ :\psi\partial_z\psi:(z)\,,\qquad \bar{T}(\bar{z}) = -\frac{1}{2}\ :\bar{\psi}\partial_{\bar{z}}\bar{\psi}:(\bar{z})\,, \tag{11.11}$$

are holomorphic and anti-holomorphic fields. The operator product expansion

$$T(z)T(z') = \frac{1}{4\,(z-z')^4} + \frac{2}{(z-z')^2}\,T(z') + \frac{1}{z-z'}\,\partial T(z') + \text{reg} \tag{11.12}$$

is directly verified using the Wick's rule; it shows that this conformal field theory has the central charge

$$c = \frac{1}{2}\,. \tag{11.13}$$

12. Spin fields

All fields local with respect to the fermions ψ, $\bar{\psi}$ are obtained as the composite fields, built from the fermi fields themselves and their derivatives at the same point. We will call this space, i.e.

$$\mathcal{F}_{\text{NS}} = \text{Span}\left\{I,\ \psi,\ \bar{\psi}, \psi\partial\psi,\ ...\bar{\psi}\partial^n\psi\,,...\right\} \tag{12.1}$$

the Neveu-Schwartz sector. The spin field $\sigma(x)$, as well as the "disorder field" $\mu(x)$ associated with the microscopic disorder parameter $\mu_{\mathbf{x}}$, obviously do not belong to this space. To understand their status, let us recall their basic property, inheritant from the microscopic definitions. Namely, if we take any correlation function which involves $\psi(x)$ (or $\bar{\psi}(x)$) as well as several σ-insertions, i.e.

$$\langle\,\psi(x)\,\sigma(x_1)\cdots\sigma(x_n)\cdots\rangle\,, \tag{12.2}$$

it is a double-valued function of the Euclidean point x, which changes the sign every time the point x is brought around any one of the points x_1, x_2, $\cdots$ x_n.

This property alone does not define the field $\sigma(x)$ uniquely. In fact, there are infinitely many local fields which satisfy this property. Indeed, within the lattice theory we could have taken a product of three spins in some neighboring points, say

$$\sigma_{\mathbf{x}}\,\sigma_{\mathbf{x}+\mathbf{\Delta}_1}\,\sigma_{\mathbf{x}-\mathbf{\Delta}_2}\,. \tag{12.3}$$

By taking the scaling limit $R_c >> \varepsilon$ we shrink all such configurations to a point, thus producing local fields. Obviously, any such field (let us call it for the moment σ_3) have the same property that the product

$$\psi(x)\,\sigma_3(x_1) \to -\psi(x)\,\sigma_3(x_1) \tag{12.4}$$

changes sign when x goes around x_1. One can throw in any odd number of the lattice σ insertions at different finite lattice separations, and all will become local fields with the same property (12.4). On top of that, there is the dual field $\mu(x)$, which also brings along an infinite number of new fields.

Thus we have an infinite-dimensional space of "spin fields", the fields whose product with $\psi(x)$ has the property (12.4). I will denote this space $\mathcal{F}_{\rm R}$ (the "Ramond sector"). By the definition, for any $O \in \mathcal{F}_{\rm R}$ the products

$$\psi(x)\, O(x_1) \to -\psi(x)\, O(x_1)\,, \qquad \bar\psi(x)\, O(x_1) \to -\bar\psi(x)\, O(x_1) \tag{12.5}$$

change sign when x is brought around x_1. $\mathcal{F}_{\rm R}$ is the vector space since the sum of any two fields satisfying the property (12.5) satisfies this property as well.

We need some tools to sort out the content of the space $\mathcal{F}_{\rm R}$ of the "spin fields". To that end, let us consider the correlation function

$$\langle\, \psi(z)\, O_{\rm R}(z_1,\bar z_1)\, \cdots \rangle\,, \tag{12.6}$$

where $O_{\rm R} \in \mathcal{F}_{\rm R}$. By definition of the spin fields, this correlation function has square-root brunching point at $z = z_1$, i.e. the analytic structure of (2.59) at z sufficiently close to z_1 can be described by the expansion

$$\psi(z)\, O_{\rm R}(z_1,\bar z_1) = \sum_{n\in\mathbf{Z}} (z-z_1)^{-n-1/2}\, O_{\rm R}^{(n)}(z_1,\bar z_1)\,. \tag{12.7}$$

The defining monodromy property of the product in the l.h.s. (the product changes sign when z is brought around z_1) is reflected in the half-integer powers in the r.h.s. This equation can be understood as the operator product expansion, with the coefficients $O_1^{(n)}$ in the r.h.s. being some fields belonging (as one easily checks) to the space

$$O_1^{(n)} \in \mathcal{F}_{\rm R}\,. \tag{12.8}$$

Given a field $O_1 \in \mathcal{F}_{\rm R}$, the expansion (12.7) provides definition of the fields $O_1^{(n)}$. The situation is best understood if one defines the operators a_n acting in $\mathcal{F}_{\rm R}$, as follows

$$a_n O_{\rm R}(x,\bar x) = \oint_x \big(z-x\big)^{n-1/2}\, \psi(z)\, O_1(x,\bar x)\, \frac{dz}{2\pi i}\,, \tag{12.9}$$

where the integration is performed over closed contour which encircles the point x. Note that the integrand in (12.9) is single-valued in the vicinity of x, and the contour is indeed a closed one. Note also that the contour can be made arbitrary small, close to the point x; this makes it clear that the r.h.s of (12.9) defines a field localized at the point $(x,\bar x)$.

Of course, the anti-holomorphic component $\bar\psi$ gives rise to similar set of operators $\bar a_n$,

$$\bar a_n\, O(x,\bar x) = \oint_{\bar x} \big(\bar z-\bar x\big)^{n-1/2}\, \bar\psi(\bar z)\, O(x,\bar x)\, \frac{d\bar z}{2\pi i}\,, \tag{12.10}$$

where the integration is over the counterclockwise contour in the $\bar{z}$ plane.

Our nearest goal is to derive the commutation relations among the operators a_n and $\bar{a}_n$; we will show that

$$\{a_n, a_m\} = \delta_{n+m,0}\,, \qquad \{\bar{a}_n, \bar{a}_m\} = \delta_{n+m,0}\,, \tag{12.11}$$

and

$$\{a_n, \bar{a}_m\} = 0\,, \tag{12.12}$$

and that the space $\mathcal{F}_{\mathrm{R}}$ has the structure of the Fock space generated by these operators.

Let us consider the field $a_n a_m O(z_0)$, the result of successive application of the operators a_m and a_n,

$$a_n a_m O(z_0) = \oint_{C_2} \frac{dw}{2\pi i}\,\psi(w)\,(w-z_0)^{n-1/2} \oint_{C_1} \frac{dz}{2\pi i}\,\psi(z)\,(z-z_0)^{m-1/2}\,O(z_0)\,. \tag{12.13}$$

The integration contours C_2 and C_1 both encircle the point z_0, but by definition, according

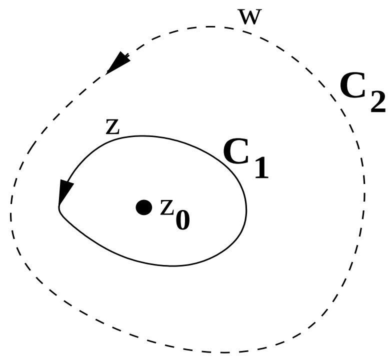


Figure 22:

to the order of the operators $a_n a_m$ in (12.13), the integration over z generating the action of a_m is performed first. In fact, the actual order of integration is not important if we assume that the contour C_1 lays inside the contour C_2 (**Fig 15**) - this is the arrangement which corresponds to order of operators written in (12.13). On the other hand

$$a_m a_n O(z_0) = \oint_{C_1} \frac{dz}{2\pi i}\,\psi(z)\,(z-z_0)^{m-1/2} \oint_{C_2} \frac{dw}{2\pi i}\,\psi(w)\,(w-z_0)^{n-1/2}\,O(z_0)\,, \tag{12.14}$$

where this time we assume that C_2 lays inside C_1, since the operator a_n acts first.

If not for the order of integrations, the expressions (12.13) and (12.14) are almost identical - (12.13) can be transformed to (12.14) by moving the contour C_2 to get it inside C_1, and also interchanging the positions of $\psi(w)$ and $\psi(z)$. The interchange of the ψ's results only in the change of sign,

$$\psi(w)\psi(z) = -\psi(z)\psi(w)\,,$$

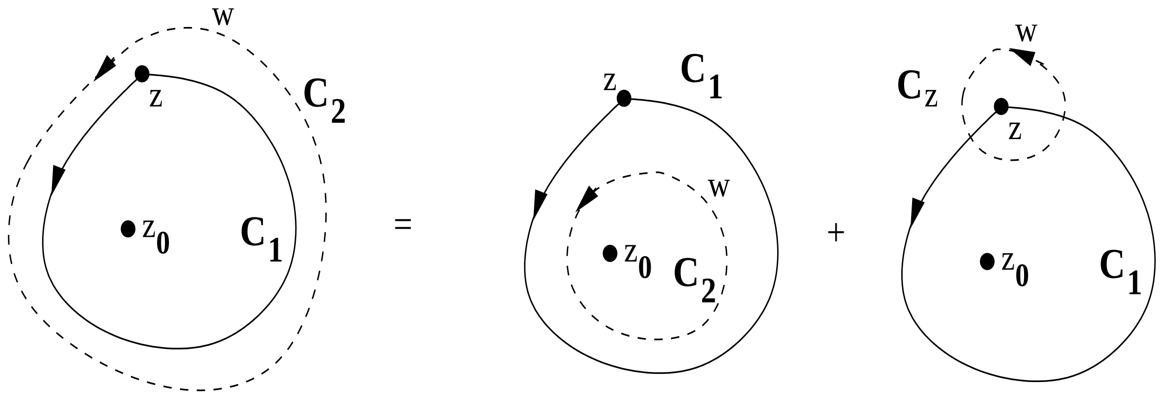


Figure 23:

but the contour deformation leaves behind an extra term. Since the operator product in the integrand in (12.13) is singular at $w = z$,

$$\psi(z)\psi(z') = \frac{1}{z - z'} + \text{reg}\,, \tag{12.15}$$

one has to take into account the contribution of the residue of the pole in (12.14) when moving C_2 inside C_1. This contribution is

$$\oint_{C_1} \frac{dz}{2\pi i} \oint_{C_z} \frac{dw}{2\pi i}\, \psi(w)\, \psi(z)\, (w - z_0)^{n-1/2}\, (z - z_0)^{m-1/2}\, O(z_0)\,, \tag{12.16}$$

where C_z is a small contour surrounding the point z (**Fig.16**). This integral is done by the residue calculation using (12.15), with the result

$$(12.16) \;=\; \oint_{C_1} \frac{dz}{2\pi i}\, (z - z_0)^{n+m-1}\, O(z_0) = \delta_{n+m,0}\, O(z_0)\,. \tag{12.17}$$

As the result,

$$(12.13) = -(12.14) + (12.17)\,, \tag{12.18}$$

which is the first of the anti-commutation relations (12.11). The rest of the anti-commutators (12.11),(12.12) are derived in a similar way.

We will show later that the space $\mathcal{F}_{\rm R}$ contains at least one "Fock vacuum", i.e. the field (which I temporarily denote $O_{\rm vac}$) such that

$$O_{\rm vac} \in \mathcal{R}: \qquad a_n O_{\rm vac} = 0\,, \quad \bar{a}_n O_{\rm vac} = 0 \qquad \text{for all} \quad n > 0\,. \tag{12.19}$$

Now, the "Fock vacuum" field $O_{\rm vac}$ cannot be unique. The fields

$$a_0 O_{\rm vac}\,, \qquad \bar{a}_0 O_{\rm vac} \tag{12.20}$$

also satisfy the vacuum conditions (12.19). At the same time the fields (12.20) cannot be just multiples of the original $O_{\rm vac}$ since the algebra

$$\{a_0, a_0\} = 2\,a_0^2 = 1\,, \qquad \{\bar a_0, \bar a_0\} = 2\,\bar a_0^2 = 1\,, \qquad \{a_0, \bar a_0\} = 0 \tag{12.21}$$

does not have one-dimensional representations. The minimal degeneracy of $O_{\rm vac}$ sufficient to support the relations is two. That is exactly what we expect. Remember, the lattice Ising model exhibits order-disorder duality: it admits two equivalent descriptions in terms of the original spins $\sigma_{\mathbf x}$ and the "dual spins" $\mu_{\tilde{\mathbf x}}$, the parameter K being replaced by $\tilde K$: $\exp\big(-2K\big) = \tanh \tilde K$. The duality transformation maps the high-T domain for the σ's to the low-T domain of the μ's, and vice versa. In particular, at the critical point $K = K_c$ the model is self-dual, $K_c = \tilde K_c$, i.e. the correlations of the dual spins are exactly the same as the correlations of the original spins. The critical point corresponds to the massless theory (11.1).

For those reasons one expects the vacuum $O_{\rm vac}$ of the space $\mathcal{F}_{\rm R}$ to be two-dimensional, spanned by the fields $\sigma(x)$ and $\mu(x)$, the continuous limits of the order and disorder variables $\sigma_{\mathbf x}$ and $\mu_{\tilde{\mathbf x}}$ of the lattice theory,

$$O_{\rm vac} = \big(\sigma, \mu\big)\,. \tag{12.22}$$

Moreover, we must have

$$a_0\sigma \sim \mu\,, \qquad a_0\mu \sim \sigma\,, \tag{12.23}$$

and similarly with $\bar a_0$. Indeed, the action of a_0 on some field $O \in \mathcal{F}_{\rm R}$ is essentially the result of the fusion of ψ and O,

$$a_0 O(z_1) = \lim_{z\to z_1} \sqrt{z - z_1}\;\psi(z)\, O(z_1)\,. \tag{12.24}$$

If one also recalls that microscopically the fermions $\psi, \bar\psi$ are products of the type $\sigma\mu$, the relations (12.23) appear the only consistent possibility. This is also consistent with the property that the product $\sigma(x)\mu(x')$ changes sign when x is brought around x' (one has to remember that $\psi(x)\sigma(x')$ and $\psi(x)\mu(x')$ change signs under such move).

Nicely symmetric representation of the anti-commutation relations (12.21) is given by the equations

$$\begin{aligned} a_0\sigma(x) &= \frac{\omega}{\sqrt 2}\,\mu(x)\,, \qquad & a_0\mu(x) &= \frac{\bar\omega}{\sqrt 2}\,\sigma(x)\,,\\ \bar a_0\sigma(x) &= \frac{\bar\omega}{\sqrt 2}\,\mu(x)\,, \qquad & \bar a_0\mu(x) &= \frac{\omega}{\sqrt 2}\,\sigma(x)\,, \end{aligned} \tag{12.25}$$

where

$$\omega = e^{\frac{i\pi}{4}}\,, \qquad \bar\omega = e^{-\frac{i\pi}{4}}\,. \tag{12.26}$$

The equations (12.25), together with

$$a_n\sigma(x) = \bar a_n\sigma(x) = a_n\mu(x) = \bar a_n\mu(x) = 0 \quad \text{for} \quad n > 0 \tag{12.27}$$

can be taken as the definition of the spin fields in the Ising field theory, and if fact I could have started my discussion of the IFT with writing down the free fermion theory and defining the spin field by (12.25),(12.27). Our previous discussion of the lattice model provides their microscopic interpretation.

Let us now determine the anomalous dimensions of the spin fields. This is done in terms of the energy-momentum tensor

$$T = -\frac{1}{2}\,\psi\partial\psi\,, \qquad \bar{T} = \bar{\psi}\bar{\partial}\bar{\psi}\,. \tag{12.28}$$

These expressions can be understood as follows. Consider again the OPE

$$\psi(z)\psi(z') = \frac{1}{z-z'} + \text{reg}\,. \tag{12.29}$$

In this case it is easy to find the structure of the regular terms as well. We have by definition

$$\psi(z)\psi(z') = \frac{1}{z-z'} + :\psi(z)\psi(z'):\,, \tag{12.30}$$

where the first term represents the wick contraction of the two fields, and the symbol : ... : stands for the Wick normal ordering (which means that all contractions inside : ... : are excluded). It is easy to see looking at arbitrary correlation function that the Wick ordered product is regular at $z = z'$. It can be expanded in Taylor series in the powers of the difference $z - z'$. Since $:\psi(z)\psi(z) := 0$, we have

$$\psi(z)\psi(z') = \frac{1}{z-z'} - (z-z')\;:\psi(z')\partial\psi(z'): + O((z-z')^2)\,. \tag{12.31}$$

The regular term explicitly written here is proportional to

$$T(z') = -\frac{1}{2}\;:\psi(z')\partial\psi(z'):\;. \tag{12.32}$$

Therefore, (12.31) reads

$$\psi(z)\psi(z') = \frac{1}{z-z'} + 2\,(z-z')\,T(z') + O((z-z')^2)\,. \tag{12.33}$$

Equivalent statement is

$$T(w) = \frac{1}{2}\oint_{C_w} \frac{dz}{2\pi i}\,(z-w)^{-2}\,\psi(z)\psi(w)\,. \tag{12.34}$$

It is now possible to express the action of L_n's on the spin states $O \in \mathcal{F}_{\text{R}}$ in terms of the operators a_n. Consider the integral

$$I_{n,m}O(0) = \oint_{C_1} \frac{dz}{2\pi i}\,z^{n+1/2} \oint_{C_z} \frac{dw}{2\pi i}\,w^{m+1/2}\,(w-z)^{-2}\,\psi(w)\psi(z)\,O(0)\,, \tag{12.35}$$

where C_z encircles the point z, as in the last term in the r.h.s in **Fig.16**.

Using (12.33) we can evaluate the integral over w

$$\oint_{C_z} \frac{dw}{2\pi i}\, w^{m+1/2}\, (w-z)^{-2}\, \psi(w)\psi(z) =$$
$$= \oint_{C_z} \frac{dw}{2\pi i}\, w^{m+1/2}\, (w-z)^{-2} \left[\frac{1}{w-z} + 2\,(w-z)\, T(z) + ... \right] = \tag{12.36}$$
$$= \frac{1}{2}\,(m^2 - 1/4)\, z^{m-3/2} + 2\, z^{m+1/2}\, T(z)\, .$$

Then the z integral evaluates to

$$I_{n,m} = 2\, L_{n+m} + \frac{1}{2}\,(m^2 - 1/4)\, \delta_{n+m,0}\, . \tag{12.37}$$

On the other hand, the contour C_z can be represented as the difference

$$\oint_{C_z} = \oint_{C_2^+} - \oint_{C_2^-}\, , \tag{12.38}$$

where C_2^+ is placed outside C_1 as is shown in the l.h.s. of the **Fig.16**, while C_2^- goes inside C_1, as in the first term of the r.h.s. of that Figure. Then

$$I_{n,m} O(0) = \oint_{C_2^+} \frac{dw}{2\pi i}\, w^{m+1/2} \oint_{C_1} \frac{dz}{2\pi i}\, z^{n+1/2}\, (w-z)^{-2}\, \psi(w)\psi(z)\, O(0) -$$
$$- \oint_{C_1} \frac{dz}{2\pi i}\, z^{n+1/2} \oint_{C_2^-} \frac{dw}{2\pi i}\, w^{m+1/2}\, (w-z)^{-2}\, \psi(w)\psi(z)\, O(0)\, . \tag{12.39}$$

In the first term we have $|w| > |z|$ and write

$$(z-w)^{-2} = \sum_{k=0}^{\infty} k\, \frac{z^{k-1}}{w^{k+1}}\, .$$

Then this integral evaluates to

$$\sum_{k=0}^{\infty} k\, a_{m-k} a_{n+k}\, O(0)\, . \tag{3.94}$$

In the second term, we have instead $|z| > |w|$, so we can expand

$$(z-w)^{-2} = \sum_{k=0}^{\infty} k\, \frac{w^{k-1}}{z^{k+1}}\, ,$$

hence the second term evaluates to

$$\sum_{k=0}^{\infty} k\, a_{n-k} a_{m+k}\, O(0)\, . \tag{3.95}$$

Finally

$$L_{n+m}\, O(0) = \frac{1}{4}\,(1/4 - m^2)\,\delta_{n+m,0}\, O(0) +$$

$$+\frac{1}{2}\sum_{k=0}^{\infty} k\left[a_{m-k}a_{n+k} + a_{n-k}a_{m+k}\right] O(0)\,. \tag{3.96}$$

Lecture 5. 2D Gravity and Liouville Theory

13. Conformal field theory in curved background

Here I consider Euclidean field theories on generic Riemann manifold (closed or with a boundary) with a metric $g_{\mu\nu}(x)$. A field theory is called *conformal field theory* (CFT) if its energy-momentum tensor $T_{\mu\nu}$ obeys the equation

$$T^{\mu}_{\mu}(x) = \alpha\, R(x)\,, \tag{13.1}$$

where $R(x)$ is the scalar curvature. Here α is a parameter usually written as

$$\alpha = -\frac{c}{12}\,, \tag{13.2}$$

where c is known as the Virasoro central charge. In (local) conformal complex coordinates

$$ds^2 = e^{\sigma(z,\bar z)}\, dz d\bar z \tag{13.3}$$

we have

$$R = -\,4\, e^{-\sigma}\, \partial_z \partial_{\bar z} \sigma\,. \tag{13.4}$$

13.1. Holomorphic pseudotensor

As the consequence of the anomaly equation (13.1), we can write the full energy-momentum tensor $T_{\mu\nu}$ in terms of two pseudo-tensors

$$T_{\mu\nu} = T^{(0)}_{\mu\nu} + \frac{\alpha}{2}\, t_{\mu\nu} \tag{13.5}$$

where $T^{(0)}$ is traceless, and $t^{\mu}_{\mu} = 2\,R$. It is conventional to denote $T = T^{(0)}_{zz}$, $\bar T = T^{(0)}_{\bar z\bar z}$; then in the components

$$\begin{aligned}
T_{z\bar z} &= \frac{\alpha}{2}\, t_{z\bar z}\,, \qquad & t_{z,\bar z} &= -\,2\,\partial_z\partial_{\bar z}\sigma\,,\\
T_{zz} &= T + \frac{\alpha}{2}\, t_{zz}\,, \qquad & t_{zz} &= -\partial_z\sigma\partial_z\sigma + 2\,\partial_z^2\sigma\,,\\
T_{\bar z\bar z} &= \bar T + \frac{\alpha}{2}\, t_{\bar z\bar z}\,, \qquad & t_{\bar z\bar z} &= -\partial_{\bar z}\sigma\partial_{\bar z}\sigma + 2\,\partial_{\bar z}^2\sigma\,.
\end{aligned} \tag{13.6}$$

Then, in virtue of the covariant continuity equations $\nabla_\mu T^{\mu\nu} = 0$, the fields T and $\bar T$ satisfy the Cauchy-Riemann equations

$$\partial_{\bar z} T = 0\,, \qquad \partial_z \bar T = 0\,, \tag{13.7}$$

so that $T = T(z)$ is the *holomorphic* field (and $\bar{T} = \bar{T}(\bar{z})$ is the *anti-holomorphic* fields).

These relations are tied to the choice of conformal coordinates (13.3). The coordinate transformations which preserve this form of the metric are the conformal analytic transformations

$$z \to w(z)\,, \qquad \bar{z} \to \bar{w}(\bar{z})\,; \tag{13.8}$$

these transformations lead to the transformation

$$\sigma(z,\bar{z}) \;\to\; \sigma(w,\bar{w}) = \sigma(z,\bar{z}) - \log\left(\partial_z w(z)\,\partial_{\bar{z}}\bar{w}(\bar{z})\right) \tag{13.9}$$

of the conformal factor in (13.3). As the result, the pseudotensor t obeys anomalous transformation law [7]

$$t_{zz} \;\to\; t_{ww}\;: \qquad t_{zz} = \left(\partial_z w\right)^2\, t_{ww} + 2\,\{w,z\}\;. \tag{13.11}$$

Since $T_{\mu\nu}$ is true tensor, the holomorphic object $T^{(0)}_{\mu\nu}$ must transform anomalously as well,

$$T(z) \;\to\; T(w)\;: \qquad T(z) = \left(\partial_z w\right)^2\, T(w) - \alpha\,\{w,z\}\;. \tag{13.12}$$

13.2. Structure of the space of fields

At this point one can temporarily forget about the background metric, and apply the standard (flat space) constructions of CFT. The infinitesimal form (under $z \to w(z) = z + \varepsilon(z)$) of the transformation law (13.12)

$$\delta_\varepsilon T = \varepsilon\,\partial_z T + 2\,(\partial_z\varepsilon)\,T + \frac{c}{12}\,\partial_z^3\,\varepsilon \tag{13.13}$$

implies the operator product expansions

$$T(z)T(z') = \frac{c}{2\,(z-z')^4} + \frac{2}{(z-z')^2}\,T(z') + \frac{1}{z-z'}\,\partial_{z'}T(z') + \text{regular terms}\,. \tag{13.14}$$

This allows one to define two sets of operators, $\{L_n\}$ and $\{\bar{L}_n\}$ ($n \in \mathbb{Z}$), acting in the space of local fields $\mathcal{F}$, as

$$\begin{aligned} L_n O(z_0,\bar{z}_0) &= \oint_{C_{z_0}} \frac{dz}{2\pi i}\,(z-z_0)^{n+1}\,T(z)\,O(z_0,\bar{z}_0)\,,\\ \bar{L}_n O(z_0,\bar{z}_0) &= \oint_{\bar{C}_{\bar{z}_0}} \frac{d\bar{z}}{2\pi i}\,(\bar{z}-\bar{z}_0)^{n+1}\,\bar{T}(\bar{z})\,O(z_0,\bar{z}_0)\,, \end{aligned} \tag{13.15}$$

[7] As usual, $\{w,z\}$ stands for the Schwartzian derivative

$$w,z = \frac{\partial_z^3 w}{\partial_z w} - \frac{3}{2}\left(\frac{\partial_z^2 w}{\partial_z w}\right)^2\,. \tag{13.10}$$

for any $O \in \mathcal{F}$. Here C_{z_0} is a small contour encircling the point z_0, in the z-plane, in the counterclockwise direction, and $\bar{C}_{\bar{z}_0}$ is similar contour in the $\bar{z}$-plane. As follows from (13.14), each of the sets of operators, $\{L_n\}$ and $\{\bar{L}_n\}$, satisfies the Virasoro commutators, e.g.

$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{c}{12}\, \delta_{n+m,0}\,, \tag{13.16}$$

while L_n and $\bar{L}_m$ commute. Furthermore, it is possible to define a metric in the space $\mathcal{F}$, such that

$$L_n^\dagger = L_{-n}\,, \tag{13.17}$$

and argue that the the Hermitian operators L_0 and $\bar{L}_0$ are bounded from below [8]. The operator $S = L_0 - \bar{L}_0$ describes the spin, and for local fields its spectrum is in $\frac{1}{2}\,\mathbb{Z}$.

As the result, the space $\mathcal{F}$ can be described in terms of irreducible highest-weight (actually, lowest weight) representations $\mathcal{V}_\Delta$ of the Virasoro algebra,

$$\mathcal{F} = \oplus_a \mathcal{F}_a\,, \qquad \mathcal{F}_a = \mathcal{V}_{\Delta_a} \otimes \bar{\mathcal{V}}_{\bar{\Delta}_a}\,, \quad \Delta_a - \bar{\Delta}_a \in \frac{1}{2}\,\mathbb{Z}\,. \tag{13.18}$$

Each of the invariant subspaces contains the *primary* field $\Phi_{\Delta_a,\bar{\Delta}_a} \in \mathcal{F}_a$, such that it is annihilated by all positive-mode operators $L_n, \bar{L}_n$ with $n > 0$. The primary field is characterized by two *conformal dimensions*, Δ_a and $\bar{\Delta}_a$, the eigenvalues of L_0 and $\bar{L}_0$, respectively. By the definition, the primary field $\Phi_{\Delta,\bar{\Delta}}$ obeys the transformation law

$$\Phi_{\Delta,\bar{\Delta}}(z,\bar{z}) = \left(\frac{dw}{dz}\right)^{\Delta} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{\bar{\Delta}} \Phi_{\Delta,\bar{\Delta}}(w,\bar{w}) \tag{13.19}$$

under the conformal coordinate transformations $z \to w(z),\ \bar{z} \to \bar{w}(\bar{z})$. I will often write simply Φ_a for the primary field $\Phi_{\Delta_a,\bar{\Delta}_a}$. The rest of the fields in $\mathcal{F}_a$ is obtained by applying the negative-mode operators $L_{-n},\ \bar{L}_{-n}$ to the primary field Φ_a. These fields - the *descendants* of Φ_a - transform in more complicated way.

Let us now recall that we are dealing with the field theory in a (fixed) background metric $g_{\mu\nu}(x)$, and try to understand true meaning of the above transformations of the fields. When we change to new coordinates $y^\mu(x)$, and correspondingly transform the the metric tensor $g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(y)$,

$$g_{\mu\nu}(x) = \frac{\partial y^{\mu'}}{\partial x^\mu}\frac{\partial y^{\nu'}}{\partial x^\nu}\, \tilde{g}_{\mu'\nu'}(y(x))\,, \tag{13.20}$$

we of course still have the same theory, just viewed from the different coordinate frame. The natural objects are then those which transform covariantly, as scalars, vectors, tensors, etc. How the above conformal fields, with the strange transformation properties like (13.19) relate

[8] One recalls that $\mathcal{F}$ is isomorphic to the $\mathcal{H}_{\text{radial}}$, the space of states in radial quantization picture. In this relation, the radial Hamiltonian is

$$H_{\text{radial}} = -\frac{c}{12} + L_0 + \bar{L}_0\,.$$

to such covariant objects? The answer is simple in the case of the primary fields Φ_a. Recall that in conformal complex coordinates, with the metric (13.3), the conformal transformation (13.8) leads to the transformation (13.9), i.e.

$$e^{\sigma(z,\bar{z})} = \frac{dw}{dz}\frac{d\bar{w}}{d\bar{z}}\, e^{\sigma(w,\bar{w})}\,. \tag{13.21}$$

If $\Phi_{\Delta,\Delta}$ is a spinless primary field, the combination

$$\Phi^{[g]}(z,\bar{z}) = e^{-\Delta\,\sigma(z,\bar{z})}\,\Phi_{\Delta,\Delta}(z,\bar{z}) \tag{13.22}$$

is a scalar field. Likewise, if we have the primary fields with spin, e.g. $\Phi_{\Delta+1,\Delta}$ and $\Delta_{\Delta,\Delta+1}$, the local objects

$$\begin{aligned} \Phi^{[g]}_z(z,\bar{z}) &= e^{-\Delta\,\sigma(z,\bar{z})}\,\Phi_{\Delta+1,\Delta}(z,\bar{z}) = A_z\,,\\ \Phi^{[g]}_{\bar{z}}(z,\bar{z}) &= e^{-\Delta\,\sigma(z,\bar{z})}\,\Phi_{\Delta,\Delta+1}(z,\bar{z}) = A_{\bar{z}} \end{aligned} \tag{13.23}$$

transform as the contravariant components of a vector A_μ. The corresponding covariant components A^μ are

$$A^z = 2\,e^{-(\Delta+1)\sigma}\;\Phi_{\Delta,\Delta+1}\,, \qquad A^{\bar{z}} = 2\,e^{-(\Delta+1)\sigma}\;\Phi_{\Delta+1,\Delta}\,. \tag{13.24}$$

Primary fields with higher spin are interpreted similarly. Given a scalar field of the form (13.22), we can construct the associated density by multiplying by $\sqrt{g}$, so that the integral

$$\int e^{(1-\Delta)\,\sigma(z,\bar{z})}\,\Phi_{\Delta,\Delta}(z,\bar{z})\,d^2z \tag{13.25}$$

is invariant under the coordinate transformations.

The descendants require a little bit more care. The simplest example is the descendant $L_{-1}\Phi$ of a spinless primary field $\Phi = \Phi_{\Delta,\Delta}$. Since in the flat case $L_{-1}\Phi = \partial_z\Phi$, the associated covariant object is

$$\partial_z\Phi^{[g]} = \partial_z\left(e^{-\Delta\,\sigma}\,\Phi_{\Delta,\Delta}\right)\,. \tag{13.26}$$

Likewise, the covariant fields associated with $L_{-1}\bar{L}_{-1}\Phi$ and $L_{-1}^2\Phi$ are, respectively

$$\partial_z\partial_{\bar{z}}\left(e^{-\Delta\,\sigma}\,\Phi_{\Delta,\Delta}\right) \qquad \text{and} \qquad e^{-\sigma}\partial_z\left[e^{\sigma}\partial_z\left(e^{-\Delta\sigma}\right)\right]\,. \tag{13.27}$$

Expression for $L_{-2}\Phi$.

13.3. Weyl anomaly and partition function

The anomaly equation

$$T^\mu_\mu = -\frac{c}{12}\,R \tag{13.28}$$

has another important implication. By the definition of the energy-momentum tensor, the field T^μ_μ describes the response of the theory to the variation of the conformal factor σ. If one changes

$$g_{\mu\nu}(x) \;\to\; (1+\delta\sigma(x))\; g_{\mu\nu}(x) \tag{13.29}$$

we have

$$\delta\mathcal{A} = \frac{1}{4\pi}\int \sqrt{g}\;\delta\sigma(x)\,T^\mu_\mu(x)\,d^2x\,. \tag{13.30}$$

In conformal field theory, this equation gives much control of the dependence of the partition function

$$Z[g] = \int e^{-\mathcal{A}[g,\phi]}\; D[\phi] \tag{13.31}$$

of the form of the metric g. Let us write the metric tensor as

$$g_{\mu\nu}(x) = e^{\sigma(x)}\,\hat{g}_{\mu\nu}(x)\,, \tag{13.32}$$

where $\hat{g}$ is a fixed "reference" metric. The scalar curvature associated with the metric g has the form

$$\sqrt{g}\;R(x) = \sqrt{\hat{g}}\;\left(\hat{R}(x) - \Delta_{\hat{g}}\sigma(x)\right)\,, \tag{13.33}$$

where $\hat{R}$ and $\Delta_{\hat{g}}$ are the scalar curvature and the Laplace operator associated with the metric $\hat{g}$. The equation

$$\delta\log Z\left[e^\sigma\,\hat{g}\right] = \frac{c}{48\pi}\int_{\mathbb{M}}\sqrt{g}\;R(x)\,\delta\sigma(x)\,d^2x \tag{13.34}$$

which follows from (13.30), can be integrated to yield

$$Z\left[e^\sigma\,\hat{g}\right] = \exp\left\{\frac{c}{48\pi}\int\sqrt{\hat{g}}\;\left[\hat{R}(x)\,\sigma(x) + \frac{1}{2}\;\hat{g}^{\mu\nu}(x)\partial_\mu\sigma\partial_\nu\sigma\right]\,d^2x\right\}\,Z\left[\,\hat{g}\,\right]\;. \tag{13.35}$$

In writing this equation I have assumed that the manifold $\mathbb{M}$ is compact. The case of the boundary will be mentioned later.

One important implication of this equation is explicit dependence of the partition function of a statistical system at criticality on the overall size of the system. Recall that if the correlation length is finite, and the size of the system $l >> R_c$, according to the usual arguments about the thermodynamic limit the partition function behaves as

$$Z \;\to\; \exp\left\{-\frac{F\,A}{kT}\right\}\,, \qquad l >> R_c \tag{13.36}$$

where $A\sim l^2$ is the volume (area) of the $\mathbb{M}$, and f is the specific free energy (the free energy $\mathbb{F}$ is an intensive quantity, $\mathbb{F} = F\,A$). At criticality $R_c = \infty$, and this behavior is modified. The above analysis implies that instead we have

$$Z \;\to\; A^\kappa\,e^{-\frac{F_c\,A}{T_c}}\,, \tag{13.37}$$

with

$$\kappa = \frac{c}{6}\,(1-\gamma)\ . \tag{13.38}$$

Here γ is the genus of $\mathbb{M}$. Let me stress that the physical partition function is dimensionless, so the factor A^κ is understood here as $(A/\varepsilon)^\kappa$, where ε is a microscopic scale (interatomic distance, lattice spacing, etc) usually referred to as the "short-scale cutoff".

Changing the size of the system from l_0 to $l = L\,l_0$ (**Fig.1**) is described by the corresponding change of the metric

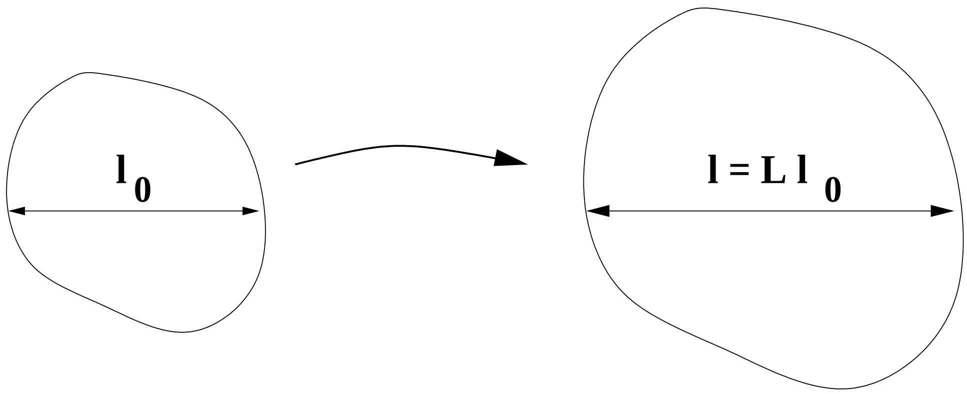


Figure 24:

$$g_{\mu\nu}(x)\ \rightarrow\ L^2\ g_{\mu\nu}(x)\ , \tag{13.39}$$

i.e. to the shift of $\sigma(x)$,

$$\sigma(x) \rightarrow \sigma(x) + 2\,\log L\ . \tag{13.40}$$

It follows from (13.35)

$$Z[L^2\,g] = \exp\left\{\frac{c}{48\pi}\,\log L^2\ \int R\,\sqrt{g}\ d^2x\right\}\ Z[g]\,. \tag{13.41}$$

Recalling that $\int R\,\sqrt{g} = 8\pi\,(1-\gamma)$, where γ is the genus of $\mathbb{M}$, we derive

$$Z[L^2\,g] = e^{\frac{c}{6}\,(1-\gamma)\,\log L^2}\ Z[g]\ , \tag{13.42}$$

confirming (13.36).

Let me add few remarks. The conformal anomaly equation can be generalized to include the cases where $\mathbb{M}$ has boundaries. The equation (13.1) acquires additional term concentrated at the boundary, such that the equation (13.35) changes as follows

$$\frac{Z\,[e^\sigma\,\hat{g}]}{Z\,[\,\hat{g}\,]} = \exp\left\{\frac{c}{48\pi}\int_{\mathbb{M}}\sqrt{\hat{g}}\ \left[\hat{R}\,\sigma(x) + \frac{1}{2}\,\hat{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\right]\ d^2x + \frac{c}{24\pi}\int_{\partial\mathbb{M}} K_{\hat{g}}\,\sigma(x)\,dl(x)\right\}\ , \tag{13.43}$$

where $K_{\hat{g}}$ is the geodesic curvature of the boundary, and $dl(x)$ is the length element of the boundary. The Gauss-Bonnet theorem [9] then assures that the equation (13.42) still holds, with $(1-\gamma)$ replaced by $\chi(\mathbb{M})/2$. For example, for a disk we have

$$Z_{\text{disk}} \sim \left(L^2\right)^{\frac{c}{12}}\, e^{-\frac{F_c A}{kT_c}} \tag{13.44}$$

where L is the typical size of the system.

The equation (13.35) is written under assumption that $\sigma(x)$ is sufficiently smooth (continuous) function on $\mathbb{M}$. However, in many problems one has to deal with manifold $\mathbb{M}$ having singularities. For example, we can have a conical point at some x_0, where the curvature $R(x)$ (rather, the associated density) has delta-function spike

$$\sqrt{g}\, R(x) = 8\pi\eta\; \delta(x-x_0) + \text{regular part}\,, \qquad \eta < 1 \tag{13.45}$$

Such singularity corresponds to the conical point, and the parameter η has simple interpretation. When sufficiently close to the point x_0 the regular terms in (13.45) are can be

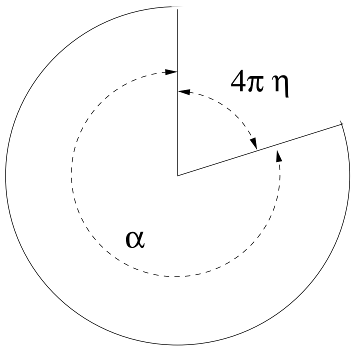


Figure 25:

disregarded, and we have a flat cone with the tip at x_0. Then $4\pi\,\eta$ is the "deficit angle" (**Fig.2**), so that the length of a small geodesic circle of radius r is $\alpha\, r$ with

$$\alpha = 2\pi\,(1-2\eta)\,. \tag{13.46}$$

If the reference metric $\hat{g}$ is taken to be flat, so that $d\hat{s}^2 = dzd\bar{z}$ in some local complex coordinates $(z,\bar{z})$ near x_0 (such that x_0 is at $z,\bar{z}=0$), the conformal factor diverges at $z=0$,

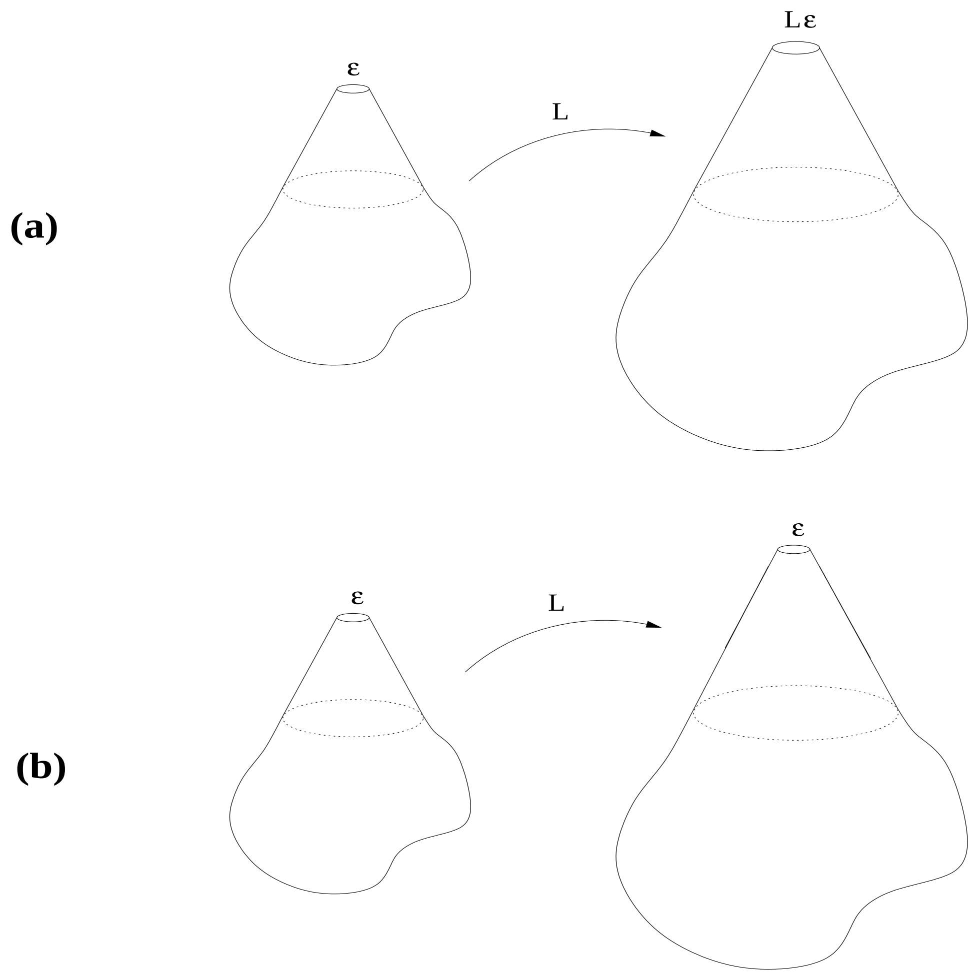
$$ds^2 = e^{\sigma(z,\bar{z})}\, dzd\bar{z} \;\to\; e^{\hat{\sigma}}\, \frac{dzd\bar{z}}{(z\bar{z})^{2\eta}} \qquad \text{as} \quad |z| \to 0\,, \tag{13.47}$$

[9]
$$\int_{\mathbb{M}} R(x)\, dA(x) + 2\int_{\partial\mathbb{M}} K_g(x)\, dl(x) = 4\pi\,\chi(\mathbb{M}) = 8\pi\,(1-\gamma)$$

where $\hat{\sigma}$ is a constant. Since with this choice of the background metric $\hat{R} = 0$, the only important term in (13.35) involves the integral of $\partial_z \sigma \partial_{\bar{z}} \sigma$, which logarithmically diverges at $|z| \to 0$. To sort this out, we replace the singular metric g with the regularized one, with the curvature bump (13.45) spread over some small domain D around x_0. Let us take D to be small disk $|z| < \epsilon$ in the above complex coordinates, and replace inside it the singular metric with the flat one,

$$e^{\sigma} \to e^{\sigma_\epsilon} = \begin{Bmatrix} e^{\hat{\sigma}}\,(z\bar{z})^{-2\eta} & \text{at} \quad |z| > \epsilon \\ e^{\hat{\sigma}}\,(\epsilon^2)^{-2\eta} & \text{at} \quad |z| < \epsilon \end{Bmatrix} \tag{13.48}$$

About conical singularities!



13.4. Correlation functions

Similar control is possible over the σ-dependence of the correlation functions

$$\langle\, O_1^{[g]}(x_1) \cdots O_N^{[g]}(x_N)\,\rangle_{[g]} \ , \tag{13.49}$$

where the subscript $[g]$ signifies that the background metric $g_{\mu\nu}$ is present, and $O^{[g]}$ are scalar or tensor fields (see §1.2 above). In the simplest case where all the field insertions correspond to spinless primaries, as in (13.22), we have for $g = e^{\sigma}\,\hat{g}$:

$$\langle\, \Phi^{[g]}_{\Delta_1}(x_1)\cdots\Phi^{[g]}_{\Delta_n}(x_n)\,\rangle_{[g]} = \left[\prod_{i=1}^{n} e^{-\Delta_i\,\sigma(x_i)}\right]\,\langle\, \Phi^{[\hat{g}]}_{\Delta_1}(x_1)\cdots\Phi^{[\hat{g}]}_{\Delta_n}(x_n)\,\rangle_{[\hat{g}]}\ . \tag{13.50}$$

14. Quantum gravity

In quantum gravity one treats the geometry, i.e. the metric $g_{\mu\nu}(x)$ as another variable to be integrated over in the functional integral. Important class of problems in QG involves calculating the "correlation functions", i.e. the functional integrals

$$Z^{-1}\int O^{[g]}_1(x_1)O^{[g]}_2(x_2)\cdots O^{[g]}_n(x_n)\,e^{-\mathcal{A}[\phi,g]}\,D[\phi]D[g]\,, \tag{14.1}$$

with Z now being the full partition function involving the integration over the geometries

$$Z = \int e^{-\mathcal{A}[\phi,g]}\,D[\phi]D[g]\,. \tag{14.2}$$

Now the action $\mathcal{A}$ depends both on the matter field(s) $\phi(x)$ and the metric tensor $g_{\mu\nu}(x)$ in a local way. The fields $O_i(x)$ are local composite objects built from $\phi(x)$ and $g(x)$ (the superscript $[g]$ in (14.1) is written to remind possible involvement of the metric; in what follows I usually omit it). Here I will usually assume that the fields $O(x)$ are *scalars* (i.e. transform trivially under coordinate transformations).

Let me briefly remind few basic things about the integration measure $D[g]$ in the above integrals. The metric tensors related by coordinate transformations represent the same geometries, therefore the variations

$$\delta g_{\mu\nu}(x) = \nabla_\mu\varepsilon_\nu(x) + \nabla_\nu\varepsilon_\mu(x)\,, \tag{14.3}$$

corresponding to the infinitesimal changes of coordinates

$$x^\mu \to x^\mu + \varepsilon^\mu(x)\,, \tag{14.4}$$

are to be treated as the gauge transformations. Therefore the measure $D[g]$ has to be defined as

$$D[g] = \frac{D[g_{\mu\nu}]}{D[\varepsilon]}\,, \tag{14.5}$$

where $D[g_{\mu\nu}]$ is local functional measure for the tensor field $g_{\mu\nu}(x)$, and the denominator represent a measure on the space of diffeomorphisms, which can be realized as the space of the vector fields $\varepsilon^\mu(x)$. The idea of Polyakov was to define both measures by introducing natural "ultra-local" metrics in each of the spaces. Thus one introduces the norms

$$||\delta g_{\mu\nu}||^2 = \int_{\mathbb{M}} \sqrt{g}\,\left[g^{\mu\alpha}g^{\nu\beta} + C\,g^{\mu\nu}g^{\alpha\beta}\right]\delta g_{\mu\nu}\delta g_{\alpha\beta}\,d^2x\,, \tag{14.6}$$

$$||\varepsilon||^2 = \int_{\mathbb{M}} \sqrt{g}\; g_{\mu\nu}\, \varepsilon^\mu \varepsilon^\nu \; d^2x \,. \tag{14.7}$$

In the first of these equations $C > 1/2$ is a constant whose exact value does not affect the result. Write the variation of the metric tensor as

$$\delta g_{\mu\nu}(x) = g_{\mu\nu}(x)\; \delta\sigma(x) + \nabla_\mu \varepsilon_\nu(x) + \nabla_\nu \varepsilon_\mu(x)\,, \tag{14.8}$$

where $\delta\sigma(x)$ represents the variation of the conformal factor, and the last two terms correspond to the gauge transformations. In fact, it will be convenient to make a shift of the "physical" variation,

$$\delta\sigma + \nabla_\alpha \varepsilon^\alpha \;\; \rightarrow \;\; \delta\sigma\,; \tag{14.9}$$

then (14.8) takes the form

$$\delta g_{\mu\nu} = g_{\mu\nu}\; \delta\sigma + \mathcal{E}_{\mu\nu}\,, \qquad \mathcal{E}_{\mu\nu} = \nabla_\mu \varepsilon_\nu + \nabla_\nu \varepsilon_\mu - g_{\mu\nu}\; \nabla_\lambda \varepsilon^\lambda \,. \tag{14.10}$$

Then the physical and gauge variations in (14.6) separate,

$$||\delta g||^2 = \int \sqrt{g}\; \Big[A \left(\delta\sigma(x) \right)^2 + \mathcal{E}_{\mu\nu} \mathcal{E}^{\mu\nu} \Big]\; d^2x \,, \tag{14.11}$$

where $A = 2(1 + 2C)$ is another constant. Therefore, the ratio in (14.5) can be written as

$$\det\left[\mathcal{E} \right]\; D[\sigma]\,, \tag{14.12}$$

where we have the determinant of the operator $\mathcal{E}(\varepsilon)$ defined in (14.10), and $D[\sigma]$ is suitably defined measure over the space of functions $\sigma(x)$. As usual, it is advantageous to write the determinant as the Gaussian functional integral

$$\det\left[\, \mathcal{E} \,\right] \equiv Z_{\text{ghost}}[\, g\,] = \int e^{-\mathcal{A}_{\text{ghost}}[B,C\,|\,g]}\; D[B, C] \tag{14.13}$$

over the fermionic (i.e anti-commutative) fields $B_{\mu\nu}(x)$ and $C^\mu(x)$ - the "ghosts", the field $B_{\mu\nu}$ being symmetric and traceless,

$$B_{\mu\nu}(x) = B_{\nu\mu}(x)\;, \qquad \text{and} \qquad g^{\mu\nu}(x) B_{\mu\nu}(x) = 0\,, \tag{14.14}$$

with the ghost action

$$\mathcal{A}_{\text{ghost}}[B, C|g] = \frac{1}{2\pi} \int_{\mathbb{M}} \sqrt{g}\; B_{\mu\nu} \nabla^\mu C^\nu \; d^2x \,. \tag{14.15}$$

In this analysis I have implicitly assumed that the conformal factor field $\sigma(x)$ exhausts all physical degrees of freedom, i.e. that any geometry g can be represented by metric tensor of the form

$$g_{\mu\nu}(x) = e^{\sigma(x)}\, \hat{g}_{\mu\nu}(x)\,, \tag{14.16}$$

where $\hat{g}_{\mu\nu}(x)$ is some fixed "reference" metric. Generally, this is not the case. While the conformal factor takes care of the bulk (functional) part of the physical degrees of freedom, there is a finite number (the number depends on the topology of $\mathbb{M}$) of additional parameters - the "conformal moduli". Roughly speaking, for the Eq.(14.16) to become the exhausting characterization of all geometries, the reference metric $\hat{g}$ must be given dependence of finite number of the parameters. Here I will not describe the situation in details. I what follows I will mostly deal with $\mathbb{M}$ having topology of a sphere; in that case there are no moduli. But in general case we should write

$$\int_{[g]} (\cdots) = \int_{\text{moduli}} d\mu(\hat{g}) \int D[\,\sigma\,]\; Z_{\text{ghost}}\,[\,e^{\sigma}\,\hat{g}\,]\;(\cdots)\,, \tag{14.17}$$

where $d\mu(\hat{g})$ is properly defined measure on the moduli space, and Z_{ghost} is the ghost determinant (14.13).

Few words about the measure $D[\sigma]$. As it comes from the above analysis, it is not the usual linear functional measure we considered before in the context of scalar field. Indeed, from (14.11) we have for the metric in the space of the functions $\sigma(x)$

$$||\delta\sigma||^2 = A\int \sqrt{g}\;\,(\delta\sigma)^2\;d^2x = A\int \sqrt{\hat{g}}\;e^{\sigma}\;(\delta\sigma)^2\;d^2x\,. \tag{14.18}$$

Formally, the factor e^{σ} violates translational symmetry of the measure. Intuitively, this unusual form of the measure is expected. In the intuitive expression $\prod_x d\sigma_x$ we must assume that the points x fill the coordinate space with the density $e^{\sigma}(x)\Delta^2 x$, which is a constant density in terms of the physical geometry. To put it differently, the microscopic "cutoff" scale ε has to be set in terms of the physical metric $e^{\varepsilon}\,\hat{g}$, so that formally, in the coordinate space the cutoff distance Δx must depend on the field $\sigma(x)$. However, as we have already discussed before (in connection with the generic local field variable transformations) one can expect that this non-linear measure can be reduced to a linear one, at the price of adding certain local terms to the Lagrangian density. We will return to this point shortly.

The ghost determinant $Z_{\text{ghost}}[\,e^{\sigma}\,\hat{g}\,]$ in (14.17) depends on the conformal factor in a relatively simple way. The ghost theory (14.15) is a conformal field theory, and the dependence is controlled by the conformal anomaly equation. I will not give detailed calculations here (Appendix?). The associated central charge has special value

$$c_{\text{ghost}} = -26\,, \tag{14.19}$$

and hence we have

$$\log\frac{Z_{\text{ghost}}\,[\,e^{\sigma}\,\hat{g}\,]}{Z_{\text{ghost}}\,[\,\hat{g}\,]} = -\frac{26}{48\pi}\int\sqrt{\hat{g}}\;\left[\hat{R}(x)\,\sigma(x) + \frac{1}{2}\;\hat{g}^{\mu\nu}\,\partial_\mu\sigma\partial_\nu\sigma\right]\;d^2x\,. \tag{14.20}$$

Let us assume that the action $\mathcal{A}[\phi, g]$ in (14.1) is such that the quantum theory of the "matter" field(s) ϕ, in fixed metric background g, is conformally invariant, with the central charge c. Then, as we have seen, the dependence of the "matter" partition function

$$Z_{\text{matter}}[\,g\,] = \int e^{-\mathcal{A}[\phi,g]}\,D[\phi] \tag{14.21}$$

on the conformal factor $\sigma(x)$ is established in closed form, the Eq.(13.35) . The gravitational partition function (14.2) then can be written as the integral

$$Z = \int_{\text{moduli}} d\mu(\hat{g})\, Z_{\text{matter}}[\hat{g}]\, Z_{\text{ghost}}[\hat{g}] \int e^{-\mathcal{A}_L[\sigma,\hat{g}]}\, D[\sigma]\,, \tag{14.22}$$

where the action $\mathcal{A}_L[\sigma]$ combines the anomaly contributions from both the matter, Eq.(13.35), and the ghosts, Eq.(14.20),

$$\mathcal{A}_L[\sigma] = \frac{26-c}{48\pi} \int_{\mathbb{M}} \sqrt{\hat{g}}\, \left[\hat{R}(x)\,\sigma(x) + \frac{1}{2}\,\hat{g}^{\mu\nu}\,\partial_\mu\sigma\partial_\nu\sigma\right] d^2x\,. \tag{14.23}$$

Similarly, one can reduce the functional integral in the numerator in the expression (14.1) for the correlation functions. Assuming for simplicity that all the insertion fields are scalars associated with conformal primaries of the matter theory, and using (13.50), we have for the correlation function (14.1)

$$Z^{-1} \int_{\text{moduli}} d\mu(\hat{g})\; Z_{\left(\substack{\text{matter,}\\ \text{ghosts}}\right)}[\,\hat{g}\,]\; \langle\, \Phi_{\Delta_1}(x_1)\cdots\Phi_{\Delta_n}(x_n)\,\rangle_{[\hat{g}]}\; \times$$
$$\int D[\,\sigma\,]\,\left(e^{-\Delta_1\,\sigma(x_1)}\;\ldots\; e^{-\Delta_n\,\sigma(x_n)}\right)\; e^{-\mathcal{A}_L[\sigma,\hat{g}]}\,, \tag{14.24}$$

The integral over the field $\sigma(x)$ will be the main subject of our attention.

So far we have ignored the possibility of having separate gravitational terms, i.e. the terms in $\mathcal{A}[\phi, g]$ which depend on the metric only. There are many possible local terms of this kind, e.g.

$$\mathcal{A}_{\text{grav}}[g] = \Lambda_c \int_{\mathbb{M}} \sqrt{g}\, d^2x + G \int_{\mathbb{M}} \sqrt{g}\, R(x)\, d^2x + \cdots\,. \tag{14.25}$$

The first term here is the cosmological term, where the cosmological constant Λ enters multiplied by the total area of $\mathbb{M}$. The second term has the form of the Einstein action. In two dimensions this term is topological invariant since $\int R\sqrt{g}\, d^2x = 8\pi\,(1-\gamma)$. Unless we are interested in summation over all topologies (as in fact we are in the string theory), we can disregard this term. Further terms could involve higher powers of the curvature, as well as other scalars of higher dimensions built from R. Introducing such terms leads to problems in defining the theory, and at this point I will not allow them into the action. We are left with the cosmological term which has important effect in the theory. Adding the cosmological term modifies the action (16.18) by adding another term to the expression in the square brackets

$$\int \sqrt{\hat{g}}\;\; [\ldots + \Lambda\, e^{\sigma}]\; d^2x\,, \tag{14.26}$$

where Λ is a constant proportional to Λ_c.

Matter contributions to the cosmological term.

15. Classical limit

The full action $\mathcal{A}_L[\sigma, \hat{g}]$, including the cosmological term (14.26), can be written as

$$\mathcal{A}_L[\sigma, \hat{g}] = \frac{1}{\hbar}\, S_L[\sigma, \hat{g}]\,, \tag{15.1}$$

where

$$S_L[\sigma, \hat{g}] = \frac{1}{2\pi}\int_{\mathbb{M}} \sqrt{\hat{g}}\;\left[\frac{1}{2}\,\hat{g}^{\mu\nu}\,\partial_\mu\sigma\partial_\nu\sigma + \hat{R}(x)\,\sigma(x) + \Lambda\, e^{\sigma}\right]\, d^2x\,, \tag{15.2}$$

and I have introduced the "Planck's constant" $\hbar$,

$$\frac{1}{\hbar} = \frac{26-c}{24}\,. \tag{15.3}$$

In the limit $\hbar \to 0$ (which corresponds to sending the matter central charge c to $-\infty$) the integral over $\sigma(x)$ is dominated by the stationary points of the classical action (15.2). In this limit it is important to understand the associated classical configurations $\sigma^{\rm cl}(x)$, the solutions of the classical equations associated with the action (15.2). That is what we are going to do next.

15.1. Liouville equation

The Euler-Lagrange equation for the action (15.2) has the form

$$-\,\Delta_{\hat{g}}\sigma(x) + \hat{R}(x) + \Lambda\, e^{\sigma(x)} = 0\,, \tag{15.4}$$

where $\Delta_{\hat{g}}$ is the Laplace operator associated with the metric $\hat{g}$. Recalling that $\sqrt{\hat{g}}\,\left(\hat{R} - \Delta_{\hat{g}}\sigma\right) = \sqrt{g}\,R$ (Eq.(13.33)), where $g = e^{\sigma}\hat{g}$ and R is the scalar curvature associated with g, one can write (15.4) as

$$R(x) + \Lambda = 0\,. \tag{15.5}$$

The classical configurations describe geometries with constant curvature equal $-\Lambda$. In what follows both cases of negative and positive curvature will be important; correspondingly, Λ can positive or negative, depending on the situation.

Concentrating attention on some domain in $\mathcal{M}$, let us take $\hat{g}$ to be the flat Euclidean metric, so that $\hat{R} = 0$. In local conformal complex coordinates the metric $g = e^{\sigma}\,\hat{g}$ then has the form (13.3), and the equation (15.4) reduces to

$$-4\,\partial_z\partial_{\bar{z}}\sigma + \Lambda\, e^{\sigma} = 0\,. \tag{15.6}$$

This is known as the Liouville equation, and it gives the name to the whole theory we are developing (including the full quantum theory).

Some general properties of the Liouville equation are readily established. Consider the local quantities

$$t = -\partial_z\sigma\partial_z\sigma + 2\partial_z^2\sigma\,, \qquad \bar{t} = -\partial_{\bar{z}}\sigma\partial_{\bar{z}}\sigma + 2\partial_{\bar{z}}^2\sigma\,, \tag{15.7}$$

It is easy to check that if σ satisfies (16.7) then these are holomorphic and anti-holomorphic functions, respectively, i.e. $t = t(z)$ and $\bar{t} = \bar{t}(\bar{z})$. Indeed, for instance

$$2\,\partial_{\bar{z}} t = -4\,\partial_z\sigma\partial_z\partial_{\bar{z}}\sigma + 4\,\partial_z^2\partial_{\bar{z}}\sigma = -\partial_z\sigma\;\Lambda\,e^{\sigma} + \partial_z(\Lambda\,e^{\sigma}) = 0\,. \tag{15.8}$$

In fact, t and $\bar{t}$ are components of classical energy-momentum tensor $t_{\mu\nu}$ defined as the variation

$$\delta S[\sigma,\hat{g}] = -\frac{1}{4\pi}\int \sqrt{\hat{g}}\,\delta\hat{g}^{\mu\nu}\,t_{\mu\nu}\,d^2x \tag{15.9}$$

of the classical action (15.2) with respect to the reference metric $\hat{g}_{\mu\nu}$. Direct variation yields in flat metric $\hat{g}$

$$t_{\mu\nu} = -\partial_\mu\sigma\partial_\nu\sigma + \hat{g}_{\mu\nu}\left(\frac{1}{2}\,(\partial\sigma)^2 + \Lambda\,e^{\sigma}\right) + 2\,\left(\partial_\mu\partial_\nu\sigma - \hat{g}_{\mu\nu}\,\partial^2\sigma\right)\,. \tag{15.10}$$

The last term in (15.10) is due to the variation of the curvature term in (15.2). The trace of this tensor

$$t_\mu^\mu = 2\,\left(\Lambda\,e^{\sigma} - \partial^2\sigma\right) \tag{15.11}$$

vanishes in virtue of the classical equation (15.4) (with $\hat{R} = 0$), and

$$t_{zz} = t\,, \qquad t_{\bar{z}\bar{z}} = \bar{t}\,. \tag{15.12}$$

The above properties manifest conformal invariance of the Liouville equation. Conformal transformations $z \to w(z)\,,\ \bar{z} \to \bar{w}(\bar{z})$ leave the form of the equation (16.7) unchanged provided one transforms the field σ according to the eq.(13.9), i.e.

$$\sigma(w,\bar{w}) = \sigma(z\bar{z}) - \log\left(\frac{dw}{dz}\frac{d\bar{w}}{d\bar{z}}\right)\,. \tag{15.13}$$

Important role plays the field $e^{-\frac{\sigma}{2}}$. Consider its second derivative

$$\partial_z^2 e^{-\frac{\sigma}{2}} = -\frac{1}{2}\,\partial_z\left(\partial_z\sigma\;e^{-\frac{\sigma}{2}}\right) = -\frac{1}{4}\,\left(-\partial_z\sigma\partial_z\sigma + 2\,\partial_z^2\sigma\right)\,e^{-\frac{\sigma}{2}}\,, \tag{15.14}$$

i.e.

$$\begin{aligned} -4\,\partial_z^2 e^{-\frac{\sigma}{2}} &= t(z)\,e^{-\frac{\sigma}{2}}\,,\\ -4\,\partial_{\bar{z}}^2 e^{-\frac{\sigma}{2}} &= \bar{t}(\bar{z})\,e^{-\frac{\sigma}{2}}\,, \end{aligned} \tag{15.15}$$

where the second equation, with the $\bar{z}$ derivatives, is derived by identical manipulations. This property suggests the following general method of integration of the Liouville equation.

Consider the ordinary differential equation

$$-4\,\partial_z^2\psi(z) = t(z)\,\psi(z)\,, \tag{15.16}$$

as well as similar differential equation with respect to $\bar{z}$,

$$-4\,\partial_{\bar{z}}^2\bar{\psi}(\bar{z}) = \bar{t}(\bar{z})\,\bar{\psi}(\bar{z})\,. \tag{15.17}$$

Before further development, let us stop to discuss the transformation properties of these differential equations. Let us change to another variable $w = w(z)$. It is easy to check that the transformation preserves the structure of the equation (15.16), i.e. brings it to the form

$$-4\,\partial_w^2\psi(w) = t(w)\,\psi(w)\,, \tag{15.18}$$

with the new function

$$\psi(w) = (\partial_w z(w))^{-\frac{1}{2}}\ \psi(z(w))\,, \tag{15.19}$$

while the coefficient function $t(w)$ transforms as it should,

$$t(w) = (\partial_w z(w))^2\ t(z(w)) + 2\ \{z, w\}\ . \tag{15.20}$$

The equations (15.16), (15.17) play central role in the classical (and, with appropriate modifications, quantum) Liouville theory. Let $\psi(z) = (\psi_1(z), \psi_2(z))^{\mathrm{t}}$ be two linearly independent solutions of the holomorphic equations (15.16) (which I will understand as a column, hence the superscript t). As usual, the Wronskian of these two solutions is a constant, which can be given any value by suitable choice of the basis in the space of the solutions. We choose

$$\psi_1\partial_z\psi_2 - \psi_2\partial_z\psi_1 = 1\,. \tag{15.21}$$

Let also $\bar{\psi}(\bar{z}) = \big(\bar{\psi}_1(\bar{z}), \bar{\psi}_2(\bar{z})\big)$ be the complex-conjugated functions (now understood as a raw), which obviously form basis in the space of solutions of the Eq.(15.17). We have

$$\bar{\psi}_1\partial_{\bar{z}}\bar{\psi}_2 - \bar{\psi}_2\partial_{\bar{z}}\bar{\psi}_1 = 1\,. \tag{15.22}$$

It is straightforward to verify that the combination

$$\sigma(z,\bar{z}) = -2\,\log\big(\bar{\psi}(\bar{z})\,\mathbf{\Lambda}\,\psi(z)\big) + \log 8 = -2\,\log\big(\bar{\psi}_a(\bar{z})\,\mathbf{\Lambda}^{ab}\psi_b(z)\big) + \log 8\,, \tag{15.23}$$

with constant matrix $\mathbf{\Lambda}$, solves the Liouville equation (16.7) provided

$$\det\mathbf{\Lambda} = -\Lambda\,. \tag{15.24}$$

Indeed, let $\bar{\chi}(z) = \bar{\psi}(\bar{z})\,\mathbf{\Lambda}$. Then

$$-4\,\partial_z\partial_{\bar{z}}\sigma = -4\,\partial_z\partial_{\bar{z}}\left\{-2\,\log\left(\bar{\chi}(\bar{z})\psi(z)\right)\right\} = 8\,\partial_z\,\frac{(\partial_{\bar{z}}\bar{\chi}\psi)}{(\bar{\chi}\psi)} =$$
$$8\,\frac{(\partial_{\bar{z}}\bar{\chi}\partial_z\psi)(\bar{\chi}\psi) - (\partial_{\bar{z}}\bar{\chi}\psi)(\bar{\chi}\partial_z\psi)}{(\bar{\chi}\psi)^2}\,. \tag{15.25}$$

The expression in the numerator reduces to the product of the Wronskian (15.21) and the Wronskian

$$\bar{\chi}_1\partial_{\bar{z}}\bar{\chi}_2 - \bar{\chi}_2\partial_{\bar{z}}\bar{\chi}_1 = \det\mathbf{\Lambda} = -\Lambda\,. \tag{15.26}$$

We find

$$-4\,\partial_z\partial_{\bar z}\sigma \;=\; -\frac{8\,\Lambda}{(\bar\chi\psi)^2} = -\Lambda\, e^{\sigma}\,. \tag{15.27}$$

Thus, starting with generic differential equation (15.16), with generic analytic function $t(z)$, one obtains formal solution (15.23) of the Liouville equation (16.7). Generally, such formal solution is not is not satisfactory for two reasons: (a) the function (15.23) is not real, and (b) its is not single-valued. While the reality condition is relatively easy to impose (one just takes Hermitian $\mathbf{\Lambda}$), the requirement that $\sigma(z,\bar z)$ must be single-valued function in the Euclidean domain (i.e. when $\bar z = z^*$) poses difficult problem. One needs to analyze the monodromy properties of the differential equation (15.16).

15.2. Monodromy problem

Given the differential equation (15.16), it is possible to take different bases (ψ_1,ψ_2), which are related by $SL(2,\mathbb{C})$ transformations. By suitable choice of the basis, one can always bring the Hermitian matrix $\mathbf{\Lambda}$ to a canonical form, which however depends on the sign of the curvature. At positive curvature the most convenient form is just the identity matrix

$$\text{Positive curvature}\quad (\Lambda<0):\qquad \mathbf{\Lambda}=\sqrt{-\Lambda}\begin{pmatrix}1&0\\0&1\end{pmatrix}, \tag{15.28}$$

while at negative curvature one can take, for instance

$$\text{Negative curvature}\quad (\Lambda>0):\qquad \mathbf{\Lambda}=\sqrt{\Lambda}\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{15.29}$$

Let us assume that the domain of analyticity of $t(z)$ contains non-contractible loop C. Let us denote $\psi(C*z)$ the result of the analytic continuation along C. Since $t(z)$ is assumed to be single-valued function (see its definition (15.7)), $\psi(C*z)$ still solves the same equation (15.16), hence

$$\psi(C*z)=\mathbb{M}(C)\,\psi(z)\,,\qquad \bar\psi(C*\bar z)=\bar\psi(\bar z)\,\mathbb{M}^\dagger(C)\,, \tag{15.30}$$

where $\mathbb{M}(C)\in SL(2,\mathbb{C})$ is some matrix, representing the monodromy C. If there are several non-contractible loops C_i, the associated matrices $\mathbb{M}(C_i)$ form representation of the monodromy group. For the solution (15.23) to be single-valued, obviously, the transformations $\mathbb{M}(C_i)$ must preserve the form of the matrix $\mathbf{\Lambda}$,

$$\mathbf{M}^\dagger(C_i)\,\mathbf{\Lambda}\,\mathbf{M}(C_i)=\mathbf{\Lambda}\,. \tag{15.31}$$

Therefore, all $\mathbf{M}(C_i)$ must belong to certain real subgroup of $SL(2,C)$, depending on the sign of the curvature, [10]

$$\text{Positive curvature } (\Lambda < 0): \qquad \mathbf{M}(C_i) \in SU(2) \ , \tag{15.32}$$

$$\text{Negative curvature } (\Lambda > 0): \qquad \mathbf{M}(C_i) \in SU(1,1) \, . \tag{15.33}$$

In fact, at negative curvature it is often more convenient to choose different canonical form of the matrix $\mathbf{\Lambda}$:

$$\text{Negative curvature} \quad (\Lambda > 0): \qquad \mathbf{\Lambda} = \sqrt{\Lambda}\, i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ; \tag{15.34}$$

in such basis the monodromy matrices must be real,

$$\text{Negative curvature } (\Lambda > 0): \qquad \mathbf{M}(C_i) \in SL(2,R) \, , \tag{15.35}$$

for the condition (16.11) to hold.

Generally, these conditions impose highly nontrivial restrictions on possible form of the function $t(z)$. Finding useful solutions of the Liouville equation amounts to solving this monodromy problem. But before going into further details, let me exhibit few elementary solutions.

[10]The elements of the subgroups $SU(2)$ and $SU(1,1)$ are 2×2 matrices of the form

$$SU(2): \ \mathbf{M} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} , \ |a|^2 + |b|^2 = 1 \quad \text{and} \quad SU(1,1): \ \mathbf{M} = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} , \ |a|^2 - |b|^2 = 1 \, ;$$

The last subgroup is equivalent to $SL(2,R)$, the group of real 2×2 matrices with unit determinant.

Lecture 6. Classical Liouville Theory

16. Classical Liouville theory

In the presence of conformal matter (with the central charge c), the integration over 2D geometries g reduces to the functional integral over $D[\sigma]$, with the action

$$\mathcal{A}_L[\sigma, \hat{g}] = \frac{1}{\hbar}\, S_L[\sigma, \hat{g}]\,, \tag{16.1}$$

where

$$S[\sigma, \hat{g}] = \frac{1}{2\pi} \int_{\mathbb{M}} \sqrt{\hat{g}}\, \left[\frac{1}{2}\, \hat{g}^{\mu\nu}\, \partial_\mu\sigma\partial_\nu\sigma + \hat{R}(x)\, \sigma(x) + \Lambda\, e^{\sigma}\right] d^2x\,, \tag{16.2}$$

and

$$\frac{1}{\hbar} = \frac{26-c}{24}\,, \tag{16.3}$$

The field $\sigma(x)$ relates to the conformal factor in the metric tensor

$$g_{\mu\nu}(x) = e^{\sigma(x)}\, \hat{g}_{\mu\mu}(x)\,, \tag{16.4}$$

where $\hat{g}$ is some fixed "background" metric. In the limit $\hbar \to 0$, the functional integral is dominated by the classical configurations - the stationary points of the action (16.2). The classical configurations describes 2D geometries of constant curvature,

$$R(x) + \Lambda = 0\,. \tag{16.5}$$

By taking $\hat{g}$ of the form

$$\hat{g}\ : \qquad d\hat{s}^2 = dz d\bar{z} \tag{16.6}$$

in some local complex coordinates $(z, \bar{z})$, this equation reduces to the famous Liouville equation

$$4\, \partial_z \partial_{\bar{z}} \sigma = \Lambda\, e^{\sigma}\,. \tag{16.7}$$

The solution of the equation (16.7) can be written in terms the solutions of the associated ordinary differential equation

$$4\, \partial_z^2 \psi(z) + t(z)\, \psi(z) = 0\,, \tag{16.8}$$

where $t(z)$ holomorphic function, the t_{zz} component of the energy-momentum tensor $t_{\mu\nu}$ associated with the Liouville action. Namely, we have

$$\sigma(z, \bar{z}) = -2\, \log\big(\bar{\psi}(\bar{z})\, \mathbf{\Lambda}\, \psi(z)\big) + \log 8\,, \tag{16.9}$$

where $\psi(z)$ stands for a column $\psi(z) = (\psi_1(z), \psi_2(z))^{\mathrm{t}}$ of two linearly independent solutions of (16.8) (with unit Wronskian), $\bar{\psi}(\bar{z}) = \big(\bar{\psi}_1(\bar{z}), \bar{\psi}_2(\bar{z})\big)$ is the raw of the corresponding complex-conjugated functions, and $\mathbf{\Lambda}$ is some (basis-dependent) Hermitian matrix. The expression (16.9) yields single-valued function $\sigma(z,\bar{z})$ if, and only if, the matrices $\mathbf{M}(C_i) \in SL(2,C)$ representing the monodromy group of the differential equation (16.8)

$$\psi(C_i * z) = \mathbb{M}(C_i)\,\psi(z)\,, \tag{16.10}$$

leave the matrix $\mathbf{\Lambda}$ invariant

$$\mathbf{M}^\dagger(C_i)\,\mathbf{\Lambda}\,\mathbf{M}(C_i) = \mathbf{\Lambda} \qquad \text{for any} \;\; C_i\,. \tag{16.11}$$

That means the representation must belong to certain real subgroup of $SL(2,C)$, depending on the sign of the curvature $-\Lambda$:

$$\text{Positive curvature } \;(\Lambda < 0): \qquad \mathbf{M}(C_i) \in SU(2)\;\;, \tag{16.12}$$

$$\text{Negative curvature } (\Lambda > 0): \qquad \mathbf{M}(C_i) \in SU(1,1) \sim SL(2,R)\,. \tag{16.13}$$

This condition poses highly nontrivial restrictions on possible form of $t(z)$. The classical Liouville problem thus reduces to finding special $t(z)$, such that the monodromy condition (16.12) is satisfied. Before going into further details, let me exhibit few elementary solutions.

16.1. Elementary solutions

Sphere is the simplest solution with positive curvature. The only function with the transformation property [11] of t_{zz}, regular everywhere on a complex sphere, is $t = 0$. The two solutions of (16.8) are

$$\psi_1(z) = 1\,, \quad \psi_2(z) = z\,, \tag{16.14}$$

and according to (16.9)

$$\sigma(z,\bar{z}) = -2\,\log(1 + z\bar{z}) + \log(-8/\Lambda) \tag{16.15}$$

(Λ is negative). The corresponding metric

$$ds^2 = e^{\sigma(z,\bar{z})}\,dzd\bar{z} = -\frac{8}{\Lambda}\,\frac{dzd\bar{z}}{(1+z\bar{z})^2} \tag{16.16}$$

describes the sphere of the area

$$A = \int e^{\sigma(z,\bar{z})}\,d^2z = -\frac{8\pi}{\Lambda}\,. \tag{16.17}$$

For the flat background (16.6) we might want to write the Liouville action (16.2) simply as

$$S[\sigma] = \frac{1}{2\pi}\int \left[2\,\partial_z\sigma\partial_{\bar{z}}\sigma + \Lambda\,e^{\sigma}\right]\,d^2z\,, \tag{16.18}$$

[11] Recall that with $zz(w)$

$$t(w) = \left(\partial_w z(w)\right)^2\, t(z(w)) + 2\,\{z,w\}\,.$$

because we have $\hat{g} = 1$, and also $\hat{R} = 0$ everywhere at finite $(z, \bar{z})$. This is a bit too naive. The "action" (16.18) diverge for the spherical solution (16.15). Indeed,

$$\int_{|z|<L} 2\,\partial_z\sigma\partial_{\bar{z}}\sigma\, d^2z = \int_{|z|<L} \frac{8\, z\bar{z}}{(1+z\bar{z})^2}\, d^2z \to 8\pi\, \left(\log L^2 - 1\right) \tag{16.19}$$

where I have restricted the integration to the disk $|z| > L$, and then sent L to infinity. Clearly, such divergence is not acceptable. The saddle-point contribution to the functional integral is proportional to $\exp\left(-\frac{S_{\rm cl}}{\hbar}\right\}$, and infinite value of the classical action would mean that the contribution is zero.

Resolution of this puzzle requires more careful treatment of the background metric $\hat{g}$. Recall that $\hat{g}$ must be taken to be a metric on a sphere, albeit an arbitrary one. Most importantly, we must have $\int \sqrt{\hat{g}}\, \hat{R}\, d^2x = 8\pi$, not zero. It is possible to choose $\hat{g}$ in such a way that all the curvature is concentrated at one point $x = x_0$,

$$\sqrt{\hat{g}}\ \hat{R}(x) = 8\pi\, \delta(x - x_0)\,. \tag{16.20}$$

In fact, the "flat" metric $g_0 :\ d\hat{s}^2 = dzd\bar{z}$ should be understood as the one of this kind, with x_0 corresponding to $z_0 = \infty$. This fact is manifest in the coordinates $w = 1/z,\ \bar{w} = 1/\bar{z}$,

$$d\hat{s}^2 = dzd\bar{z} = \frac{dwd\bar{w}}{(w\bar{w})^2}\,, \tag{16.21}$$

so that

$$\sqrt{\hat{g}}\ \hat{R}(w, \bar{w}) = -4\,\partial_w\partial_{\bar{w}}\,(-2\,\log(w\bar{w})) = 8\pi\, \delta^{(2)}(w)\,. \tag{16.22}$$

But we have already seen in the Section 1 that such curvature singularities produce short-distance divergent terms in the partition function. Recall that for the metric $\hat{g}$ having the curvature singularity at some point x_0, such that

$$\sqrt{\hat{g}}\ \hat{R}(x) = 8\pi\, \eta\, \delta(x - x_0)\,, \tag{16.23}$$

the partition function develops divergent factor

$$Z[\,\hat{g}\,] \sim \exp\left\{-\frac{1}{\hbar}\,\left(-\,4\eta^2\ \log\varepsilon^2\right)\right\} \tag{16.24}$$

where again ε is the microscopic "size" of the singularity; it can be understood as the size of small domain over which the curvature bump (16.23) is spread. In the coordinates $(z, \bar{z})$ in the Eq.(16.21) this "small" domain is represented by the region outside the circle $|z| = 1/\varepsilon \equiv L$. To simplify things, let us take the "cutoff" spherical metric $\hat{g}_L$,

$$\hat{g}_L\ :\qquad d\hat{s}^2 = \left\{\begin{array}{ll} dzd\bar{z} & \text{for}\quad |z| < L \\ L^4\,\frac{dzd\bar{z}}{(z\bar{z})^2} & \text{for}\quad |z| < L \end{array}\right\} \tag{16.25}$$

instead of (16.21). Then, according to (16.24)

$$Z[\hat{g}_L] \sim \exp\left\{-\frac{1}{\hbar}\left(4\,\log L^2\right)\right\} \tag{16.26}$$

The curvature $\hat{R}$ now is distributed evenly along the circle $|z| = L$, and the action (16.2) has the form

$$S[\sigma, \hat{g}_L] = \frac{1}{2\pi}\int_{|z|<L}\left[2\,\partial_z\sigma\partial_{\bar{z}}\sigma + \Lambda\, e^{\sigma}\right]\, d^2z + \frac{1}{2\pi}\,\frac{4}{L}\int_{|z|=1}\sigma\, dl\,, \tag{16.27}$$

which differs from (16.18) by the last term involving the values of σ at the circle $|z| = L$; this term comes from the the $\hat{R}$-term in (16.2). The spherical solution (16.15) should now be modified as follows

$$e^{\sigma} \;=\; \left\{\begin{array}{ll} -\frac{8}{\Lambda}\,\frac{1}{(1+z\bar{z})^2} & \text{for}\quad |z| < L \\ -\frac{8}{\Lambda}\,\frac{1}{L^4} & \text{for}\quad |z| > L \end{array}\right\} \tag{16.28}$$

Repeating the calculation which has led to (16.19) we have now

$$S[\sigma, \hat{g}_L] = 4\,\left(\log L^2 - 1\right) + 4\,\left(-2\,\log L^2 + \log(-8/\Lambda)\right) + O(1/L^2)\,. \tag{16.29}$$

The divergent term $-4\,\log L^2$ here is exactly of what is needed to compensate the divergent factor in (16.26). It is convenient to redefine the action, adding the "counter-term" $4\,\log L^2$ associated with this divergent factor in (16.26),

$$S^{\text{reg}}[\sigma, \hat{g}_L] = S[\sigma, \hat{g}_L] + 4\,\log L^2\,. \tag{16.30}$$

In general, we take large domain $D_L \subset \mathbb{R}^2$ of the overall linear size L (which can be the disk $|z| < L$ used in the above analysis), and understand the action as the $L \to \infty$ limit of the expression

$$S[\sigma, \hat{g}_{\text{flat}}] = \frac{1}{2\pi}\int_{D_L}\left[2\partial_z\sigma\partial_{\bar{z}}\sigma + \Lambda\, e^{\sigma}\right]\, d^2z + \frac{2}{\pi}\int_{\partial D_L} K_{\text{flat}}\,\sigma\, dl + 4\,\log L^2\,, \tag{16.31}$$

where K_{flat} is the curvature of the curve ∂D_L in the flat metric $d\hat{s}^2 = dzd\bar{z}$ ($K_{\text{flat}} = 1/L$ for the disk). This "regularized" modification of the action (16.18) is finite at $L \to \infty$.

Pseudosphere (or Poincaré disk) is elementary solution of negative curvature. Again, we set $t(z) = 0$, so that $\psi(z)$ still has the form (16.14), but this time use

$$\mathbf{\Lambda} = \sqrt{\Lambda}\,\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \tag{16.32}$$

in (16.9), so that

$$\sigma(z, \bar{z}) = -2\,\log\left(1 - z\bar{z}\right) + \log\left(8/\Lambda\right)\,. \tag{16.33}$$

The metric

$$ds^2 = \frac{8}{\Lambda}\,\frac{dzd\bar{z}}{(1 - z\bar{z})^2} \tag{16.34}$$

diverges at the unit circle on the z-plane. The metric (16.34) defines geometry of a hyperboloid of two sheets, the sheets corresponding to the inner and outer parts of the plane with the circle $|z| = 1$ cut out. Of course these parts are geometrically identical, and we can concentrate attention on the disk $|z| \leq 1$. The unit circle $|z| = 1$ (often referred to as the "boundary") represents the "absolute" - the set of asymptotic directions of geodesics. Geometrically, the points on the circle a infinitely far from each other, and from any point inside the disk. I will return to this well-known geometry later.

16.2. Local solutions

It is important to understand the role of isolated singularities, i.e. poles of the function $t(z)$. The most important role is played by *regular singularities*, the second-order poles of $t(z)$. The first order poles are not really singularities, since they can be eliminated by suitable analytic variable transformation

$$z \to w(z); \tag{16.35}$$

one can check that the solutions $\psi(z)$ are regular at such points. Poles of the order higher then two - the so-called *irregular singularities* - give rise to rather complicated behavior of the solutions. It is good idea to avoid such singularities, unless absolutely necessary. Let us assume that near some point (which I take to be $z = 0$) $t(z)$ behaves as

$$t(z) \to \frac{r}{z^2} \quad \text{as} \quad z \to 0\,, \tag{16.36}$$

with some real [12] residue r. It is easy to check that analytic transformations (16.35) preserve the character of the singularity, including the value of the residue r. The solutions of (16.8) have power-like behavior $\psi(z) \sim z^\eta$ near the point $z = 0$; in fact, the two solutions behave as

$$\psi_1(z) \sim z^\eta\,, \quad \psi_2(z) \sim z^{1-\eta} \qquad \text{as} \quad z \to 0\,, \tag{16.37}$$

where η is one of the solutions of the quadratic equation

$$r = 4\,\eta\,(1-\eta)\,. \tag{16.38}$$

One may regard this relation as the useful parametrization of the residue r in terms of η. Sometimes it is more convenient to use the related parameter λ related to η,

$$\eta = \frac{1}{2} - \frac{\lambda}{2}\,, \quad 1 - \eta = \frac{1}{2} + \frac{\lambda}{2}\,, \tag{16.39}$$

then

$$r = 1 - \lambda^2\,. \tag{16.40}$$

Obviously, the situation is different depending on the value of r. We have three types of monodromy to consider.

[12] Here we restrict attention to real r; this is because in the Liouville theory $t(z) = -(\partial_z\sigma)^2 + 2\partial_z^2\sigma$ is built from real-valued function $\sigma(z, \bar{z})$.

(i) **Elliptic** type of monodromy appears at $r<1$ (in which case the regular singularity (16.36) is said to be the "elliptic" one). In this case λ (and η) is real. Since the sign of λ does not affect (16.40), I will always assume that $\lambda>0$. We can choose the basis $(\psi_1,\psi_2)^t$ such that

$$\psi(z)=\begin{pmatrix}\psi_1(z)\\ \psi_2(z)\end{pmatrix}\ \to\ \frac{1}{\sqrt{\lambda}}\begin{pmatrix}z^{\frac12-\frac{\lambda}{2}}\\ z^{\frac12+\frac{\lambda}{2}}\end{pmatrix}\qquad \text{as}\quad z\to 0\,. \tag{16.41}$$

The monodromy matrix $\mathbf{M}(C_0)$ associated with C_0 going around the point $z=0$ is easily obtained by taking the contour C_0 sufficiently close to 0; we have

$$\mathbf{M}(C_0)=\begin{pmatrix}-e^{-i\pi\lambda} & 0\\ 0 & -e^{\,i\pi\lambda}\end{pmatrix}\,,\qquad \operatorname{tr}\mathbf{M}(C_0)=-2\,\cos(\pi\lambda)\in[-2:2]\,. \tag{16.42}$$

Since the matrix (16.42) simultaneously belongs to $SU(2)$ and $SU(1,1)$, hence this type of singularity is consistent with both cases of positive and negative curvature. The expression (16.9) takes the form

$$\sigma(z,\bar z)=-2\,\log\left(\psi_1(z)\bar\psi_1(\bar z)\pm\psi_1(z)\bar\psi_1(\bar z)\right)+\log\left(\pm(-8\lambda^2/\Lambda)\right)\,, \tag{16.43}$$

where the sign plus (minus) applies to the case of positive (negative) curvature. In both cases the first term in the argument of the logarithm dominates at $z\to 0$,

$$\sigma(z,\bar z)\to -2\eta\,\log(z\bar z)\qquad \text{as}\quad z\to 0\,, \tag{16.44}$$

and we see that this type of singularity corresponds to conical point [13] at $z=0$,

$$\sqrt{g}\,R(z,\bar z)=-4\,\partial_z\partial_{\bar z}\sigma(z,\bar z)=8\pi\eta\,\delta^{(2)}(z)+\text{regular terms}\,. \tag{16.45}$$

The expression (16.43) can be regarded as global solution of the constant curvature problem. The global character depends on the sign of the curvature.

Positive curvature. In this case the metric corresponding to (16.43)

$$ds^2=\left(-\frac{8\lambda^2}{\Lambda}\right)\frac{dz d\bar z}{(z\bar z)^{1-\lambda}\,(1+(z\bar z)^{\lambda})^2}\,. \tag{16.46}$$

is smooth at all finite nonzero z. The metric is invariant with respect to the transformation $z\to w=1/z$, so that there is also the conical singularity at $z=\infty$,

$$\sqrt{g}\,R(w,\bar w)=8\pi\eta\,\delta^{(2)}(w)+\text{regular terms}\,. \tag{16.47}$$

At $\lambda<1$ this geometry can be visualized as the BEPETEHO with two conical tips (**Fig.1**). The total area is finite, and the total curvature is 8π,

$$A=\int e^{\sigma}\,d^2z=\frac{8\pi\,\lambda}{(-\Lambda)}\,,\qquad \int_{\mathbb{R}^2/\{0,\infty\}}\sqrt{g}\,R\,d^2z=8\pi\,\lambda\,. \tag{16.48}$$

[13] At $\lambda<1$ (i.e. $\eta>0$) this geometry looks like the tip of a cone of the deficit angle $2\pi(1-\lambda)$. At $\lambda>1$ (i.e. $\eta<0$) the interpretation is similar but harder to imagine since it does not admit imbedding into three-dimensional space.

The Liouville action however diverges at $|z| \to 0$ and at $|z| \to \infty$. This is expected, as we have already know that this kind of the curvature singularities lead to the divergent factors (16.24) in the partition function, which in turn manifest themselves as the divergences of the action. It is straightforward to build "regularized" action by adding certain counter-terms, along the lines we have done that in the case of sphere, but I will postpone detailed discussion of this point.

Negative curvature. In this case we have

$$ds^2 = \left(\frac{8\lambda^2}{\Lambda}\right) \frac{dzd\bar{z}}{(z\bar{z})^{1-\lambda}\,(1-(z\bar{z})^{\lambda})^2}\,. \tag{16.49}$$

Considering the domain $|z| < 1$ we observe the geometry of the Poicaré disk with the conical point somewhere inside.

(ii) **Hyperbolic** monodromy is realized at $r > 1$. In this case we use the parametrization (16.40) with pure imaginary

$$\lambda = i\,p\,. \tag{16.50}$$

Taking for definiteness p to be positive, we can use the basis (with unit Wronskian)

$$\psi(z) = \begin{pmatrix} \psi_1(z) \\ \psi_2(z) \end{pmatrix} \quad \to \quad \frac{1}{\sqrt{p}} \begin{pmatrix} z^{\frac{1}{2}-\frac{ip}{2}} \\ -i\,z^{\frac{1}{2}+\frac{ip}{2}} \end{pmatrix} \quad \text{as} \quad z \to 0\,, \tag{16.51}$$

in which the monodromy matrix $\mathbf{M}(C_0)$ has the form

$$\mathbf{M}(C_0) = \begin{pmatrix} -e^{\pi p} & 0 \\ 0 & -e^{-\pi p} \end{pmatrix} \in SL(2,R)\,, \quad \mathrm{tr}\,\mathbf{M}(C_0) = -2\,\cosh(\pi p) < -2\,. \tag{16.52}$$

Since this matrix belongs to $SL(2,R)$ but not $SU(2)$, this situation is realized only at negative curvature. The matrix $\mathbf{\Lambda}$ for this basis gas to be taken in the form

$$\mathbf{\Lambda} = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \tag{16.53}$$

so that we have

$$\begin{aligned}\sigma(z,\bar{z}) - \log\left(8p^2/\Lambda\right) = -2\,\log\left[(z\bar{z})^{\frac{1}{2}-\frac{ip}{2}} + (z\bar{z})^{\frac{1}{2}-\frac{ip}{2}}\right] + \log\frac{8p^2}{\Lambda} = \\ -2\,\log\left[|z|\,\cos\left(p\,\log|z|\right)\right] + \log\frac{2p^2}{\Lambda}\end{aligned} \tag{16.54}$$

To make this expression more clear, introduce the polar coordinates

$$z = e^{\tau + i\theta}\,, \qquad \bar{z} = e^{\tau - i\theta}\,, \tag{16.55}$$

in which the metric looks as follows

$$ds^2 = \frac{2p^2}{\Lambda}\,\frac{d\tau^2 + d\theta^2}{\cos^2(p\tau)}\,. \tag{16.56}$$

The metric tensor is singular at $\tau = \tau_n = p^{-1}(\pi/2 + n\pi)$. The singularities are at the concentric circles in the z-plane, $|z| = e^{\tau_n}$. Clearly, this can not be $z \to 0$ asymptotic of anything useful. Still, there are many important reasons to be interested in this type of monodromy. Let me mention few.

Taking the metric (16.56) literally, we can concentrate attention on one of the regular regions, say

$$-\pi/2 < p\tau < \pi/2\,. \tag{16.57}$$

In the z-plane this corresponds to annulus between two circles $|z| = e^{-\frac{\pi}{2p}}$ and $|z| = e^{\frac{\pi}{2p}}$. The circles has geometric structure similar to the Poincaré disk's "absolute". This is the geometry of the hyperboloid of one sheet. This geometry is sometimes referred to as the Euclidean AdS_2.

Let us note that this geometry has interesting real-time continuation. Setting

$$\tau = it \tag{16.58}$$

we obtain the Minkowski space-time pseudo-metric

$$ds^2 = \frac{2p^2}{\Lambda}\,\frac{-dt^2 + d\theta^2}{\cosh^2(pt)}\,. \tag{16.59}$$

It can be interpreted as the "universe" which evolves from very small spatial circle at $t \to -\infty$, expands to large size $\sim \sqrt{p/\Lambda}$, and then shrinks back to zero size again at $t \to \infty$. Another interesting real-time continuation appears if one takes instead pure imaginary values of θ, but I will not explore it here.

If we understand the expression

$$t(z) \simeq \frac{1+p^2}{z^2} \tag{16.60}$$

as just the asymptotic form, which can have correction terms less singular at small $|z|$, it is possible (and often happens) that the resulting geometry outside of some "outmost" circle is smooth, or otherwise relevant. The hyperbolic monodromy then describes a "opening" inside a smooth geometry, the "boundary" circle having the structure of the absolute. Even more important are the cases when the hyperbolic monodromy appears in relation to the closed path going around more them one regular singularities; we are going to encounter this situation often.

Parabolic monodromy occurs at $r = 1$. The equation

$$4\,\partial_z^2\psi(z) + \frac{1}{z^2}\,\psi(z) = 0 \tag{16.61}$$

has two solutions $z^{\frac{1}{2}}$ and $z^{\frac{1}{2}}\,\log z$. Take the basis

$$\psi(z) = \begin{pmatrix} \psi_1(z) \\ \psi_2(z) \end{pmatrix} \;\to\; \begin{pmatrix} z^{\frac{1}{2}}\,\log z \\ i\,z^{\frac{1}{2}} \end{pmatrix} \quad \text{as} \quad z \to 0\,. \tag{16.62}$$

The monodromy matrix $\mathbf{M}(C_0)$ now has the triangular form

$$\mathbf{M}(C_0) = \begin{pmatrix} -1 & -2\pi \\ 0 & -1 \end{pmatrix} \;\in\; SL(2,R)\,. \tag{16.63}$$

The associated metric is found in the standard way, using again the matrix $\mathbf{\Lambda}$ as in (16.53),

$$ds^2 = \frac{8}{\Lambda}\;\frac{dz d\bar{z}}{(z\bar{z})\,\log^2(z\bar{z})}\,. \tag{16.64}$$

This is the "parabolic point", the limiting case of a conic singularity with the curvature bump containing one half of the curvature of a sphere,

$$\sqrt{g}\;R(z,\bar{z}) = 4\pi\,\delta^{(2)}(z) + \text{regular part} \tag{16.65}$$

This geometry is interesting in many respects, as we will see. In particular, the geodesic distance from the parabolic point at $z = 0$ to ant other point is infinite, as simple calculation shows. The geometry can be visualized as infinitely long "leg" growing from the surface (**Fig.2**).

17. Punctured sphere

The following problem has direct relevance to the problem of determining the correlation functions in 2D quantum gravity, in the classical limit $\hbar \to 0$. Let $\mathbf{M}$ be a topological sphere, with n marked points $x_1, ..., x_n$ - the "punctures". The problem is to find a metric of constant curvature, with prescribed curvature singularities at the pints x_i,

$$\sqrt{g}\;\;(R(x) + \Lambda) = \sum_{i=1}^{n} 8\pi\,\eta_i\,\delta(x - x_i)\,. \tag{17.1}$$

We assume that all the parameters η_i are real, so that the points x_i are either conical or parabolic singularities. In fact, we will be mostly interested in calculating the value of the action (16.2) on such configuration. The reason why its is important to find the classical action is in its relation to the classical limit $\bar{h}$ of the correlation functions of 2D quantum gravity.

17.1. Correlation functions and "regularized" action

Recall that calculating the correlation function

$$\langle\, O_1(x_1)...O_n(x_n)\,\rangle \tag{17.2}$$

with $O_i(x) = e^{-\Delta_i\sigma(x)}\,\Phi_{\Delta_i}(x)$, the $\Phi_{\Delta_i}(x)$ being spinless conformal primaries, involves the functional integral

$$Z[\hat g]\int \left(e^{\mathrm{A}_1\sigma(x_1)}...\,e^{\mathrm{A}_n\sigma(x_n)}\right)\; e^{-\frac{1}{\hbar}\,S[\sigma,\hat g]}\; D[\sigma]\,, \tag{17.3}$$

where, at this point $\mathrm{A}_i = -\Delta_i$ (we will see soon that in properly defined theory this relation requires modification). In the limit $\hbar \to 0$ the integral is dominated by the classical configurations of $\sigma(x)$. If the dimension Δ is kept finite in this limit, it suffices to evaluate the value of the exponential $e^{A\,\sigma(x)}$, with $\sigma(x)$ associated with the constant-curvature metric $g = e^{\sigma}\hat g$. I will refer to such situation as the "light" insertion - inserting such exponential does not affect the stationary-point configuration. However, it is often important to consider the case when $\Delta \sim \frac{1}{\hbar}$, so that

$$A_i = \frac{4\eta_i}{\hbar} \tag{17.4}$$

with η_i having finite limit at $\hbar \to 0$. Then such insertions have to be treated as the part of the functional to be extremized. I will call such insertions the "heavy" ones. Assuming that all the we have n heavy insertions at the points $x_1, ..., x_n$, the stationary-point equation reads

$$\sqrt{\hat g}\;\left(\hat R(x) - \Delta_{\hat g}\sigma(x) + \Lambda\, e^{\sigma(x)}\right) = \sum_{i=1}^{n}\, 8\pi\eta_i(x - x_i)\,, \tag{17.5}$$

which is exactly the equation (17.1). Thus he classical limit of the correlation function with the heavy insertions is determined by the Liouville action calculated on the solution of this equation.

There is a problem here. We already know that for such metric the action (16.2) has divergences associated with the tips of the conical singularities. The functional integral thus has inherent divergence, which can be removed by "spreading" the insertions over small domains D_i containing the points x_i. We write

$$\sigma(x_i) \to \int_{D_i}\,\sigma(x)\,d\mu_i(x)\,, \qquad \int_{D_i}\,d\mu_i(x) = 1\,, \tag{17.6}$$

with some sufficiently regular measures $d\mu_i(x)$. Then, the integral is finite, but depends on the domains D_i (together with the measures $d\mu_i$). In particular, if ε_i is the g-geodesic size (i.e. taken with respect to the actual metric $g = e^{\sigma}\,\hat g$) of the domain D_i, then each heavy insertion $e^{A_i\,\sigma(x)}$ in (17.3) produces the factor (16.24), i.e.

$$(\varepsilon_i)^{-\frac{8\eta_i^2}{\hbar}}\;. \tag{17.7}$$

We can define the renormalized insertions

$$Y^{(\eta_i)}(x) = \left[e^{\frac{4\eta_i}{\hbar}\,\sigma(x)}\right]_{\rm r} \sim (\varepsilon_i)^{-\frac{8\eta_i^2}{\hbar}}\ e^{\frac{4\eta_i}{\hbar}\,\sigma(x)}\,; \tag{17.8}$$

then the functional integral (17.3) with such renormalized insertions is expected to have finite limit at $\varepsilon_i \to 0$, which we also expect (really, hope) to be independent (up to finite renormalizations of the insertion field (17.8)) on precise shapes of D_i, as well as the measures $d\mu_i$.

The parameters ε_i are the g-geodesic sizes of the domains D_i. In practice, it is much advantageous to think in terms of the fixed background metric $\hat{g}$. If the field $\sigma(x)$ is smooth (continuous) [14], for sufficiently small D_i the g-geodesic size ε_i relates to its $\hat{g}$-size ϵ_i (which I will usually call the "coordinate size") in a simple way

$$\varepsilon_i \sim \epsilon_i\, e^{\frac{1}{2}\,\sigma(x_i)}\,. \tag{17.9}$$

Thus we have

$$\left[e^{\frac{4\eta_i}{\hbar}\,\sigma(x)}\right]_{R} \sim (\epsilon_i)^{-\frac{8\eta_i^2}{\hbar}}\ e^{\frac{4\eta_i(1-\eta_i)}{\hbar}\,\sigma(x)}\,. \tag{17.10}$$

Note that the last exponential involves the quantity

$$r_i = 4\eta_i(1-\eta_i)\,. \tag{17.11}$$

The expression (17.10) may look somewhat confusing because of the different factors in the exponentials in the left- and the right- hand sides. It can be understood as follows. The right-hand side is suitable for simple interpretation in terms of the fixed metric $\hat{g}$, which I assume for simplicity to be the flat one,

$$\hat{g}: \qquad d\hat{s}^2 = dzd\bar{z}\,. \tag{17.12}$$

When we make the conformal coordinate transformation $z \to w(z)$, we keep the the metric $\hat{g}$ unchanged, to retain the form $dwd\bar{w}$, but instead transform the conformal factor e^{σ}, so that the actual metric

$$g\,:ds^2 = \qquad e^{\sigma(z,\bar{z})}\,dzd\bar{z} \tag{17.13}$$

transforms properly. Under such transformations the "coordinate" sizes ϵ_i of the domains D_i change, and the extra factor $\exp\{-\frac{4\eta_i^2}{\hbar}\,\sigma(x_i)\}$ in the right-hand side of (17.10) is designed to compensate for this effect. As the result, if we keep ϵ_i fixed, the "heavy" exponential field (17.10) transforms as the conformal primary with the left and right conformal dimensions equal to

$$\tilde{\Delta}_i = \frac{r_i}{\hbar} = \frac{4\eta_i(1-\eta_i)}{\hbar}\,. \tag{17.14}$$

[14] This can be arranged in the classical case, as we have seen above, but it is not at all obvious if, and in what sense, this condition holds in the full-fledged quantum theory. However, I will proceed with this assumption made.

Correspondingly, if have a spinless primary field $\Phi_{\Delta_i}(x)$ with the "heavy" dimension $\Delta_i \sim \frac{1}{\hbar}$, in order to form a scalar we have to multiply Φ_{Δ_i} by the exponent (17.10) with η_i adjusted in such a way that

$$\Delta_i = -\frac{4\eta_i(1-\eta_i)}{\hbar}\,. \tag{17.15}$$

At the same time, the right-hand side of (17.8), if misunderstood, may convey wrong transformation property of this field under the Weyl transformation of the metric $g(x) \to (1+\delta\sigma(x))\,g(x)$. If we want to keep the "physical" short-distance scale ε_i fixed, its coordinate size ϵ_i must be appropriately changed with the Weyl transformation. It is the left-hand side of (17.10) which faithfully represents the Weyl transformation property of this field. In what follows I will mostly make explicit use of the form in the left-hand side; however, in dealing with this expression it is important to remember that the shape of D_i in the z-plane changes under conformal transformations, in particular ...

With this interpretation in mind, let us now come back to the classical problem (17.1). Let us take the flat background, $\hat{g}:\ d\hat{s}^2 = dzd\bar{z}$; the equation (17.1) then reads,

$$-4\,\partial_z\partial_{\bar{z}}\sigma(z,\bar{z}) + \Lambda\, e^{\sigma(z,\bar{z})} = \sum_{i=1}^{n} 8\pi\,\eta_i\,\delta^{(2)}(z-z_i)\,. \tag{17.16}$$

We can define the action through the following limiting procedure. Start with the large domain $D_L \subset \mathbb{R}^2$ of the complex plane, of the size L (say, disk of the radius L), such that it contains all the points z_i. Cut out small domains $D_i \subset D_L$ around the points z_i, of the (coordinate) sizes ε_i (again, I will take D_i to be the disks $|z-z_i| < \varepsilon_i$). Define the regularized action

$$S^{(\mathrm{reg})}[\sigma] = \frac{1}{2\pi}\int_{D_L\setminus\bigcup_i D_i} \left[2\,\partial_z\sigma\partial_{\bar{z}}\sigma + \Lambda\, e^{\sigma}\right] d^2z - \sum_{i=1}^{n}\left\{\frac{2\eta_i}{\pi}\int_{\partial D_i} K\,\sigma\, dl + 4\eta_i^2\,\log\varepsilon_i^2\right\} + \frac{2}{\pi}\int_{\partial D_L} K\,\sigma\, dl + 4\,\log L^2\,. \tag{17.17}$$

The first term is the "cutoff" version of the naive action (16.18). The second term involves integrals over the boundaries ∂D_i of the domains D_i; these represent the "smoothed" versions of the terms $\frac{4\eta_i}{\hbar}\,\sigma(x_i)$ due to the heavy insertions. There, I have taken $d\mu_i(x)$ to be concentrated at the boundary of D_i; K stands for the flat curvature of the boundary ∂D_i ($K = 1/\varepsilon_i$ for the case of D_i being the disk). The terms $4\eta^2\,\log\varepsilon^2$ come from the renormalization factors in (17.8). The last two terms in (17.17) are the same as in the spherical action (16.31); they take care of the hidden curvature singularity at $z=\infty$ of the "flat" metric $dzd\bar{z}$. The solution of the equation (17.16) has the following asymptotic behavior near the singularities and at large $|z|$

$$\sigma(z,\bar{z}) = -2\eta_i\,\log|z-z_i|^2 + \hat{\sigma}_i + \ldots \qquad \text{as} \quad |z-z_i| \to 0\,, \tag{17.18}$$

and

$$\sigma(z,\bar{z}) = -2\,\log|z|^2 + \hat{\sigma}_\infty + \ldots \qquad \text{as} \quad |z| \to \infty\,. \tag{17.19}$$

Here I have written down the sub-leading constant terms in the asymptotic forms (the dots stand for terms vanishing in the limit); I would like to stress that unlike η_i, which are the "input" parameters supplied with the problem, the values of constants $\hat{\sigma}_i$ and $\hat{\sigma}_\infty$ are not (actually, can not be) prescribed in advance but determined through the solution, i.e. they are rather part of the "output".

Using these equations it is not difficult to check that the regularized action (17.17) has finite limit as $\varepsilon_i \to 0$ and $L \to \infty$. I will use the notation

$$S_{(\mathrm{cl})}\left(\{z_i, \eta_i\}\right) = \lim S^{(\mathrm{reg})}[\sigma_{(\mathrm{cl})}] \tag{17.20}$$

(sometimes simply $S\left(\{z_i, \eta_i\}\right)$ for the limiting value of the action (17.17) evaluated on the classical configuration $\sigma_{(\mathrm{cl})}$ - the solution of the classical Liouville problem (17.16). Then, the classical limit of the functional integral in (17.3), with the (renormalized) heavy insertions (17.8) we have

$$\int \left(Y^{(\eta_1)}(z_1, \bar{z}_1) \ldots Y^{(\eta_n)}(z_n, \bar{z}_n) \right) \, e^{-\frac{1}{\hbar} S[\sigma]} \, D[\sigma] \, \sim \, \exp\left\{ -\frac{1}{\hbar} \, S_{(\mathrm{cl})}\left(\{z_i, \eta_i\}\right) \right\} . \tag{17.21}$$

Let us establish the following useful relations

$$\frac{\partial S\left(\{z_i, \eta_i\}\right)}{\partial \eta_i} = -4 \, \hat{\sigma}_i \, , \tag{17.22}$$

where $\hat{\sigma}_i$ are the constant terms in the asymptotic form (17.18). In view of the Eq.(17.21) this relation is almost obvious. Indeed, taking the η_i derivative of the left-hand side, under the sign of the functional integral, we bring down the quantity

$$\frac{1}{\hbar} \left(4 \, \sigma(x_i) - 8\eta_i \, \log \varepsilon_i^2 \right) \, . \tag{17.23}$$

In the stationary-point approximation this has to be evaluated at $\sigma(x) = \sigma_{(\mathrm{cl})}(x)$, and in view of the asymptotic form (17.18) it evaluates to $X_i/\hbar$. Comparing to the η_i derivative of the right-hand side we arrive at (17.22).

This "derivation" is not mathematically acceptable - it appeals to the notion of the functional integral, which at the moment is not rigorously defined object. The relation (17.22) itself concerns with quantities which are mathematically well defined. Of course in this case it is not difficult to derive (17.22) directly, without the reference to the relation (17.21). Take the η_i-derivative of $S\left(\{z_i, \eta_i\}\right)$, using its definition (17.17). In fact, it is useful to rewrite the action (17.17) in the form

$$S[\sigma] = \frac{1}{2\pi} \int_{D_L \setminus \bigcup_i D_i} \left[2 \, \partial_z \sigma \partial_{\bar{z}} \sigma + \Lambda \, e^{\sigma} \right] \, d^2 z - \sum_{i=1}^{n} \left(4\eta_i \, \hat{\sigma}_i - 4\eta_i^2 \, \log \varepsilon_i^2 \right) + \\ 4 \, \hat{\sigma}_\infty - 4 \, \log L^2 \, , \tag{17.24}$$

which applies to the limit of sufficiently small ε_i and $1/L$, where the asymptotics (17.18) and (17.19) allow one to express the boundary integrals through the constants $\hat{\sigma}_i$ and $\hat{\sigma}_\infty$. The

η_i-derivative of the first term in here can be integrated by parts, and then, with the help of the classical equation (17.16), reduced to the boundary terms at ∂D_i and ∂D_L. When combined with the η_i derivative of the remaining part of the action (17.17), the divergent parts cancel, and the limit $\varepsilon_i \to 0$ and $L \to \infty$ can be taken, leading to (17.22).

There is also very useful interpretation of the derivatives of $S\left(\{z_i, \eta_i\}\right)$ with respect to the positions z_i, in terms of the so-called accessory parameters.

17.2. Accessory parameters

It is possible to prove that solution of the problem (17.16) exists and is unique, for any positions z_i of the singularities, and for any set of real parameters η_i [15] (Picard?). Let $\sigma(x)$ be this solution. Consider again the form

$$t(z) = \partial_z\sigma\partial_z\sigma + 2\,\partial_z^2\sigma\,. \tag{17.25}$$

It is the holomorphic function on $\mathbb{C}\backslash\{z_i\}$, with the second-order poles at $z = z_i$. We can write

$$t(z) = \sum_{i=1}^{n}\left[\frac{r_i}{(z-z_i)^2} + \frac{c_i}{z-z_i}\right]\,, \tag{17.26}$$

where the coefficients at the second-order poles are fixed by the asymptotic conditions (17.18)

$$r_i = 4\eta_i(1-\eta_i) = 1 - \lambda_i^2\,, \tag{17.27}$$

while the parameters c_i are not predetermined, but are to be found from the solution of the Liouville problem; they are known as the *accessory parameters*. The representation of the monodromy group acting on the solution $\psi(z) = (\psi_1(z), \psi_2(z))$, i.e. the collection of matrices $\mathbf{M}(C_i) \in SL(2,C)$ associated with the closed paths on $\mathbb{C}$, depend on the parameters c_i. We already know that it is possible to construct single-valued function $\sigma(z,\bar{z})$ in terms of the solution $\psi(z)$, if (and only if) all the matrices $\mathbf{M}(C_i)$ are equivalent to unitary matrices, or all are equivalent to pseudo-unitary matrices. In other words, if a basis $(f_1(z), f_2(z))$ exists, such that either

$$\forall\; C_i \quad \mathbf{M}(C_i) \;\in\; SU(2)\,, \tag{17.28}$$

or

$$\forall\; C_i \quad \mathbf{M}(C_i) \;\in\; SU(1,1)\,. \tag{17.29}$$

In the first case we have solution of the Liouville problem with positive curvature, and the second possibility corresponds to the case of negative curvature (in fact, the sign of the curvature can be red out directly from (17.26), see below). Since the Eq.(17.16) admits unique solution $\sigma(x)$, we can be sure that there exists unique choice of the accessory parameters c_i which fulfills one of the above monodromy conditions, (17.28) or (17.29). These special

[15] For trivial reason, the sign of the curvature $-\Lambda$ must be taken positive of $\sum_i \eta_i < 1$, and negative otherwise.

values of c_i depend parameters depend on the coordinates z_i of the singular points, as well as on the parameters η_i. I will write

$$\hat{c}_i = \hat{c}_i\left(\{z_i, \eta_i\}\right) \tag{17.30}$$

for the accessory parameters solving the problem. The dependence is not holomorphic (i.e. $\hat{c}_i$ depend on z_i and $\bar{z}_i$), and usually highly transcendental.

It is possible to show that these functions are the derivatives of the classical action (17.17), this time with respect to the coordinates z_i,

$$\frac{\partial S\{z_i, \eta_i\})}{\partial z_i} = -\,\hat{c}_i\,. \tag{17.31}$$

Again, from the point of view of quantum field theory (i.e. the functional integral (17.3) is very natural. Indeed, if we assume that the functional integral over $\sigma(x)$ defines a full-fledged quantum conformal field theory, it is natural to interpret the classical field $t(z)$ in terms of the classical limit of the holomorphic field T, the zz-component of the quantum $T_{\mu\nu}$,

$$t(z) = \hbar\, T(z)\,. \tag{17.32}$$

Recall the basic Ward identity valid in quantum conformal field theory,

$$\langle\, T(z)\, V_{\tilde{\Delta}_1}(z_1, \bar{z}_1) \ldots V_{\tilde{\Delta}_n}(z_n, \bar{z}_n)\,\rangle = \sum_{i=1}^{n}\left[\frac{\tilde{\Delta}_i}{(z-z_i)^2} + \frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right]\langle\, V_{\tilde{\Delta}_1}(z_1, \bar{z}_1) \ldots V_{\tilde{\Delta}_n}(z_n, \bar{z}_n)\,\rangle\,, \tag{17.33}$$

where $V_{\tilde{\Delta}_i}$ stand for any primary fields of the (left) conformal dimensions $\tilde{\Delta}_i$. Take these fields to be the renormalized exponentials (17.8),

$$V_{\tilde{\Delta}_i} = Y_{\eta_i} = \left[e^{\frac{4\eta_i}{\hbar}\,\sigma}\right]_{\mathrm{r}}\,, \qquad , \tag{17.34}$$

and consider the classical case $\hbar \to 0$. In this limit we have $\tilde{\Delta}_i \simeq 4\eta_i(1-\eta_i)/\hbar$, and the correlation function in the right-hand side of the equation (17.33) is expected to assume the exponential form (17.21). At the same time, the field $T(z) = \frac{1}{\hbar}\,t(z)$ is the "light" one - in the classical limit its insertion does not affect the classical background $\sigma_{(\mathrm{cl})}(x)$ - and so the insertion results in just multiplying the "heavy" correlator by $\frac{1}{\hbar}\,t(z)$ computed in this background, i.e.

$$\langle\, T(z)\, Y_{\eta_1}(z_1, \bar{z}_1) \ldots Y_{\eta_n}(z_n, \bar{z}_n)\,\rangle \quad\to\quad \frac{1}{\hbar}\; t(z)\,\exp\left\{-\frac{1}{\hbar}\,S\left(\{z_i, \eta_i\}\right)\right\} \tag{17.35}$$

as $\hbar \to 0$. Then comparison of (17.33) with (17.26) leads to (17.31).

It is possible to give proof of the relation (17.31) using only "clean" mathematics (Zograf and Takhtadjan). Let me give you the idea. First, it is easy to check that the accessory coefficients $\hat{c}_i$ show up in the higher terms of the expansion (17.18) of $\sigma(z,\bar{z})$ near the points z_i,

$$\sigma(z,\bar{z}) = -2\eta_i \log|z-z_i|^2 + \hat{\sigma}_i + \frac{\hat{c}_i}{4\eta_i}(z-z_i) + \frac{\bar{\hat{c}}_i}{4\eta_i}(\bar{z}-\bar{z}_i) + O(|z|^2, |z|^{2-4\eta_i}), \quad (17.36)$$

so that

$$\partial_z \sigma(z,\bar{z}) \to -\frac{2\eta_i}{z-z_i} + \frac{\hat{c}_i}{4\eta_i} + ... \quad (17.37)$$

where again the dots stand for terms vanishing in the limit $z \to z_i$. Keeping ε_i and $1/L$ sufficiently small but finite, take the derivative of (17.24) with respect to one if the coordinates z_i, say z_1. Denote $\partial_i\sigma$ the derivative of the classical configuration with respect to the parameter z_i,

$$\partial_i \sigma(z,\bar{z}) = \frac{\partial\sigma}{\partial z_i}(z,\bar{z}). \quad (17.38)$$

We have from (17.36)

$$\begin{aligned} \partial_i \sigma(z,\bar{z}) &= \frac{\partial\hat{\sigma}_j}{\partial z_i} + ... \qquad\qquad \text{for} \quad z \to z_j \quad \text{with} \quad j \neq i, \\ \partial_i \sigma(z,\bar{z}) &= \frac{2\eta_i}{z-z_i} + \frac{\partial\hat{\sigma}_i}{\partial z_i} - \frac{\hat{c}_i}{4\eta_i} + ... \qquad \text{for} \quad z \to z_j. \end{aligned} \quad (17.39)$$

Taking the z_i derivative of the bulk term in (17.24) amounts to variation around the classical solution with

$$\delta\sigma \sim \partial_i\sigma, \quad (17.40)$$

and, in addition, taking the z_i-derivative of the position of the disk D_i. Since the variation is around the stationary point, the integrand is a total derivative, and the integral reduces to boundary terms. Moreover, the variation (17.40) is regular near the points z_j with $j \neq i$, and the stationarity guarantees that the terms corresponding to the ∂D_j, $j \neq i$ parts of the boundary cancel the z_i derivatives of associated terms in the second part of the action (17.24). Only the boundary ∂D_i requires special care. Assuming for simplicity that D_i is a disk, the boundary term from the above variation (17.40) at ∂D_i reads

$$-\frac{1}{2\pi}\int_{\partial D_i} \partial_i\sigma\, \partial_r\sigma\, dl, \quad (17.41)$$

where $\partial_r\sigma$ is the radial derivative, with respect to $r = |z|$. Using the asymptotics (17.36) and (17.39) one finds

$$(17.41) \;=\; 4\eta_i \frac{\partial\hat{\sigma}_i}{\partial z_i} - \frac{3}{2}\hat{c}_i. \quad (17.42)$$

The first term cancels the z_i derivative of the remaining term $-4\eta_i\,\hat{\sigma}_i$ in the second part of the action (17.24). It remains to take into account the z_i derivative of the position of the disk D_i. The deletion of the domain D_i is equivalent to the insertion of the cutoff factor

$\Theta(z,\bar z) = \theta(|z-z_i|^2 - \varepsilon_i^2)$ into the integrand of the bulk part of the action, so we need to add the term

$$-\frac{1}{2\pi} \int \frac{\partial\Theta(z,\bar z)}{\partial z_i}\, 2\,\partial_z\sigma\partial_{\bar z}\sigma\, d^2 z\,. \tag{17.43}$$

which evaluates to $\hat{c}_i$.

17.3. More on accessory parameters

Generally, I will assume that $z=\infty$ is a regular point; in view of the transformation property of $t(z)$ this is equivalent to the statement

$$t(z) \sim \frac{1}{z^4} \qquad \text{as} \quad z \to \infty\,. \tag{17.44}$$

For the form (17.26) this condition implies three linear equations for the parameters c_i,

$$\sum_{i=1}^{n} c_i = 0$$

$$\sum_{i=1}^{n} (r_i + z_i\, c_i) = 0 \tag{17.45}$$

$$\sum_{i=1}^{n} \left(2r_i + z_i^2\, c_i\right) = 0 \tag{17.46}$$

Two points

$$r_1 = r_2 = r\,, \qquad c_1 = -c_2 = -\frac{2r}{z_1 - z_2}\,. \tag{17.47}$$

so that

$$S(\{z_1, z_2\}, \eta) = 2r\, \log(z_1 - z_2) + S_0(\eta) \tag{17.48}$$

Three points

$$c_1 = \frac{r_3 - r_1 - r_2}{z_1 - z_2} + \frac{r_2 - r_1 - r_3}{z_1 - z_3}\,, \tag{17.49}$$

so that

$$S(\{z_i, \eta_i\}) = S_0(\{\eta_i\}) + (r_1 + r_2 - r_3)\, \log(z_1 - z_2) + (r_1 + r_3 - r_2)\, \log(z_1 - z_3) + (r_2 + r_3 - r_1)\, \log(z_2 - z_3) \tag{17.50}$$

Four points

$$c_i = -\frac{\partial S}{\partial z_i} \tag{17.51}$$

with

$$S(\{z_i;\eta_i\}) = r_1 \log \frac{(z_1-z_4)(z_1-z_3)}{(z_4-z_3)} + r_2 \log \frac{(z_1-z_2)(z_2-z_4)}{(z_4-z_1)} + \\ r_3 \log \frac{(z_1-z_3)(z_3-z_4)}{(z_4-z_1)} + r_4 \log \frac{(z_1-z_4)(z_3-z_4)}{(z_3-z_1)} + S_0(x;\{\eta_i\}) \tag{17.52}$$

where

$$x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)} \tag{17.53}$$

In fact, I would prefer to use related function

$$S_0(x;\{\eta_i\}) = f(x) - r_2 \log(x)\,. \tag{17.54}$$

These forms are suited to setting

$$z_1 = 0\,, \quad z_2 = x\,, \quad z_3 = 1\,, \quad z_4 = \infty\,. \tag{17.55}$$

Then we have in this limit

$$\begin{aligned} c_1(x) &= r_1 + r_2 + r_3 - r_4 - (1-x)\,c(x)\,, \qquad & c_2(x) &= c(x)\,, \\ c_3(x) &= r_4 - r_1 - r_2 - r_3 - x\,c(x)\,, & c_4(x) &= 0\,. \end{aligned} \tag{17.56}$$

where

$$c(x) = -f'(x)\,. \tag{17.57}$$

Note that (17.56) is simply the general solution of the $SL(2)$ equations specialized to this case,

$$\begin{aligned} & c_1 + c_2 + c_3 = 0 \\ , \qquad & x\,c_2 + c_3 + r_1 + r_2 + r_3 - r_4\,. \end{aligned} \tag{17.58}$$

17.4. Elementary solutions

Two punctures

$$t(z) = \frac{r}{(z-z_1)^2} + \frac{r}{(z-z_2)^2} - \frac{2r}{(z-z_1)(z-z_2)} \tag{17.59}$$

Three punctures

$$c_1 = \frac{r_3 - r_1 - r_2}{z_1 - z_2} + \frac{r_2 - r_1 - r_3}{z_1 - z_3}\,, \tag{17.60}$$

so that

$$S = (r_1+r_2-r_3)\,\log(z_1-z_2)+(r_1+r_3-r_2)\,\log(z_1-z_3)+(r_2+r_3-r_1)\,\log(z_2-z_3)+S_0 \tag{17.61}$$

Four punctures

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \tag{17.62}$$

18. Classical conformal block

18.1. Monodromy around two points

Let $f^{(i)}(z)$ be local solutions, such that (unit Wronskian!)

$$f^{(i)}(z) = \frac{1}{\sqrt{\lambda_i}} \begin{pmatrix} (z - z_i)^{\frac{1-\lambda_i}{2}}\; p_1^{(i)}(z) \\ (z - z_i)^{\frac{1+\lambda_i}{2}}\; p_2^{(i)}(z) \end{pmatrix} \tag{18.1}$$

Then

$$f^{(i)}(C_i * z) = \mathbf{U}_i\, f^{(i)}(z)\,, \qquad \mathbf{U}_i = -\begin{pmatrix} e^{-i\pi\lambda_i} & 0 \\ 0 & e^{\,i\pi\lambda_i} \end{pmatrix} \tag{18.2}$$

We define also the transition matrices $\mathbf{L}^{ji} \in SL(2, C)$ as

$$f^{(i)}(z) = \mathbf{L}^{(ij)}\, f^{(j)}(z)\,. \tag{18.3}$$

Obviously $\mathbf{L}^{(ik)}\mathbf{L}^{(kj)} = \mathbf{L}^{(ij)}$,

$$\mathbf{L}^{(ik)}\mathbf{L}^{(ki)} = \mathbf{I} \tag{18.4}$$

Take arbitrary basis $g(z)$, and concentrate attention on two points, say z_1 and z_2. Let $\mathbf{M}(C_{12})$ be the monodromy matrix associated with the continuation around the contour encircling both the points z_1 and z_2, and

$$g(C_{12} * z) = \mathbf{M}(C_{1,2})\, g(z)\,. \tag{18.5}$$

The trace of the monodromy matrix of course is independent on the choice of the basis.

Theorem: If the trace of this monodromy is real

$$\mathrm{tr}\,\mathbf{M}(C_{12}) \in \mathbb{R}\,, \tag{18.6}$$

then the basis $f(z)$ exists in which either

$$\mathbf{M}(C_1)\,,\ \mathbf{M}(C_2)\ \in SU(2)\,, \tag{18.7}$$

or

$$\mathbf{M}(C_1)\,,\ \mathbf{M}(C_2)\ \in SU(1,1)\,, \tag{18.8}$$

Proof: Let us look for the basis $f(z)$ in the form

$$f(z) = \mathbf{G}^{-1}\, f^{(1)}(z)\,, \qquad \mathbf{G} = \begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix}, \tag{18.9}$$

where s is a complex number to be determined. In this basis

$$\mathbf{M}(C_1) = \mathbf{U}_1\,, \tag{18.10}$$

which belongs to both $SU(2)$ and $SU(1,1)$. Using the notation $\mathbf{L} = \mathbf{L}^{(21)}$ we have

$$\mathbf{M}(C_2) = \mathbf{G}^{-1}\,\mathbf{L}^{-1}\,\mathbf{U}^{(2)}\,\mathbf{L}\ \mathbf{G} \qquad \text{and} \qquad \mathbf{M}(C_{21}) = \mathbf{M}(C_2)\,\mathbf{U}^{(1)}\,. \tag{18.11}$$

Write explicitly

$$\mathbf{M}(C_2) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{18.12}$$

The freedom in the choice of the real parameter G can be used to make elements B and C to have equal absolute values,

$$|B| = |C|\,. \tag{18.13}$$

so that

$$B = R\,e^{i\beta}\,, \qquad C = R\,e^{i\gamma}\,, \tag{18.14}$$

with real positive r.

Since this matrix is similar to $\mathbf{U}^{(2)}$, its trace is real,

$$A + D = -2\,\cos\pi\lambda_2\,. \tag{18.15}$$

By assumption, the trace

$$\mathrm{tr}\,\mathbf{M}(C_{21}) = \mathrm{tr}\,\left(\mathbf{M}(C_2)\,\mathbf{U}^{(1)}\right) = -A\,e^{-i\pi\lambda_1} - D\,e^{i\pi\lambda_1} \tag{18.16}$$

is also real. It follows [16]

$$D = \bar{A}\,. \tag{18.17}$$

Since $BC = AD - 1 = A\bar{A} - 1$ the product BC is real. That means that in (18.14) we must have $e^{i(\beta+\gamma)} = \pm 1$, and hence

$$C = \pm\,\bar{B}\,. \tag{18.18}$$

[16] Denote $\omega = \exp(i\pi\lambda_1)\,,\ \bar{\omega} = \exp(-i\pi\lambda_1)$. We have

$$\begin{aligned} A +\ D &= \bar{A} + \bar{D} \\ \bar{\omega}\,A + \omega\,D &= \omega\,\bar{A} + \bar{\omega}\bar{D}\,. \end{aligned}$$

Multiplying the first of these equations by ω, and then subtracting from it the second one, we have ...

18.2. Local analysis of two punctures

Let us select two sufficiently close singular points, say z_0 and z_1. We can write

$$t(z) = \frac{r_1}{(z-z_1)^2} + \frac{r_2}{(z-z_2)^2} + \frac{c_1}{z-z_1} + \frac{c_2}{z-z_2} + t_{\rm r}(z)\,, \tag{18.19}$$

where $t_{\rm r}(z)$ is analytic in some domain which includes both z_0 and z_1, but no other singularities. Without loss of generality, we can set $z_1 = 0$ and $z_2 = x$. Then we can write

$$t_{\rm r}(z) = a_0 + a_1\, z + a_2\, z^2 + ... \tag{18.20}$$

This converges in some domain $|z| < X$, with $|x| < X$. We would like to adjust the parameters c_1 and c_2, and the coefficients a_k, so that the diff has prescribed monodromy around z_0, z_1, i.e.

$$\mathrm{tr}\,\mathbf{M}(C_{12}) = m_{12}\,. \tag{18.21}$$

The trace m_{12} may be a complex number, but I will parameterize it as

$$m_{12} = -2\,\cos(\pi\lambda)\,. \tag{18.22}$$

For real m_{12} the parameter λ can be taken real or pure imaginary (depending on the type of monodromy), and we use

$$r = 1 - \lambda^2 \tag{18.23}$$

If all a_k are zero, the problem is simple:

$$c_1 = r_1 + r_2 - r\,, \quad c_2 = -c_1\,. \tag{18.24}$$

Take

$$t(w) = \frac{r_1}{w^2} + \frac{r_2}{(w-x)^2} + \frac{\rho}{w(x-w)}\,, \qquad \rho = r_1 + r_2 - r\,. \tag{18.25}$$

Let

$$w = w(z) = z\,\frac{v(z)}{v(x)}\,, \qquad v(z) = 1 + b_1\, z + b_2\, z^2 + b_3\, z^3 + \tag{18.26}$$

so that $w(0) = 0$ and $w(x) = x$, and assume that the series for $v(z)$ converges at $|z| < X$.

We have

$$t(z) = (w')^2\left(\frac{r_1}{w^2} + \frac{r_2}{(w-x)^2} + \frac{\rho}{w(x-w)}\right) + 2\,\{w, z\}\,. \tag{18.27}$$

This of course has pole terms

$$t(z) = \frac{r_0}{z^2} + \frac{r_1}{(z-x)^2} + \frac{c_1}{z} + \frac{c_2}{z-x} + t_r(z)\,. \tag{18.28}$$

with

$$\begin{aligned} c_1 = {} & \frac{\rho}{x} + (r + r_1 - r_2)\, b_1 + \rho\,(b_1^2 - b_2)\, x + \rho\,(2b_1 b_2 - b_1^3 - b_3)\, x^2 + O(x^3)\,, \\ c_2 = {} & -\frac{\rho}{x} + (r + r_2 - r_1)\, b_1 + (2b_2\,(3r_2 - \rho) - b_1^2\,(4r_2 - \rho))\, x + \\ & (3b_3\,(4r_2 - \rho) - 3b_1 b_2\,(6r_2 - \rho) + b_1^3\,(8r_2 - \rho))\, x^2 + O(x^3)\,, \end{aligned} \tag{18.29}$$

and the regular part expands as (18.20) with

$$a_0 = (2b_2\,(2r+3) - b_1^2\,(r+6)) + (5b_3 - 6b_1 b_2 + 2b_1^3)\,x + O(x^2)$$
$$a_1 = (6b_3\,(r+4) - 2b_1 b_2\,(r+24) + 24b_1^3) + O(x)\,, \qquad (18.30)$$
$$\text{etc} \qquad (18.31)$$

18.3. Four-point conformal block

Take the four-point $t(z)$, and set again

$$z_1 = 0\,, \quad z_2 = x\,, \quad z_3 = 1\,, \quad z_4 = \infty\,. \qquad (18.32)$$

then

$$t(z) = \frac{r_1}{z^2} + \frac{r_2}{(z-x)^2} + \frac{c_1}{z} + \frac{c_2}{z-x} + t_{34}(z)\,, \qquad (18.33)$$

with

$$t_{34}(z) = \frac{r_3}{(z-1)^2} + \frac{c_3}{z-1}\,, \qquad (18.34)$$

where c_i must obey the above equations (17.58), i.e.

$$c_1 + c_2 + c_3 = 0 \qquad (18.35)$$
$$,\quad x\,c_2 + c_3 + r_1 + r_2 + r_3 - r_4 = 0\,,$$

in particular

$$c_1 + (1-x)\,c_2 = r_1 + r_2 + r_3 - r_4\,. \qquad (18.36)$$

The term $t_{34}(z)$ in (18.33) plays the role of the regular part $t_r(z)$ in (18.28). We find

$$t_{34}(z) = a_0 + a_1\,z + ... \qquad (18.37)$$

with

$$a_0 = r_3 - c_3\,, \qquad a_1 = 2r_3 - c_3\,, \qquad \text{etc} \qquad (18.38)$$

It is consistent to look for the parameters $b_k = b_k(x)$ in (18.26) as the power series

$$b_k(x) = b_k^{(0)} + b_k^{(1)}\,x + b_k^{(2)}\,x^2 + \dots\,. \qquad (18.39)$$

We can use the freedom in choosing the first of these series, $b_1(x)$, to take care of the Equation (18.36), order by order in x. Thus

$$b_1^{(0)} = \frac{r + r_3 - r_4}{2r}\,, \qquad (18.40)$$
$$b_1^{(1)} = \frac{(r + r_2 - r_1)((r_3 - r_4)^2 + 2r^2 + 3r\,(r_3 - r_4) - 6r^2\,b_2^{(0)})}{4r^3}\,, \qquad (18.41)$$
$$\text{etc} \qquad (18.42)$$

The remaining coefficients in $b_2(x)$, $b_3(x)$, etc, are determined, again order by order in x, from the equations (18.38). We find

$$b_2^{(0)} = \frac{5}{16} + \frac{r_3 - r_4}{4r} + \frac{(r_3 - r_4)(4(r_3 - r_4) - r)}{16r^2} + \frac{(r_3 - r_4)^2 + 2(r_3 + r_4) - 3}{16(r+3)} \tag{18.43}$$

and then

$$c_1 = \frac{\rho}{x} + \frac{(r + r_1 - r_2)(r + r_3 - r_4)}{2r} + ... \tag{18.44}$$

$$c_2 = -\frac{\rho}{x} + \frac{(r + r_2 - r_1)(r + r_3 - r_4)}{2r} + ... \tag{18.45}$$

For the generating function

$$f(x) = -\rho \log(x) + \frac{(r + r_2 - r_1)(r + r_3 - r_4)}{2r} + \tag{18.46}$$

19. Appendix

19.1. Hypergeometric function

19.1.1. Integrals

:

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,(1-zt)^{-a}\,dt\,. \tag{19.1}$$

Relations:

$$F(a, b, c; z) = (1-z)^{c-a-b}\,F(c-a, c-b, c; z)\,, \tag{19.2}$$

$$F(a, b, c; z) = (1-z)^{-a}\,F\left(a, c-b, c; \frac{z}{z-1}\right)\,. \tag{19.3}$$

19.2. Differential equation

$$z(1-z)\,u_{zz} + [c - (1+a+b)z]\;u_z - ab\,u = 0\,. \tag{19.4}$$

19.3. Solutions

We define three bases.

Canonical near $z = 0$:

$$\begin{aligned} f_1(z) &= F(a, b, c; z) \\ f_2(z) &= z^{1-c}\,F(1+a-c, 1+b-c, 2-c; z) \end{aligned} \tag{19.5}$$

with Wronskian $W[f_1, f_2] = (1-c)$.

Canonical near $z = 1$:

$$\begin{aligned} g_1(z) &= F(a, b, 1 + a + b - c; 1 - z) \\ g_2(z) &= (1 - z)^{c-a-b}\, F(c - a, c - b, 1 + c - a - b; 1 - z) \end{aligned} \tag{19.6}$$

with Wronskian $W[g_1, g_2] = (a + b - c)$.

Canonical near $z = \infty$:

$$\begin{aligned} h_1(z) &= (-z)^{-a}\, F(a, 1 + a - c, 1 + a - b; 1/z) \\ h_2(z) &= (-z)\, F(b, 1 + b - c, 1 + b - a; 1/z) \end{aligned} \tag{19.7}$$

with Wronskian $W[h_1, h_2] = (b - a)$.

19.4. Transformations

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \mathbf{L} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \qquad \det \mathbf{L} = \frac{1 - c}{a + b - c} \tag{19.8}$$

$$\begin{aligned} L_{11} &= \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, & L_{12} &= \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} \\ L_{21} &= \frac{\Gamma(2 - c)\Gamma(c - a - b)}{\Gamma(1 - a)\Gamma(1 - b)}, & L_{22} &= \frac{\Gamma(2 - c)\Gamma(a + b - c)}{\Gamma(1 + b - c)\Gamma(1 + a - c)} \end{aligned} \tag{19.9}$$

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \mathbf{K} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \qquad \det \mathbf{K} = \frac{1 - c}{a - b} \tag{19.10}$$

$$\begin{aligned} K_{11} &= \frac{\Gamma(c)\Gamma(b - a)}{\Gamma(b)\Gamma(c - a)}, & K_{12} &= \frac{\Gamma(c)\Gamma(a - b)}{\Gamma(a)\Gamma(c - b)} \\ K_{21} &= \frac{\Gamma(2 - c)\Gamma(b - a)}{\Gamma(1 - a)\Gamma(1 + b - c)}, & K_{22} &= \frac{\Gamma(2 - c)\Gamma(a - b)}{\Gamma(1 - b)\Gamma(1 + a - c)} \end{aligned} \tag{19.11}$$

20. Groups

$\mathbf{SU(2)}$:

$$\mathbf{M} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1\,. \tag{20.1}$$

$\mathbf{SU(1,1)}$:

$$\mathbf{M} = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad |a|^2 - |b|^2 = 1\,. \tag{20.2}$$

Lecture 7. Quantum Liouville theory

21. Basic properties

21.1. The functional measure

As we have observed, substantial part of the problem of 2D quantum gravity coupled to conformal matter can be reduced to the functional integral over the field $\sigma(x)$, interpreted as the conformal factor in the metric

$$g_{\mu\nu}(x) = e^{\sigma(x)}\,\hat{g}_{\mu\nu}(x)\,, \tag{21.1}$$

with the action

$$\mathcal{A}[\sigma] = \frac{26 - c_{\mathrm{M}}}{48\pi}\int\sqrt{\hat{g}}\,\left[\frac{1}{2}\,\hat{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + \hat{R}\,\sigma + \Lambda\,e^{\sigma}\right]\,d^2x\,, \tag{21.2}$$

where c_{M} is the Virasoro central charge of the conformal "matter".

We have also noticed the problem with the integration measure in this functional integral. As it comes out from the gauge-fixed measure over the geometries, it is not the standard linear measure, but rather the measure associated with the metric

$$||\delta\sigma||^2 = \int\sqrt{\hat{g}}\;e^{\sigma(x)}\;(\delta\sigma(x))^2\;d^2x \tag{21.3}$$

in the space of the fields $\sigma(x)$.

As I have mentioned before in different context, there is a general argument (not a proof!) that a local nonlinear measure, like the one above ((21.3)), can be replaced by the linear one, at the expense of adding some (may be complicated) local terms to the Lagrangian density. The linear measure here is the one corresponding to the metric

$$||\delta\sigma||^2 = \int\sqrt{\hat{g}}\;\;(\delta\sigma(x))^2\;d^2x\,; \tag{21.4}$$

unlike (21.3) it is obviously invariant with respect to the translations in the field space,

$$\sigma(x) \to \sigma(x) + C(x)\,, \tag{21.5}$$

with arbitrary but fixed function $C(x)$. I will adopt this assumption. Moreover, I will assume (following Distler and Kawai) that the additional terms in the action are of the same form as those already present in the "bare" actoin, i.e. the price for the change to the linear measure (21.4) is just some change in the coefficients in front of different terms in the action. Thus, we look for the "renormalized" action in the form

$$\mathcal{A}_{\mathrm{r}}[\sigma] = \frac{1}{8\pi\,b^2}\int\sqrt{\hat{g}}\;\left[\frac{1}{2}\,\hat{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + q\,\hat{R}\,\sigma + \tilde{\Lambda}\,e^{\sigma}\right]\,d^2x\,, \tag{21.6}$$

where b, q, and $\tilde{\Lambda}$ are parameters, to be adjusted for consistency of the theory (see below). As the renormalization is the quantum phenomenon, we expect that in the classical limit

$$\frac{1}{b^2} \to -\frac{c}{6} + O(1)\,, \qquad q \to 1 \qquad \text{as} \quad c \to -\infty\,. \tag{21.7}$$

I will try to show that these assumptions lead to a beautiful quantum theory, which (for certain range of the parameters) exhibits all the features expected from the quantized version of the dynamical gravity.

First of all, let me change to the notations commonly used in the literature on the subject. We introduce the renormalized field $\varphi(x)$,

$$\sigma(x) = 2b\,\varphi(x)\,, \tag{21.8}$$

so that the action (21.6) takes the form

$$\mathcal{A}_L[\hat{g},\varphi] = \frac{1}{4\pi}\int \sqrt{\hat{g}}\,\left[\hat{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + Q\,\hat{R}\,\varphi + 4\pi\mu\, e^{2b\varphi}\right]\, d^2x\,, \tag{21.9}$$

where $Q = q/b$ and $\mu = \tilde{\Lambda}/(8\pi b^2)$ are new constants. Just like I said, it is assumed that now the functional integration is to be performed with the linear measure $D[\varphi(x)]$, invariant with respect to the shifts of the functional variable,

$$D[\varphi(x) + C(x)] = D[\varphi(x)]\,. \tag{21.10}$$

In what follows we will be mostly interested in the un-normalized correlation functions of the exponential fields, i.e. the functional integrals of the form

$$\langle\, V_{a_1}(x_1)\,\cdots\, V_{a_n}(x_n)\,\rangle = \int V_{a_1}(x_1)\,\cdots\, V_{a_n}(x_n)\; e^{-\mathcal{A}_L[\varphi]}\, D[\varphi]\,, \tag{21.11}$$

where I use the notation

$$V_a(x) = e^{2a\,\varphi(x)} \tag{21.12}$$

(at this point we do not divide by the partition function because there is certain problem with definition of this quantity, see below). First of all, let me discuss some simple properties of the functional integrals (21.11) with the action (21.9).

21.2. Background independence

The auxiliary metric $\hat{g}$ in (21.6) is just that - an auxiliary one: it can be chosen at will, the physical results should not depend on that choice. In other words, if we write

$$\hat{g}_{\mu\nu}(x) = e^{\hat{\sigma}(x)}\, g^{(0)}_{\mu\nu}(x)\,, \tag{21.13}$$

where $g^{(0)}$ is yet another arbitrary but fixed metric, it must be possible to absorb $\hat{\sigma}(x)$ by suitable shift of the Liouville field $\varphi(x)$; then, in view of the assumed linear property of the measure (21.10), the dependence on $\hat{\sigma}(x)$ would disappear.

Let us first address the problem in the case when the exponential term in (21.9) is absent, i.e. at $\mu = 0$. Then the only term in the action (21.9) which explicitly depends on $\hat{\sigma}$ is the one involving the curvature $\hat{R}$. Since

$$\sqrt{\hat{g}}\ \hat{R} = \sqrt{g_0}\ \left(R_0 - \Delta_0 \hat{\sigma}\right) . \tag{21.14}$$

we have

$$\mathcal{A}_L[e^{\hat{\sigma}} g_0, \varphi]_{\mu=0} = \frac{1}{4\pi} \int \sqrt{g_0}\ \left[{g_0}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + Q\, R_0\, \varphi + Q\, g_0^{\mu\nu} \partial_\mu \hat{\sigma} \partial_\nu \varphi \right]\, d^2x\,, \tag{21.15}$$

where I have transformed by parts

$$-\int \sqrt{g_0}\, \varphi(x) \Delta_0 \hat{\sigma}(x)\, d^2x = \int \sqrt{g_0}\, g_0^{\mu\nu} \partial_\mu \varphi \partial_\nu \hat{\sigma}\, d^2x\,. \tag{21.16}$$

The expression (21.15) can be rewritten as

$$(21.15) = \frac{1}{4\pi} \int \sqrt{g_0}\ \left[{g_0}^{\mu\nu} \partial_\mu \hat{\varphi} \partial_\nu \hat{\varphi} + Q\, R_0\, \hat{\varphi} - \frac{Q^2}{2} \left(R_0\, \hat{\sigma} + \frac{1}{2}\, g_0^{\mu\nu} \partial_\mu \hat{\sigma} \partial_\nu \hat{\sigma} \right)\right]\, d^2x\,, \tag{21.17}$$

where $\hat{\varphi}$ is the shifted Liouville field

$$\hat{\varphi}(x) = \varphi(x) + \frac{Q}{2}\, \hat{\sigma}(x)\,. \tag{21.18}$$

Now, recall that what actually enters the functional integral is the combination

$$Z_{\text{matter,ghosts}}[\hat{g}]\, e^{-\mathcal{A}_L[\hat{g},\varphi]}\, D_{\hat{g}}[\varphi]\,, \tag{21.19}$$

where the first factor is the combined partition function of the matter fields and the ghosts in the background metric $\hat{g}$. We already know that if $\hat{g}$ is of the form (21.13),

$$Z_{\text{matter,ghosts}} = \exp\left\{ \frac{c_{\text{M}} - 26}{48\pi} \int_{\mathbb{M}} \sqrt{g_0}\ \left[R_0(x)\, \hat{\sigma}(x) + \frac{1}{2}\ g_0^{\mu\nu}\ \partial_\mu \hat{\sigma} \partial_\nu \hat{\sigma} \right]\ d^2x \right\}\ . \tag{21.20}$$

In addition, there is anomalous dependence of the functional measure $D[\varphi]_{\hat{g}}$, due to the trace anomaly. Note that now we are dealing with the linear measure, and the standard analysis of the anomaly applies. The contribution is identical to a single scalar field (it is possible to go back to the calculation of the anomaly, and see that the curvature term does not affect the result, and in fact the exponential term $\mu\, e^{2b\varphi}$ is irrelevant as well). Thus, the measure $D_{\hat{g}}[\varphi]$ produces additional factor of the form (21.20) with the factor $\frac{1}{48\pi}$ in front of the integral in the exponential. Then, the combination (21.19) is indeed independent of the choice of $\hat{g}$ (within given conformal class!) provided these factors cancel the additional term in in (21.17), i.e. if

$$Q^2 = \frac{25 - c_{\text{M}}}{6}\,. \tag{21.21}$$

Let us turn to the exponential term

$$\mu \int \sqrt{\hat{g}}\; e^{2b\,\varphi(x)}\, d^2x\,. \tag{21.22}$$

There are three sources of the dependence on the conformal factor in $\hat{g}$ here. First, the factor $\sqrt{\hat{g}}$ itself carries explicit dependence on $\hat{\sigma}$,

$$\sqrt{\hat{g}}\; = e^{\hat{\sigma}}\,\sqrt{g_0}\,. \tag{21.23}$$

Next, changing to the new integration variable $\hat{\varphi}$, Eq.(21.18) yields

$$e^{2b\varphi(x)} = e^{-b\,Q\,\hat{\sigma}(x)}\, e^{2b\,\hat{\varphi}(x)}\,. \tag{21.24}$$

Finally, as we already know, defining the exponential field requires regularization (Appendix), which leads to additional dependence on the background metric,

$$\left[e^{2a\,\varphi(x)}\right]_{e^{\hat{\sigma}}g_0} = e^{a^2\,\hat{\sigma}(x)}\;\left[e^{2a\,\varphi(x)}\right]_{g_0}\,. \tag{21.25}$$

Combining these contributions, we find that the exponential term is background independent provided

$$1 - b\,(Q-b) = 0\,, \tag{21.26}$$

or, in more symmetric writing,

$$Q = \frac{1}{b} + b\,. \tag{21.27}$$

We thus conclude that the requirement of the background independence almost completely fixes the parameters Q and b of the action action (21.9) in terms of the central charge of the matter c_M, through the equations (21.21) and (21.27). To fix them completely we will assume that b is positive, and that b becomes small in the classical limit, when Q goes to infinity.

Note that in order to have real b we must have $Q \geq 2$, i.e.

$$c_{\mathrm{M}} < 1\,. \tag{21.28}$$

This relation restricts validity of the present approach to very special matter theories; it essentially limits it to special $c < 1$ "minimal models" of CFT. The theory of such "minimal matter" coupled to gravity is often referred to as the "minimal gravity". It is the minimal gravity which is related to solvable matrix models. There are several proposals on how to go beyond the bound (5.5). Most of them involve supersymmetry. I will not discuss this topic here.

The exponential insertions $V_a(x)$ in (21.11) bring in their own dependence on the background metric $\hat{g}$. Repeating the above analysis we find

$$\left[e^{2a\,\varphi(x)}\right]_{e^{\hat{\sigma}}g_0} = e^{-\Delta_a\,\hat{\sigma}(x)}\;\left[e^{2a\,\tilde{\varphi}(x)}\right]_{g_0}\,. \tag{21.29}$$

where

$$\Delta_a = a\,(Q - a)\,. \tag{21.30}$$

What is important, is invariance of the scalar combinations

$$e^{a\,\varphi(x)}\,\Phi_\Delta(x) \tag{21.31}$$

with primary fields Φ_Δ of the conformal matter (or more complicated scalars). Since for the primary fields in a background metric

$$\Phi_\Delta^{[e^{\hat\sigma} g_0]}(x) = e^{-\Delta\hat\sigma(x)}\,\Phi_\Delta^{[g_0]}(x)\,, \tag{21.32}$$

we understand that a in (21.31) should be chosen in such a way that $\Delta_a + \Delta = 0$.

Densities: More often

$$\int \sqrt{\hat g}\,V_a(x)\,\Phi_\Delta(x)\,d^2x\,, \tag{21.33}$$

In this case

$$\Delta + \Delta_a = 1\,. \tag{21.34}$$

21.3. Conformal invariance

The action (21.9) is conformally invariant. The conformal transformation

$$z \to w(z)\,, \qquad \bar z \to \bar w(\bar z) \tag{21.35}$$

can be regarded as special case of the transformation (21.13), with $\hat\sigma(z,\bar z) = -\log|\partial_z w|^2$, so that the conformal change of coordinates, accompanied by the field shift

$$\varphi(w,\bar w) = \varphi(z,\bar z) - \frac{Q}{2}\,\log\left|\frac{dw}{dz}\right|^2\,, \tag{21.36}$$

leaves the quantum action (21.9) invariant (about transformation of the exponential). As usual, the most useful bookkeeping device here is the energy-momentum tensor. Defining $T_{\mu\nu}$ as usual, as the variation of the quantum action (i.e. including the effects of the measure) with respect to the background metric $\hat g_{\mu\nu}$, we find in the flat background

$$T_{\mu\nu} = -\partial_\mu\varphi\partial_\nu\varphi + \frac{\hat g_{\mu\nu}}{2}\,\left[(\partial\varphi)^2 + (4\pi\mu\;bQ)\,e^{2b\,\varphi}\right] + Q\,\left(\partial_\mu\partial_\nu\varphi - \hat g_{\mu\nu}\,\partial^2\varphi\right)\,. \tag{21.37}$$

As before, the last term (linear in φ) is from the variation of the curvature term in the (21.9). The coefficient $(4\pi\mu\,bQ) = 4\pi\mu\,(1+b^2)$ in front of the exponential term includes the correction $4\pi\mu\,b^2$, due to the transformation law (21.25) [17]. In view of the equation of motion

$$-\partial^2\varphi + (4\pi\mu\,b)\,e^{2b\varphi} \simeq 0\,, \tag{21.38}$$

[17] In deriving this energy-momentum tensor we make variation of the action (21.9) keeping the field φ fixed. If one adds the shift (21.18) to the definition of the variation, the resulting energy-momentum tensor is (21.41)

the tensor (21.37) is traceless, in the sense that T^μ_μ is a redundant field,

$$T^\mu_\mu \simeq 0\,. \tag{21.39}$$

Also, it satisfies the standard continuity equation

$$\partial_\mu T^{\mu\nu} \simeq 0\,. \tag{21.40}$$

The last statement may not be directly evident from the expression (21.37), but we explain it in the Appendix A. Note that up to a redundant field the energy-momentum tensor can be written as

$$T_{\mu\nu} = -\partial_\mu\varphi\partial_\nu\varphi + \frac{\hat{g}_{\mu\nu}}{2}\,(\partial\varphi)^2 + Q\left(\partial_\mu\partial_\nu\varphi - \frac{\hat{g}_{\mu\nu}}{2}\,\partial^2\varphi\right)\,. \tag{21.41}$$

This form comes out directly if we redefine variation over $\hat{g}_{\mu\nu}$ by adding to it the variation (21.18) of the field φ.

As usual in CFT, combination of equations (21.39) and (21.40) guarantee that the complex-coordinate components

$$\begin{aligned} T_{zz} &= T = -(\partial_z\varphi)^2 + Q\,\partial_z^2\varphi\,, \\ T_{\bar{z}\bar{z}} &= \bar{T} = -(\partial_{\bar{z}}\varphi)^2 + Q\,\partial_{\bar{z}}^2\varphi\,. \end{aligned} \tag{21.42}$$

are holomorphic and anti-holomorphic fields, respectively, i.e. $T = T(z)$ and $\bar{T} = \bar{T}(\bar{z})$. They satisfy the standard Virasoro algebra OPE, with the central charge

$$c_{\rm L} = 1 - 6\,Q^2\,. \tag{21.43}$$

This equation has simple meaning: In virtue of the relation (21.21), we have

$$c_{\rm tot} = \underbrace{c_M}_{\text{Matter}} \; - \; \underbrace{26}_{\text{Ghosts}} \; + \; \underbrace{c_{\rm L}}_{\varphi-\text{field}} \; = \; 0 \tag{1.43a}$$

i.e. the combined central charge of the full matter plus ghosts plus Liouville systems vanishes. This is exactly what we expect in consistent quantum theory of gravity. Indeed, the total energy-momentum tensor describes reaction of the system to the change of the "background" metric $\hat{g}$. Since the metric in quantum gravity is the integration variable, no dependence is allowed. Therefore, in some sense, the basic equation of quantum gravity is

$$T^{(\rm tot)}_{\mu\nu} = 0\,. \tag{21.44}$$

In this form it is too abstract. We split the total into constituents, and then careful synthesis yields zero.

Moreover, the above exponential fields $V_a(x)$ are primary fields with respect to this Virasoro algebra, with the dimensions Δ_a in Eq.(21.30),

$$T(z)\,V_a(w,\bar{w}) = \frac{\Delta_a}{(z-w)^2}\,V_a(w,\bar{w}) + \frac{1}{z-w}\,\partial_w V_a(w,\bar{w})\,. \tag{21.45}$$

The relation (21.34) then guarantees that the integrand in

$$\int V_a(z,\bar{z})\,\Phi_\Delta(z,\bar{z})\,d^2z \tag{21.46}$$

is a density, so that the integral is invariant under the conformal analytic transformations (21.35).

Two and three-point functions.

Seiberg's bound?

21.4. Flat background

In what follows, in most cases

$$\hat{g}\,: \qquad d\hat{s}^2 = dz d\bar{z}\,, \tag{21.47}$$

interpreted as the metric of a sphere, flat everywhere except for the curvature bump hidden at infinity. As in the classical case we need some way to handle that singularity. The transformation law (21.36) suggests that for the physical metric to be smooth at the infinity the Liouville field must grow as

$$\varphi(z,\bar{z}) = -Q\,\log|z|^2 + \text{finite}\,, \tag{21.48}$$

i.e. the correctly constructed action should be finite on exactly such configurations. This prescribes the boundary term

$$\mathcal{A}_L[\varphi] = \frac{1}{\pi}\int_{D_L}\left[\partial_z\varphi\partial_{\bar{z}}\varphi + \pi\mu\,e^{2b\varphi}\right]\,d^2z + \frac{Q}{\pi}\int_{\partial D_L} K\,\varphi\,dl + 2Q^2\,\log L \tag{21.49}$$

in the "cutoff" action. The limit $L\to\infty$ is to be taken. This regularization is similar to what we had in the classical case; it can be alternatively obtained by cutting out "small" domain $\mathbb{R}^2\backslash D_L$ around the infinite point, and "smoothing" the background curvature along the boundary of that domain. (What about $2Q^2\,\log L$ here?).

Anyway, the correlation functions (21.11) are then defined through the functional integrals

$$\int V_{a_1}(z_1,\bar{z}_1)\cdots V_{a_n}(z_n,\bar{z}_n)\,e^{-\mathcal{A}_L[\varphi]}\,D[\varphi]\,, \tag{21.50}$$

where finiteness of the action (presumably) automatically selects the field configurations with the asymptotic behavior (21.48).

21.5. Flat case

The most simple case is that of zero μ. The integral is Gaussian, and is evaluated by shift of the functional variable

$$\varphi(z,\bar{z}) = -\sum_{i=1}^{n} a_i\,\log|z-z_i|^2 + \tilde{\varphi}(z,\bar{z})\,, \tag{21.51}$$

where $\tilde{\varphi}$ must be regular everywhere (Fourier-transformable) [18]. Note that this is consistent with the asymptotic condition (21.48) only if

$$\sum_i a_i = Q\,. \tag{21.53}$$

The correlation functions (21.50) exist (nonzero) only if the condition (21.53) is satisfied. This is interpreted as follows. The physical geometry is flat (the expectation of the curvature is zero) everywhere except for the points z_i, where the exponential insertions in (21.50) create the curvature singularities. In view of the identification

$$\sqrt{g}\;R(x) = -\frac{2}{Q}\,\Delta\varphi(x)\,, \tag{21.54}$$

or

$$\sqrt{g}\;R(z,\bar{z}) = -\frac{8}{Q}\,\partial_z\partial_{\bar{z}}\varphi(z,\bar{z})\,, \tag{21.55}$$

(following from (21.18)) we have

$$\frac{1}{8\pi}\,\sqrt{g}\;R(x) = \sum_i \frac{a_i}{Q}\;\delta(x-x_i)\,. \tag{21.56}$$

The condition (21.53) is just the topological statement about the surface. We have at $\mu = 0$

$$\langle\, V_{a_1}(z_1,\bar{z}_1)\cdots V_{a_n}(z_n,\bar{z}_n)\,\rangle = \delta\big(\sum_i a_i - Q\big)\;\prod_{i<j}^{n}\,|z_i - z_j|^{-4a_i a_j}\,, \tag{21.57}$$

where the delta-function appears as the result of integration over the zero mode (see below).

General: Insertion of $V_a(x_0)$ creates a curvature singularity

$$\sqrt{g}\;R(x) = \frac{8\pi\,a}{Q}\,\delta(x-x_0)\,. \tag{21.58}$$

21.6. "Coulomb" integrals

Naively, one could try to evaluate the functional integral (21.11) by expanding in the parameter μ; thus, formally

$$\langle\, V_{a_1}(x_1)\,\cdots\,V_{a_n}(x_n)\,\rangle = \sum_{N=0}^{\infty}\frac{(-\mu)^N}{N!}\,C_N(x_1,x_2,\cdots,x_n)\,, \tag{21.59}$$

[18] Equivalently, one uses the Wick rules with

$$\varphi(z_1,\bar{z}_1)\varphi(z_1,\bar{z}_2) = -\log|z_1 - z_2| + :\varphi(z_1,\bar{z}_1)\varphi(z_1,\bar{z}_2): \tag{21.52}$$

with

$$C_N(x_1,\cdots,x_n)=\int\langle\, V_{a_1}(x_1)\,\cdots\, V_{a_n}(x_n)\; V_b(y_1)\cdots V_b(y_N)\,\rangle_0\, d^2y_1\cdots d^2y_N$$

where $\langle\cdots\rangle_0$ are the integrals

$$\langle\, V_{a_1}(x_1)\,\cdots\, V_{a_n}(x_n)\,\rangle_0=\int V_{a_1}(x_1)\,\cdots\, V_{a_n}(x_n)\; e^{-\mathcal{A}_0[\varphi]}\,D[\varphi] \tag{21.60}$$

with the "free" action

$$\mathcal{A}_0=\frac{1}{\pi}\int\,(\,\partial_z\varphi\partial_{\bar z}\varphi)\;d^2z\,. \tag{21.61}$$

Since we are dealing with the free field, we have

$$(21.60)\sim\prod_{i<j}^{n}|z_i-z_j|^{-4a_ia_j}\,, \tag{21.62}$$

and (21.59) lead to the "Coulomb integrals",

$$C_N(x_1,\cdots,x_n)=\prod_{i<j}^{n}|z_i-z_j|^{-4a_ia_j}\int\prod_{i=1}^{n}\prod_{k=1}^{N}|z_i-w_k|^{-4ba_i}\prod_{k<l}^{N}|w_k-w_l|^{-4b^2}\prod_{k=1}^{N}d^2w_k\,, \tag{21.63}$$

where $(z_i,\bar z_i)$ are the complex coordinates of the points x_i, and $(w_k,\bar w_k)$ are the coordinates of the integration points y.

In reality, the representation (21.59) can not be taken literally. The most important reason is that the field configurations in the free-field integrals involved in C_N generally fail to meet the necessary asymptotic conditions (21.48). As the result, different terms in (21.59) in fact represent contributions to different correlation functions, with extra insertions hidden at infinity. The true role of the coulomb integrals will be clarified in the next subsection.

Another problem, starting from some orders, the integrals develop infrared and/or ultraviolet divergences. While the ultraviolet divergences will be discussed later, let me say few words about the infrared situation.

The standard way to deal with the infrared problems is to introduce the large-scale cutoff, say bind the integrations over z_i to a large domain D_L of $\mathbb{R}^2$ ...

21.7. Scale dependence and the zero mode

The only dimensional parameter in (21.11) is the renormalized cosmological constant μ. The dependence of the correlation functions on μ therefore represents their scale dependence. It can be easily isolated, because the parameter μ in (21.9) can be absorbed by a constant shift of the field, $\varphi(x)\to\varphi(x)-\log\mu/2b$. Thus we find,

$$(21.11)\;\sim\;\mu^{(Q-\sum_i a_i)/b}\,. \tag{21.64}$$

Here I assume for simplicity that $\mathbb{M}$ has the topology of a sphere, i.e.[19]

$$\int \sqrt{\hat{g}}\, \hat{R}(x)\, d^2x = 8\pi\,. \tag{21.65}$$

It is instructive to repeat the same analysis in slightly different manner. Let us write the correlation function (21.11) as the Laplace transform

$$\langle\, V_{a_1}(x_1)\, \cdots\, V_{a_n}(x_n)\, \rangle = \int_0^\infty \langle\, V_{a_1}(x_1)\, \cdots\, V_{a_n}(x_n)\, \rangle_A\; e^{-\mu A}\, dA\,, \tag{21.66}$$

where the "fixed area" correlation function $\langle \cdots \rangle_A$ is defined as follows

$$\langle\, V_{a_1}(x_1)\, \cdots\, V_{a_n}(x_n)\, \rangle_A = \int V_{a_1}(x_1)\, \cdots\, V_{a_n}(x_n)\; e^{-\mathcal{A}_0[\varphi]}\, \delta\left(A - \int \sqrt{\hat{g}}\; e^{2b\varphi(x)}\, d^2x\right) D[\varphi]\,, \tag{21.67}$$

where $\mathcal{A}_0[\varphi]$ is the action (21.9) without the exponential term,

$$\mathcal{A}_0[\varphi] = \frac{1}{4\pi} \int \sqrt{\hat{g}}\; \left[\hat{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + Q\,\hat{R}\,\varphi\right]\, d^2x\,. \tag{21.68}$$

The "fixed area" correlation functions depends on A instead of μ, and have the following scaling power-like behavior

$$(21.67)\;\sim\; A^{-\frac{Q}{b}-1+\sum_i \frac{a_i}{b}}\,. \tag{21.69}$$

This dependence follows directly from the properties of the functional integral (21.67), as we will derive few lines below.

The scale dependence (21.69) is interpreted as follows. The factor

$$Z_{\text{grav}}(A) \sim A^{-\frac{Q}{b}-1} \tag{21.70}$$

is attributed to the fixed-area partition function. Compare it with the scale dependence of the CFT partition function on a manifold with fixed (not dynamical) metric,

$$Z_{\text{matter}}(A) \sim A^{\frac{c_{\rm M}}{6}} \tag{21.71}$$

(see Lecture 5). Promoting the metric to the quantum degree of freedom modifies the exponent as follows (check!)

$$\frac{c_{\rm M}}{6}\;\to\;-\frac{Q}{b} - 1 = \frac{c_{\rm M}}{6} - \frac{25}{6} + O\left(-\frac{1}{c_{\rm M}}\right)\,. \tag{21.72}$$

The gravitational exponent in (21.70) reduces to the fixed-geometry exponent (21.71) in the classical limit $c_{\rm M} \to -\infty$. Next, the factors $A^{\frac{a_i}{b}}$ in are associated with the insertions V_{a_i}

[19] For generic compact $\mathbb{M}$ one has to replace $Q \to (1-\gamma)Q$.

in (21.67). Recall that such insertions appear in the combinations (21.33) with the primary fields Φ_{Δ_i} of the "matter" CFT, to form the corresponding densities integrated over $\mathbb{M}$. Therefore, we call the quantities

$$\Delta_i^{(\text{grav})} = -\frac{a_i}{b} + 1 \tag{21.73}$$

the "gravitational dimensions" of the fields Φ_{Δ_i}: insertion of Φ_{Δ_i}, properly "dressed" to form covariant quantity, generates the scale factor $A^{1-\Delta_i^{(\text{grav})}}$ in the fixed-area correlation function. Then (21.34), together with (21.30) yields the following relation between the conformal gravitational dimensions of any primary field

$$\Delta = \Delta^{(\text{grav})} - b^2\, \Delta^{(\text{grav})}\, \left(1 - \Delta^{(\text{grav})}\right)\, . \tag{21.74}$$

Again, in the classical case c_{M} the gravitational dimension reduces to the fixed-metric dimension Δ.

To handle the integral (21.67), we write

$$\varphi(x) = \varphi_0 + \tilde{\varphi}(x)\,, \tag{21.75}$$

where φ_0 is the "zero mode", and $\tilde{\varphi}(x)$ is "centered" around zero, one way or another; for instance, let

$$\int \sqrt{\hat{g}}\; \tilde{\varphi}(x)\, d^2x = 0\,. \tag{21.76}$$

The integration over φ_0 eliminates the delta-function in (21.67), so that

$$V_a(x) = e^{2a\,\varphi_0}\; \tilde{V}_a(x) \;\;\to\;\; A^{a/b}\; S[\,\tilde{\varphi}\,]^{-a/b}\; \tilde{V}_a(x)\,, \tag{21.77}$$

where I use the notation $\tilde{V}_a(x) = e^{2a\,\tilde{\varphi}(x)}$, and $S[\,\tilde{\varphi}\,]$ is the integral

$$S[\,\tilde{\varphi}\,] = \int \sqrt{\hat{g}}\; \tilde{V}_b(x)\, d^2x \tag{21.78}$$

The integral (21.67) then acquires the form

$$\langle\, V_{a_1}(x_1)\, \cdots\, V_{a_n}(x_n)\,\rangle_A = \frac{1}{2b}\, A^{\frac{\sum_i a_i - Q}{b} - 1}\; G_{a_1,\ldots,a_n}(x_1,\ldots,x_n)\,, \tag{21.79}$$

with the A-independent function F is given by the following functional integral

$$G_{a_1,\ldots,a_n}(x_1,\ldots,x_n) = \int \tilde{V}_{a_1}(x_1)\, \cdots\, \tilde{V}_{a_n}(x_n)\; S[\,\tilde{\varphi}\,]^{\frac{Q-a}{b}}\; e^{-\mathcal{A}_0[\,\tilde{\varphi}\,]}\; D[\tilde{\varphi}]\,. \tag{21.80}$$

Let us get back to the Laplace representation (21.66). With the form (21.79), the area integration reduces to the Euler integral, and we find

$$\langle\, V_{a_1}(x_1)\, \cdots\, V_{a_n}(x_n)\,\rangle = \frac{\mu^{(Q-\sum_i a_i)/b}}{2b}\; \Gamma\left(\frac{\sum_i a_i - Q}{b}\right)\; G_{a_1,\ldots,a_n}(x_1,\ldots,x_n) \tag{21.81}$$

To be more precise, the integral (21.66) converges only if

$$Q - \sum_{i=1}^{n} a_i \; > \; 0 \, . \tag{21.82}$$

The divergence at small A which appears if this equation is not satisfied is directly related to the divergence of the original integral (21.11) at large negative φ. The poles of the gamma-function in (21.81) appear as the result of this divergence.

Poles at

$$\sum_i a_i + N\, b = Q \, . \tag{21.83}$$

Residues: denote

$$s = \sum_i a_i \, . \tag{21.84}$$

$$\mathrm{res}_{s=Q-Nb}\, G_{a_1,\cdots,a_n}(x_1,\cdots,x_n) = \frac{\mu^N}{N!}\, C_N(x_1,\cdots,x_n) \tag{21.85}$$

with the above Coulomb integrals.

22. Hamiltonian approach

Important insight into the structure of the Liouville theory can be gained by the following analysis. Consider a surface of the topology of a sphere with two punctures, at $x = x_s$ and $x = x_n$ (to be regarded as the "south" and the "north" poles of the sphere). Since $\hat{g}_{\mu\nu}(x)$ can be chosen arbitrarily, let us take special metric which is flat everywhere except for at these two points, where it has the delta-function peaks,

$$\sqrt{\hat{g}}\;\hat{R}(x) = 4\pi\, \delta(x - x_s) + 4\pi\, \delta(x - x_n) \;, \tag{22.1}$$

so that each of the two points x_s and x_n carries exactly half of the total curvature of the sphere. Then the Q-term in the action becomes

$$\frac{1}{4\pi} \int \sqrt{\hat{g}}\; \left[Q\, \hat{R}(x)\varphi(x)\right] d^2x = Q\, \varphi(x_s) + Q\, \varphi(x_n) \;. \tag{22.2}$$

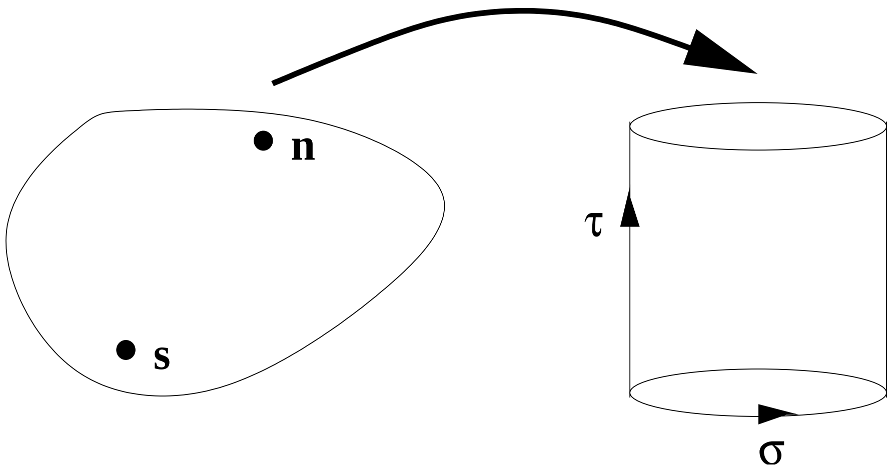
One way to look at it is to say that we can ignore the Q-term at the expense of adding two extra insertions, at $x = x_s$ and $x = x_n$,

$$e^{Q\,\varphi(x_s)}\, e^{Q\,\varphi(x_n)} \equiv V_{Q/2}(x_s) V_{Q/2}(x_n) \;. \tag{22.3}$$

Yet another way to understand this is to make a conformal transformation which maps this surface onto an infinitely long cylinder with the ends corresponding to x_s and x_n,

To be specific, let us introduce complex coordinates $z, \bar{z}$ such that

$$\begin{aligned} x_s \; : \qquad & (z, \bar{z}) = \; 0 \;, \\ x_n \; : \qquad & (z, \bar{z}) = \infty \;. \end{aligned} \tag{22.4}$$



Then we make conformal transformation (the exponential map)

$$z = e^{-iu} \ , \qquad \bar{z} = e^{i\bar{u}} \ . \tag{22.5}$$

The real variables (σ, τ)

$$u = \sigma + i\tau \ , \qquad \bar{u} = \sigma - i\tau \tag{22.6}$$

are Cartesian coordinates on the cylinder. We will interpret σ as the spatial coordinate which runs a circle,

$$\sigma \sim \sigma + 2\pi \ , \tag{22.7}$$

and τ as the Euclidean time.

The holomorphic and antiholomorphic components of the EM tensor can be written as (all the signs?)

$$T(u) = \frac{c_{\rm L}}{24} - \sum_{n=-\infty}^{\infty} e^{inu} L_n \ , \tag{5.16a}$$

$$\bar{T}(u) = \frac{c_{\rm L}}{24} - \sum_{n=-\infty}^{\infty} e^{-inu} \bar{L}_n \ , \tag{5.16b}$$

where L_n are the same Virasoro generators which appeared before in the expansion (...), i.e.

$$T(z) = \sum_{-\infty}^{\infty} \frac{L_n}{z^{n+2}} \ , \tag{5.17}$$

and the terms $\frac{c_L}{24}$ come from the Schwartzian term in the transformation law of T under specific transformation $z = e^{iu}$.

The Hamiltonian, which by definition is

$$H = \frac{1}{2\pi} \int_0^{2\pi} T_{\tau\tau} \, d\sigma = -\frac{1}{2\pi} \int (T + \bar{T}) \, d\sigma \ , \tag{22.8}$$

i.e.

$$H = -\frac{c_L}{12} + L_0 + \bar{L}_0 \ . \tag{22.9}$$

To get better idea about the structure of the space of states of the theory, take the zero mode of φ,

$$\varphi_0 = \int_0^{2\pi} \varphi(\sigma)\, \frac{d\sigma}{2\pi} \tag{22.10}$$

and consider the limit

$$\varphi_0 \ \to \ -\infty \ . \tag{22.11}$$

In this domain the term $e^{2b\varphi}$ in (5.2) is negligible, and $\varphi(\sigma, \tau)$ becomes a free field. Then, restricting attention to the asymptotic domain (5.21), we can write

$$\varphi(\sigma, \tau) = \varphi_0 + 2\,\hat{P}\,\tau + \sum_{n \neq 0} \left(\frac{i a_n}{n}\, e^{-inu} + \frac{i \bar{a}_n}{n}\, e^{in\bar{u}} \right) , \tag{5.22}$$

where $\hat{P}$ is the zero-mode momentum

$$\hat{P} = -\frac{i}{2}\, \frac{\partial}{\partial \varphi_0} \ , \tag{5.23}$$

and $a_n,\ \bar{a}_n$ are the standard oscillators

$$[a_n, a_m] = \frac{m}{2}\, \delta_{n+m,0} \ , \qquad [\bar{a}_n, \bar{a}_m] = \frac{m}{2}\, \delta_{n+m,0} \ . \tag{5.24}$$

As before, using the equation

$$T(u) = -\partial_u \varphi \partial_u \varphi + Q\, \partial_u^2 \varphi \tag{5.25}$$

one finds

$$L_n = \sum_{k \neq 0, n} a_k a_{n-k} + (2\,\hat{P} + in\,Q)\, a_n \qquad \text{for} \quad n \neq 0\,, \tag{22.12}$$

$$L_0 = \frac{Q^2}{4} + \hat{P}^2 + 2 \sum_{k>0} a_{-k} a_k \, . \tag{22.13}$$

The operators $\bar{L}_n$ are given by the same expressions, with $\bar{a}_k$ replacing a_k.

The oscillator Fock vacuum is defined as usual,

$$a_k \mid vac\rangle = 0 \qquad \text{for} \quad k > 0 \ , \tag{22.14}$$

so that the Fock space $\mathcal{F}_{\rm osc}$ is obtained by applying the negative mode operators,

$$\mathcal{F}_{\rm osc} = \text{Span}\left\{ a_{-k_1} \cdots a_{-k_n} \bar{a}_{-\bar{k}_1} \cdots \bar{a}_{-\bar{k}_m} \mid vac\rangle \right\} \ . \tag{22.15}$$

To take into account the zero mode we define the full Hilbert space

$$\mathcal{H} = \mathcal{L}_2\big(-\infty < \varphi_0 < \infty \big) \ \otimes \ \mathcal{F}_{\rm osc} \ . \tag{22.16}$$

22.1. Liouville Reflection Operator

Since the exponential term can be neglected, in the asymptotic domain $\varphi_0 \to -\infty$ we are dealing with the free boson field, hence the eigenstates of the Hamiltonian (22.9) are of the form

$$e^{\pm 2iP\varphi_0}\; a_{-k_1}a_{-k_2}\dots a_{-k_n}\,\bar{a}_{-\bar{k}_1}a_{-\bar{k}_2}\dots a_{-\bar{k}_m}\;\mid vac\rangle\;. \qquad (22.17)$$

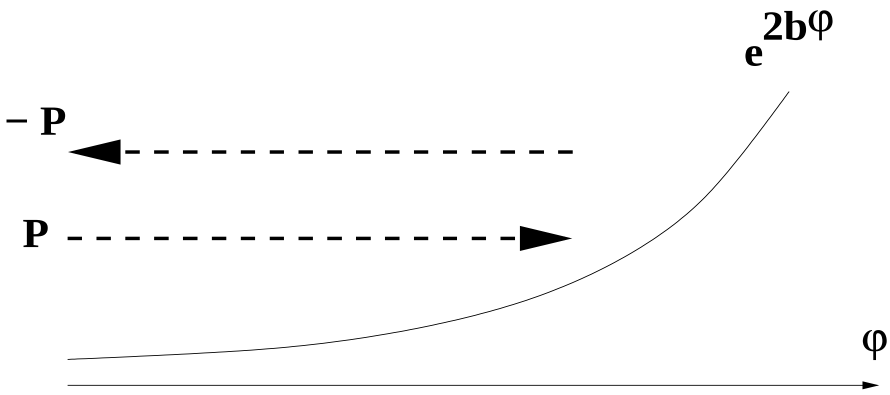
The associated eigenvalues of L_0 and $\bar{L}_0$ are

$$\Delta_{P,\{k_i\}} = \frac{Q^2}{4} + P^2 + \sum_i k_i\;,$$
$$\bar{\Delta}_{P,\{k_i\}} = \frac{Q^2}{4} + P^2 + \sum_i \bar{k}_i\;, \qquad (22.18)$$

Of course, the interaction occurs in the domain

$$\mu\, e^{2b\varphi_0} \approx 1\;, \qquad (22.19)$$

where the Liouville potential acts as a reflecting wall. Therefore the true stationary state, when considered in the above asymptotic domain $\varphi_0 \to -\infty$, must be certain combination of the incoming ($\sim\, e^{2iP\varphi_0}$) and the reflected ($\sim\, e^{-2iP\varphi_0}$) waves. For instance, the Liouville



state associated with the Fock vacuum $\mid 0\,\rangle$ must have the following $\varphi_0 \to -\infty$ asymptotic form

$$\mid \Psi_P\,\rangle = \Big(e^{2iP\varphi_0} + S(P)\, e^{-2iP\varphi_0}\Big) \otimes \mid vac\rangle\;. \qquad (22.20)$$

Here the coefficient $S(P)$ is certain phase factor often referred to as the "Liouville reflection amplitude" (or, more precisely, the Liouville vacuum reflection amplitude). It is easy to show that for real P we have $S^*(P) = S(-P)$, and hence the unitarity condition can be written in the form

$$S(P)S(-P) = 1\,, \qquad (22.21)$$

now valid at all complex P.

The notion of the reflection amplitude can be extended to the excited Fock states like

$$\mid s\rangle = a_{-k_1}a_{-k_2}\dots a_{-k_n}\bar{a}_{-\bar{k}_1}\bar{a}_{-\bar{k}_2}\dots\bar{a}_{-\bar{k}_m}\mid vac\rangle\,, \tag{22.22}$$

where we assume the symbol s to be a short-hand for the Fock state labels,

$$s = \{k_1, k_2, \cdots, k_n; \bar{k}_1, \bar{k}_2, \cdots, \bar{k}_m\}\,. \tag{22.23}$$

As usual, the Fock space splits to the level subspaces, according to the eigenvalues of the operators L_0 and $\bar{L}_0$, Eq. (22.12),

$$\begin{aligned}
&\mathcal{F}_{\rm osc} = \oplus_{N,\bar{N}}\,\mathcal{F}^{(N,\bar{N})}_{\rm osc}\,,\\
&\mathcal{F}^{(N,\bar{N})}_{\rm osc} = \text{Span}\left\{a_{-k_1}\cdots a_{-k_n}\,\bar{a}_{-\bar{k}_1}\dots\bar{a}_{-\bar{k}_m}\mid vac\rangle\,;\ \sum_{i=1}^{n}k_i = N, \sum_{j=1}^{m}\bar{k}_j = \bar{N}\right\}
\end{aligned} \tag{22.24}$$

By the same arguments as used above, the $\varphi_0 \to -\infty$ asymptotic form of the associated Liouville state has the form

$$\mid\ \Psi_{s,P}\,\rangle \to \left[e^{2iP\,\varphi_0} + e^{-2iP\,\varphi_0}\,\hat{S}(P)\right]\,\mid s\rangle \qquad \text{as} \quad \varphi_0 \to -\infty\,, \tag{22.25}$$

where $\hat{S}(P)$ is certain operator acting in the Fock space $\mathcal{F}_{\rm osc}$, Eq.(22.15). Since the The operators L_0 and $\bar{L}_0$ are exact integrals of motion of the Liouville theory, the operator $\hat{S}$ does not mix between the states at different levels,

$$\hat{S}(P)\,: \qquad \mathcal{F}^{(N,\bar{N})}_{\rm osc} \ \to\ \mathcal{F}^{(N,\bar{N})}_{\rm osc}\,. \tag{22.26}$$

We call the operator $\hat{S}(P)$ the "Liouville S-matrix". One of the ways to give it physical interpretation is to consider real-time evolution of the "universe" with the metric $e^{2b\varphi(\sigma,\tau=it)}\,(d\sigma^2 - dt^2)$. The "universe" is created very small at the infinite past. At that time φ_0 is very large negative, and the initial states are characterized as the states of the free boson theory, i.e. we can have some zero-mode momentum, and some, perhaps excited, state in the Fock space (22.15); this is the first term in the Eq.(22.25). The "universe" expands, reaching some maximal size, and then shrinks back to very small size in the infinite future, where again $\varphi_0 \to -\infty$. The "outgoing wave" in the second term of the Eq.(22.25) is the final state of this evolution.

For lower levels, it is rather straightforward to calculate the matrix elements of the operator $\hat{S}(P)$. More precisely, it is easy to express these matrix elements through the basic vacuum amplitude $S(P)$. Take, for example, the operator L_{-1},

$$L_{-1} = \sum_{k\neq 0,-1} a_k a_{-1-k} + \left(-i\,\frac{\partial}{\partial\varphi_0} - i\,Q\right)\,a_{-1}\,, \tag{22.27}$$

and apply it to the primary state (22.20). All terms in the quadratic part of (22.27) annihilate the Fock vacuum, therefore

$$L_{-1}\mid\Psi_P\rangle = e^{2iP\varphi_0}\,(2P-iQ)\;a_{-1}\mid vac\rangle + e^{-2iP\varphi_0}\,(-2P-iQ)\;a_{-1}\mid vac\rangle\,, \tag{22.28}$$

i.e.

$$\hat{S}(P)a_{-1}\mid vac\rangle = \frac{Q-2iP}{Q+2iP}\,S(P)\;\,a_{-1}\mid vac\rangle\,. \tag{22.29}$$

More examples? Eigenvalues and integrals of motion?

This procedure allows one to reconstruct $\hat{S}(P)$ level by level. But becomes difficult for higher levels.

22.2. Canonical quantization

Let us start again with the sphere with two punctures, at $z=0$ and at $z=\infty$, where the insertions $V_{Q/2}$ are added. Let us change to the polar coordinates (22.5), (22.6), and replace the background metric as follows

$$\hat{g}\;:\qquad dzd\bar{z} = e^{i(u-\bar{u})}\,dud\bar{u}\;\;\rightarrow\;\;dud\bar{u}\,. \tag{22.30}$$

The bulk part of the action becomes

$$\mathcal{A}_L[\varphi] = \frac{1}{4\pi}\int\left[(\partial_\tau\varphi)^2 + (\partial_\sigma\varphi)^2 + (4\pi\mu)\,e^{2bQ\,\tau}\,e^{2b\varphi}\right]\,d\sigma\,d\tau\,, \tag{22.31}$$

where I have taken into account the transformation (21.25) of the exponential field in the action. It is convenient to eliminate the "time" dependence by making a shift (the same as suggested in Eq. (21.18))

$$\varphi + Q\,\tau\;\;\rightarrow\;\;\varphi\,, \tag{22.32}$$

so that the action becomes

$$\mathcal{A}_L[\varphi] = \frac{1}{4\pi}\int\left[(\partial_\tau\varphi - Q)^2 + (\partial_\sigma\varphi)^2 + (4\pi\mu)\,e^{2b\varphi}\right]\,d\sigma\,d\tau\,, \tag{22.33}$$

The canonical quantization is straightforward. We introduce the canonical momentum

$$\hat{\pi}(\sigma) = \frac{i}{2\pi}\,(\partial_\tau\varphi - Q)\,, \tag{22.34}$$

and postulate the canonical commutators

$$[\hat{\pi}(\sigma),\hat{\varphi}(\sigma)] = -i\,\delta(\sigma-\sigma')\,. \tag{22.35}$$

The Hamiltonian operator has the form

$$\hat{H} = \frac{1}{4\pi}\int_0^{2\pi}\left[\alpha\,\hat{\pi}^2 + (\partial_\sigma\hat{\varphi})^2 + i\,(4\pi Q)\,\hat{\pi} + (4\pi\mu)\,e^{2b\varphi}\right]\,d\sigma\,,\qquad \alpha = (2\pi)^2\,. \tag{22.36}$$

Hermiticity!

Finally, the canonical transformation

$$\hat{\pi} \;\rightarrow \hat{\pi} - \frac{iQ}{2\pi} \tag{22.37}$$

eliminates the linear term, bringing the hamiltonian to the conventional form

$$\hat{H} = \frac{Q^2}{2} + \frac{1}{4\pi} \int_0^{2\pi} \left[\alpha\, \hat{\pi}^2 + (\partial_\sigma \hat{\varphi})^2 + (4\pi\mu)\, e^{2b\varphi}\right]\, d\sigma\,. \tag{22.38}$$

Note that canonical transformation corresponds to the change in the definition of the norm in the space of states,

$$||\Psi||^2 = \int |\Psi[\varphi(\sigma)]|^2\, D[\varphi(\sigma)] \;\rightarrow\; \int e^{-2Q\,\varphi_0}\, |\Psi[\varphi(\sigma)]|^2\, D[\varphi(\sigma)] \tag{22.39}$$

where φ_0 is the zero-mode (22.10). The additional factor $e^{-2Q\varphi_0}$ eliminates the effect of the insertion

$$V_{Q/2}(x_s)V_{Q/2}(x_n) = V_{Q/2}(\tau = +\infty)V_{Q/2}(\tau = -\infty)\,. \tag{22.40}$$

As we have seen before, the stationary states of the Hamiltonian (22.38) is the space of "scattering states" $\mid \Psi_{s,P}\rangle$.

Operator-state correspondence:

$$\begin{array}{lcl} \mid \Psi_P\rangle & \leftrightarrow & V_{Q/2+iP}(0) \\ \langle \Psi_P \mid & \leftrightarrow & V_{Q/2-iP}(\infty) \end{array} \tag{22.41}$$

Normalization: for $P, P' > 0$

$$\langle\, V_{Q/2+iP}(0)\, V_{Q/2-iP'}(\infty)\,\rangle = \pi\, \delta(P - P') \tag{22.42}$$

22.3. Minisuperspace approximation

Weak-coupling limit $b \rightarrow 0$. Suppose P is small, $P \sim b$.

Minisuperspace Hamiltonian

$$H_0 = \frac{Q^2}{12} - \frac{1}{2}\frac{\partial^2}{\partial\varphi_0^2} + 2\pi\mu\, e^{2b\varphi_0}\,. \tag{22.43}$$

Schrödinger equation

$$H_0\Psi(\varphi_0) = 2P^2\Psi(\varphi_0) \tag{22.44}$$

is solved in terms of the MacDonald function

$$\Psi_P(\varphi_0) = \frac{2\, M^{-iP/b}}{\Gamma(-2iP/b)}\, K_{2iP/b}\left(2\sqrt{M}\, e^{b\varphi_0}\right) \tag{22.45}$$

with

$$M = \frac{\pi\mu}{b^2} \tag{22.46}$$

Normalized

$$\int_{-\infty}^{\infty} \Psi^*_{P'}(\varphi_0)\Psi_P(\varphi_0)\, d\varphi_0 = \pi\, \delta(P - P')\,. \tag{22.47}$$

At large $-\varphi_0$

$$\Psi_p(\varphi_0) \to e^{2iP\varphi_0} + S^{(\mathrm{cl})}(P)\, e^{-2iP\varphi_0}\,, \tag{22.48}$$

with

$$S^{(\mathrm{cl})}(P) = -M^{-2iP/b}\, \frac{\Gamma(1+2iP/b)}{\Gamma(1-2iP/b)} \tag{22.49}$$

Adiabatic approximation?

22.4. Liouville reflection S-matrix and two-point functions

The Liouville reflection S-matrix can be interpreted as the collection of two-point correlation functions. As before, choose two arbitrary points x_s and x_n, and consider the correlation function

$$\langle\, V_a(x_s)\, V_{a'}(x_n)\,\rangle\,. \tag{22.50}$$

By the conformal invariance, it can differ from zero only if $\Delta_a = \Delta_{a'}$. This leaves two possibilities,

$$a' = a \qquad \text{or} \qquad a' = Q - a\,. \tag{22.51}$$

The second case is easier: in this case the zero-curvature condition (21.53) is satisfied, and the problem reduces to free field. The correlation function is easily evaluated, except that the zero-mode integral diverges. We will comment more on this case later. For the first case, $a' = a$, I will use the notation

$$\langle\, V_a(x_s)\, V_a(x_n)\,\rangle = \frac{R(a)}{|x_s - x_n|^{4\Delta_a}}\,; \tag{22.52}$$

for reason to become clear later I will call $R(a)$ the "reflection coefficient". It is easy to see that analytic continuation of $R(a)$ to $a = Q/2 + iP$ with real P coincides with the vacuum element $S(P)$ of the Liouville S-matrix. Indeed, take the two-point function $\langle\, V_{Q/2+iP}(x_s)\, V_{Q/2+iP}(x_n)\,\rangle$, and choose again the complex coordinates in which x_s is at $z = 0$ and x_n is at $z = \infty$. We have

$$\langle\, V_{Q/2+iP}(0)\, V_{Q/2+iP}(\infty)\,\rangle = R(Q/2 + iP)\,. \tag{22.53}$$

Changing, as before, to the cylindrical coordinates $(u, \bar u)$ through the logarithmic map

22.5. Reflection relation

23. Degenerate fields

Recall that in the classical Liouville theory the central role was played by the field $e^{-\frac{\sigma}{2}}$, and the equation

$$\left[4\,\partial_z^2 + t(z)\right]\, e^{-\frac{\sigma}{2}} = 0\,, \qquad t(z) = -(\partial_z\sigma)^2 + 2\partial_z^2\sigma\,, \tag{23.1}$$

which this field satisfies, along with similar equation with respect to $\bar{z}$. There is remarkable quantum counterparts to these equations.

Consider the field $V_{-b/2} = e^{-\frac{b}{2}\varphi}$. Its conformal dimension

$$\Delta_{-b/2} = -\frac{1}{2} - \frac{3b^2}{4} \tag{23.2}$$

is precisely the dimension of the "degenerate" field $\psi_{1,2}$ of the Kac set. Specifically, with $c_{\rm L} = 1 + 6Q^2$, the descendant field

$$\left[L_{-1}^2 + b^2\, L_{-2}\right]\, V_{-b/2} \tag{23.3}$$

is the null-vector. Recall that $L_{-1}V_a = \partial_z V_a$ and that

$$T(\zeta)\, V_a(z,\bar{z}) - \frac{\Delta_a}{(\zeta - z)^2}\, V_a(z,\bar{z}) - \frac{1}{\zeta - z}\,\partial_z V_a(z,\bar{z}) + L_{-2}V_a(z,\bar{z}) + O(z) \tag{23.4}$$

so that the field $L_{-2}V_a$ is naturally interpreted as the regularized product $: T(z)V_a(z,\bar{z}) :$. Note that in classical limit $b \to 0$ we have $4b^2\, T(z) \to t(z)$. Therefore the null-vector equations

$$\begin{aligned} \left[L_{-1}^2 + b^2\, L_{-2}\right]\, V_{-b/2} = 0 \\ \left[\bar{L}_{-1}^2 + b^2\, \bar{L}_{-2}\right]\, V_{-b/2} = 0 \end{aligned} \tag{23.5}$$

are regarded as the quantum version of the classical equations (23.1).

More generally, the dimensions of the fields $V_{-(n-1)b/2}$,

$$\Delta_{-(n-1)b/2} = \frac{1-n}{2} + \frac{n(1-n)b^2}{2} \tag{23.6}$$

are the dimensions of the degenerate fields $\psi_{1,n}$; such fields exhibit null vectors on the level n, for example

$$\left[L_{-1}^3 + 4b^2\, L_{-1}L_{-2} - 2b^2\,(1-2b^2)\, L_{-3}\right]\, V_{-b} = 0\,. \tag{23.7}$$

It is easy to verify the classical counterpart of this equation

$$\partial_z^3 e^{-\sigma} + \partial_z\left(t\, e^{-\sigma}\right) - \frac{1}{2}\,\partial_z t\, e^{-\sigma} = 0\,, \qquad t = -(\partial_z\sigma)^2 + 2\,\partial_z^2\sigma\,. \tag{23.8}$$

There are other degenerate fields. The dimension of the field $V_{-1/2b}$

$$\Delta_{-1/2b} = -\frac{3}{4b^2} - \frac{1}{2} \tag{23.9}$$

equals $\Delta_{2,1}$, i.e. it has the null-vector on the level two. The corresponding equation

$$\left[L_{-1}^2 + \frac{1}{b^2}\,L_{-2}\right]\,V_{-1/2b} = 0 \tag{23.10}$$

has no obvious counterpart in classical theory (?). Other degenerate fields are

$$\begin{aligned} &V_{-(n-1)b/2-(m-1)/2b}\;: \\ &\Delta_{-(n-1)b/2-(m-1)/2b} = \frac{Q^2}{4} - \frac{\left(nb+mb^{-1}\right)^2}{4} \end{aligned} \tag{23.11}$$

It is well known in CFT that the correlation functions involving the degenerate fields satisfy linear differential equations. The most important for our purposes is the basic equations involving the fields $V_{-b/2}$ and $V_{-1/2b}$

$$\left\{-\frac{1}{b^2}\frac{\partial^2}{\partial z^2} - \sum_{i=1}^{n}\left[\frac{\Delta_{a_i}}{(z-z_i)^2} + \frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right]\right\}\langle\, V_{-b/2}(z,\bar z)\, V_{a_1}(z_1,\bar z_1)\cdots V_{a_n}(z_n,\bar z_n)\,\rangle \tag{23.12}$$

and

$$\left\{-b^2\frac{\partial^2}{\partial z^2} - \sum_{i=1}^{n}\left[\frac{\Delta_{a_i}}{(z-z_i)^2} + \frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right]\right\}\langle\, V_{-1/2b}(z,\bar z)\, V_{a_1}(z_1,\bar z_1)\cdots V_{a_n}(z_n,\bar z_n)\,\rangle \tag{23.13}$$

The first of these equations is the quantum version of the equation

$$\left\{4\,\frac{\partial^2}{\partial z^2} + \sum_{i=1}^{n}\left[\frac{r_i}{(z-z_i)^2} + \frac{c_i}{z-z_i}\right]\right\}\psi(z) = 0\,. \tag{23.14}$$

where

$$c_i = -\frac{\partial S}{\partial z_i}\,, \tag{23.15}$$

and the limit is taken as follows

$$a_i = \frac{\eta_i}{b}\,, \qquad \Delta_{a_i} \to \frac{\eta_i(1-\eta_i)}{b^2} = \frac{r_i}{4b^2}\,. \tag{23.16}$$

23.1. Fusion rules

It is not difficult to show that the OPE involving the degenerate fields contain discrete, indeed finite, number of terms. Consider for instance

$$V_{-b/2}(z,\bar{z})\,V_a(0,0) = \sum_{a'} (z\bar{z})^{\kappa}\,\left[V_{a'}(0,0) + \cdots\right]\,, \tag{23.17}$$

where the parameter κ must be equal to

$$\kappa = \Delta_{a'} - \Delta_a - \Delta_{-b/2}\,. \tag{23.18}$$

Substituting into the differential equation (23.12) one finds that the balance of the most singular terms requires that

$$-\frac{\kappa(\kappa-1)}{b^2} + \Delta_a + \kappa = 0\,. \tag{23.19}$$

This quadratic equation has two solutions for $\Delta_{a'}$,

$$\Delta_{a'} = \Delta_{a-b/2} \qquad \text{and} \qquad \Delta_{a'} = \Delta_{a+b/2} \tag{23.20}$$

Therefore, there are two terms in the right-hand side of the OPE (23.17) (we will see soon that there is no need to distinguish between a and $Q-a$),

$$V_{-b/2}(z,\bar{z})\,V_a(0,0) = C^{(+)}_{-b/2}(a)\,\left[V_{a-b/2}(0,0) + \cdots\right] + C^{(-)}_{-b/2}(a)\,\left[V_{a+b/2}(0,0) + \cdots\right]\,, \tag{23.21}$$

where $C^{(\pm)}_{-b/2}(a)$ are constants, which we are going to determine shortly. Likewise, the equation (23.13) implies

$$V_{-1/2b}(z,\bar{z})\,V_a(0,0) = C^{(+)}_{-1/2b}(a)\,\left[V_{a-1/2b}(0,0) + \cdots\right] + C^{(-)}_{-1/2b}(a)\,\left[V_{a+1/2b}(0,0) + \cdots\right]\,. \tag{23.22}$$

More generally, the OPE involving the degenerate field (23.11) with V_a contains nm terms,

$$V_{n,m}(z,\bar{z})\,V_a(0,0) = ... \tag{23.23}$$

We will find soon

$$\frac{C^{(-)}_{-b/2}(a)}{C^{(+)}_{-b/2}(a)} = -\frac{\pi\mu\,\gamma(2ba-1-b^2)}{\gamma(-b^2)\,\gamma(2ba)} \tag{23.24}$$

We will choose normalization such that $C^{(+)}_{-b/2} = 1$. In this case

$$C^{(-)}_{-b/2}(a) = -\mu\int d^2z\,\langle\, V_a(0)\,V_{-b/2}(1)\,V_b(z)\,V_{Q-a-b/2}(\infty)\,\rangle \tag{23.25}$$

23.2. Two plus one correlation functions

Consider the three-point function, with one of the insertions being $V_{-b/2}$,

$$\langle\, V_{-b/2}(z)\, V_{a_1}(z_1)V_{a_2}(z_2)\,\rangle = C(a,a')\ \times \\ |z-z_1|^{\Delta_2-\Delta_1-\Delta_{-b/2}}\,|z-z_2|^{\Delta_1-\Delta_2-\Delta_{-b/2}}\,|z_1-z_2|^{\Delta_{-b/2}-\Delta_1-\Delta_2}\,. \tag{23.26}$$

Substituting into (23.12), we have

24. Bootstrap and singular vector decoupling

To fix the normalization we first take the standard normalization for the primary Liouville fields and denote them Φ_a to distinguish from the "conventionally" normalized fields V_a i.e

$$\langle \Phi_a(x)\Phi_a(0)\rangle_{\rm L} = \frac{1}{(x\bar{x})^{2\Delta_a}}$$

where, as before

$$\Delta_a = a(Q-a)$$

Our point is now to find the three-point function

$$\langle \Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\Phi_{a_3}(x_3)\rangle_{\rm L} = \frac{\mathbb{C}_{a_1a_2a_3}}{(x_{12}\bar{x}_{12})^{\Delta_1+\Delta_2-\Delta_3}\,(x_{23}\bar{x}_{23})^{\Delta_2+\Delta_3-\Delta_1}\,(x_{31}\bar{x}_{31})^{\Delta_3+\Delta_1-\Delta_2}}$$

In this normalization the tree point function literally coincides with the structure constants

$$\Phi_{a_1}(x)\Phi_{a_2}(0) = \sum_{a_3}\mathbb{C}_{a_1a_2a_3}\,(x\bar{x})^{\Delta_3-\Delta_1-\Delta_2}\,[\Phi_{a_3}(0)]$$

Singular operators
appear at

$$a = a_{m,n} = Q/2 - \lambda_{m,n}$$

where

$$\lambda_{m,n} = \frac{mb^{-1}+nb}{2}$$

and (m,n) a pair of natural numbers. Singular vector appears at level nm e.g.

$$\begin{array}{l} L_{-1}\Phi_0 \\ \left(L_{-1}^2 + b^2 L_{-2}\right)\Phi_{-b/2} \\ \left(L_{-1}^2 + b^{-2} L_{-2}\right)\Phi_{-b^{-1}/2} \\ \left(L_{-1}^3 + 4b^2 L_{-2}L_{-1} + 2b^2\left(1+2b^2\right)L_{-3}\right)\Phi_{-b} \qquad \textit{sign?} \\ \ldots \end{array}$$

Decoupling

$$\langle L_{-1}\Phi_0(x_1)\Phi_{a_2}(x_2)\Phi_{a_3}(x_3)\rangle = \frac{\partial}{\partial x_1}\frac{\mathbb{C}_{0a_2a_3}}{(x_{12}\bar{x}_{12})^{\Delta_2-\Delta_3}(x_{23}\bar{x}_{23})^{\Delta_2+\Delta_3}(x_{31}\bar{x}_{31})^{\Delta_3-\Delta_2}}$$

$$= \frac{(\Delta_3-\Delta_2)\,x_{32}}{x_{31}x_{12}}\frac{\mathbb{C}_{0a_2a_3}}{(x_{12}\bar{x}_{12})^{\Delta_2-\Delta_3}(x_{23}\bar{x}_{23})^{\Delta_2+\Delta_3}(x_{31}\bar{x}_{31})^{\Delta_3-\Delta_2+1}} = 0$$

Non-vanishing structure constant only for $\Delta_2 = \Delta_3$ and apparently $\mathbb{C}_{0aa} = 1$. Similarly non-trivial $\mathbb{C}_{-b/2a_2a_3}$ requires

$$a_3 = a_2 \pm b/2$$
$$a_3 = Q/2 - a_2 \pm b/2$$

i.e.,

$$\Phi_{-b/2}(x)\Phi_a(0) = \mathbb{C}_-(a)\left[\Phi_{a-b/2}\right] + \mathbb{C}_+(a)\left[\Phi_{a+b/2}\right]$$
$$\Phi_{-b^{-1}/2}(x)\Phi_a(0) = \tilde{\mathbb{C}}_-(a)\left[\Phi_{a-b^{-1}/2}\right] + \tilde{\mathbb{C}}_+(a)\left[\Phi_{a+b^{-1}/2}\right]$$

Consider

$$g(x,\bar{x}) = \left\langle \Phi_{-b/2}(x)\Phi_{a_1}(0)\Phi_{a_2}(1)\Phi_{a_3}(\infty)\right\rangle$$

Decoupling leads to the following differential equations

$$b^{-2}g_{xx} + \frac{2x-1}{x(1-x)}g_x + \left(\frac{\Delta_1}{x^2} + \frac{\Delta_2}{(1-x)^2} + \frac{(\Delta_{1,2}+\Delta_1+\Delta_2-\Delta_3)}{x(1-x)}\right)g = 0$$

where $\Delta_{1,2} = -1/2 - 3b^2/4$ and similar equation w.r.t. $\bar{x}$. Relevant independent solutions are either "s blocks"

$$\mathcal{F}\left(\begin{array}{cc} -b/2 & a_2 \\ a_1 & a_3 \end{array}|\alpha_1 - b/2|x\right) = \mathcal{F}_-^{(s)}(x) = x^{b\alpha_1}(1-x)^{\alpha_2 b}{}_2F_1(A,B,C,x)$$
$$\mathcal{F}\left(\begin{array}{cc} -b/2 & a_2 \\ a_1 & a_3 \end{array}|\alpha_1 + b/2|x\right) = \mathcal{F}_+^{(s)}(x) = x^{1+b^2-b\alpha_1}(1-x)_2^{1+b^2-ba_2}F_1(1-A,1-B,2-C,x)$$

or "u blocks"

$$\mathcal{F}_-^{(u)}(x) = x^{ba_1}(1-x)^{ba_2}{}_2F_1(A,B,1+A+B-C,1-x)$$
$$\mathcal{F}_-^{(u)}(x) = x^{1-ba_1+b^2}(1-x)^{1-ba_2+b^2}{}_2F_1(1-A,1-B,1+C-A-B,1-x)$$

where

$$A = -1 + b(a_1+a_2+a_3-3b/2)$$
$$B = b(a_1+a_2-a_3-b/2)$$
$$C = b(2a_1-b)$$

Standard transformations relate these two pairs of solutions as

$$\begin{pmatrix} \mathcal{F}_-^{(s)}(x) \\ \mathcal{F}_+^{(s)}(x) \end{pmatrix} = \begin{pmatrix} \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} & \frac{\Gamma(C)\Gamma(A+B-C)}{\Gamma(A)\Gamma(B)} \\ \frac{\Gamma(2-C)\Gamma(C-A-B)}{\Gamma(1-A)\Gamma(1-B)} & \frac{\Gamma(2-C)\Gamma(A+B-C)}{\Gamma(1+A-C)\Gamma(1+B-C)} \end{pmatrix} \begin{pmatrix} \mathcal{F}_-^{(u)}(x) \\ \mathcal{F}_+^{(u)}(x) \end{pmatrix}$$

Then

$$g = \mathbb{C}_-(a_1)\mathbb{C}_{a_1-b/2,a_2a_3}\mathcal{F}_-^{(s)}(x)\mathcal{F}_-^{(s)}(\bar{x}) + \mathbb{C}_+(a_1)\mathbb{C}_{a_1+b/2,a_2a_3}\mathcal{F}_+^{(s)}(x)\mathcal{F}_+^{(s)}(\bar{x})$$

lead to

$$\frac{\mathbb{C}_-(a_1)\mathbb{C}_{a_1-b/2,a_2a_3}}{\mathbb{C}_+(a_1)\mathbb{C}_{a_1+b/2,a_2a_3}} = \\ -\frac{(1+b^2-2a_1b)^2\gamma(-b^2/2+(a_3+a_1-a_2)b)\gamma(-b^2/2+(a_1+a_2-a_3)b)}{\gamma^2(-b^2+2a_1b)\gamma(-b^2/2+(a_2+a_3-a_1)b)\gamma(2+3b^2/2-(a_1+a_2+a_3)b)}$$

Similarly

$$\frac{\tilde{\mathbb{C}}_-(a_1)\mathbb{C}_{a_1-b^{-1}/2,a_2a_3}}{\tilde{\mathbb{C}}_+(a_1)\mathbb{C}_{a_1+b^{-1}/2,a_2a_3}} = \\ -\frac{(1+b^{-2}-2a_1b^{-1})^2\gamma(-b^{-2}/2+(a_3+a_1-a_2)b^{-1})\gamma(-b^{-2}/2+(a_1+a_2-a_3)b^{-1})}{\gamma^2(-b^{-2}+2a_1b^{-1})\gamma(-b^{-2}/2+(a_2+a_3-a_1)b^{-1})\gamma(2+3b^{-2}/2-(a_1+a_2+a_3)b^{-1})}$$

Special structure constants

$$\left(\frac{\mathbb{C}_-(a)}{\mathbb{C}_+(a)}\right)^2 = \frac{\gamma(2ab)\gamma(2+b^2-2ab)}{\gamma(2+2b^2-2ab)\gamma(-b^2+2ab)}$$

Thus

$$\frac{\mathbb{C}_{a_1-b/2,a_2a_3}}{\mathbb{C}_{a_1+b/2,a_2a_3}} = \left[\frac{\gamma(2+2b^2-2a_1b)\gamma(2+b^2-2a_1b)}{\gamma(2a_1b)\gamma(2a_1b-b^2)}\right]^{1/2} \\ \frac{\gamma((a_3+a_1-a_2)b-b^2/2)\gamma((a_1+a_2-a_3)b-b^2/2)}{\gamma((a_2+a_3-a_1)b-b^2/2)\gamma(2+3b^2/2-(a_1+a_2+a_3)b)}$$

Together with similar relation with $b \to b^{-1}$ the solution reads

$$\mathbb{C}_{a_1,a_2a_3} = \frac{A^3\left[\Upsilon(2a_1)\Upsilon(Q-2a_1)\Upsilon(2a_2)\Upsilon(Q-2a_2)\Upsilon(2a_3)\Upsilon(Q-2a_3)\right]^{1/2}}{\Upsilon(a_1+a_2+a_3-Q)\Upsilon(a_1+a_2-a_3)\Upsilon(a_2+a_3-a_1)\Upsilon(a_3+a_1-a_2)}$$

Conventional normalization

$$\Phi_a = A\left[\frac{\gamma(-2ab^{-1}+b^{-2}+2)}{\gamma(2ab-b^2)}\right]^{1/2}\left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{-Qb^{-1}/3+a/b} V_a$$

so that

$$C_{a_1a_2a_3} = \frac{\left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{(Q-a_1-a_2-a_3)/b}\Upsilon(b)\Upsilon(2a_1)\Upsilon(2a_2)\Upsilon(2a_3)}{\Upsilon(a_1+a_2+a_3-Q)\Upsilon(a_1+a_2-a_3)\Upsilon(a_2+a_3-a_1)\Upsilon(a_3+a_1-a_2)}$$

Reflection relations
follow from our previous identification $\Phi_a = \Phi_{Q-a}$ and read

$$V_a = D(a)V_{Q-a}$$

where

$$D(a) = \frac{(\pi\mu\gamma(b^2))^{(Q-2a)/b}}{b^2}\frac{\gamma(2ab-b^2)}{\gamma(2-2ab^{-1}+b^{-2})}$$

25. Double gamma and Υ

1. Definition. The Barnes double gamma-function $\Gamma_2(x|\omega_1,\omega_2)$, is defined as the analytic continuation of the two-fold series

$$\log\Gamma_2(x|\omega_1,\omega_2) = \frac{d}{dz}\sum_{m,n=0}^{\infty}(x+m\omega_1+n\omega_2)^{-z}\Bigg|_{z=0} \tag{25.1}$$

(convergent at $\operatorname{Re} z > 2$) to the point $z = 0$.

2. Contour integral. $\Gamma_2(x|\omega_1,\omega_2)$ admits the following integral representation

$$\log\Gamma_2(x|\omega_1,\omega_2) = \frac{C}{2}B_{2,2}(x|\omega_1,\omega_2) + \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{e^{-xt}\log(-t)}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})}\frac{dt}{t} \tag{25.2}$$

where the contur $\mathcal{C}$ goes from $+\infty$ to $+\infty$ encircling 0 counterclockwise, C is the Euler's constant and

$$B_{2,2}(x|\omega_1,\omega_2) = \frac{(2x-\omega_1-\omega_2)^2}{4\omega_1\omega_2} - \frac{\omega_1^2+\omega_2^2}{12\omega_1\omega_2} \tag{25.3}$$

The integral is well defined if $\operatorname{Re}\omega_1 > 0$, $\operatorname{Re}\omega_2 > 0$ and $\operatorname{Re} x > 0$. The fuction can be analytically continued for all complex values of the periods ω_1 and ω_2, excluding the case ω_1/ω_2 is a real negative number (the cases $\omega_1 = 0$ or $\omega_2 = 0$ are of course also excluded). Otherwise (25.2) continues as a meromophic function of x with no zeros and simple poles at $x = -m\omega_1 - n\omega_2$, where m and n are non-negative integers.

3. Shift relations

$$\Gamma_2(x+\omega_1|\omega_1,\omega_2) = \frac{\sqrt{2\pi}\omega_2^{1/2-x/\omega_2}}{\Gamma(x/\omega_2)}\Gamma_2(x|\omega_1,\omega_2) \tag{25.4}$$

$$\Gamma_2(x+\omega_2|\omega_1,\omega_2) = \frac{\sqrt{2\pi}\omega_1^{1/2-x/\omega_1}}{\Gamma(x/\omega_1)}\Gamma_2(x|\omega_1,\omega_2)$$

are readily verified.

3. Scaling. Function Γ_2 scales as follows

$$\Gamma_2(\lambda x|\lambda\omega_1,\lambda\omega_2) = \lambda^{-B_{2,2}(x|\omega_1,\omega_2)/2}\Gamma_2(x|\omega_1,\omega_2) \tag{25.5}$$

This is verified e.g., from the integral representation (25.2).

4. Function $\Gamma_b(x)$. In Liouville applications it's particularly convenient to take $\omega_1 = b$ and $\omega_2 = b^{-1}$ and define [?]

$$\Gamma_b(x) = \Gamma_2(x|b,b^{-1}) \tag{25.6}$$

This function is invariant under the replacement $b \to b^{-1}$

$$\Gamma_b(x) = \Gamma_{b^{-1}}(x) \tag{25.7}$$

and well defined in the whole complex plane of

$$\tau = \omega_2/\omega_1 = b^{-2} \tag{25.8}$$

except for the negative part of the real axis. For definiteness we'll always suppose that $\operatorname{Im}\tau \geq 0$, taking advantage of (25.7) otherwise. An expample of the location of the poles of $\Gamma_b(x)$ is plotted in fig.26.

The scaling (25.5) always allows to express $\Gamma_2(x|\omega_1,\omega_2)$ through $\Gamma_b(x)$

$$\Gamma_2(x|\omega_1,\omega_2) = (\omega_1\omega_2)^{-B_{2,2}(x|\omega_1,\omega_2)/4}\,\Gamma_b((\omega_1\omega_2)^{-1/2}x) \tag{25.9}$$

with $b = (\omega_1/\omega_2)^{1/2}$.

5. The Υ-function. Υ is defined through Γ_2 as follows

$$\Upsilon(x|\omega_1,\omega_2) = \frac{\Gamma_2^2((\omega_1+\omega_2)/2|\omega_1,\omega_2)}{\Gamma_2(x|\omega_1,\omega_2)\Gamma_2(\omega_1+\omega_2-x|\omega_1,\omega_2)} \tag{25.10}$$

It admits the following line integral representation

$$\log\Upsilon(x|\omega_1,\omega_2) = \int_0^\infty \frac{dt}{t}\left[\frac{(\omega_1+\omega_2-2x)^2}{4\omega_1\omega_2}e^{-2t} - \frac{\sinh^2((\omega_1+\omega_2-2x)t/2)}{\sinh(\omega_1 t)\sinh(\omega_2 t)}\right]$$

From (25.5) it follows that

$$\Upsilon(\lambda x|\lambda\omega_1,\lambda\omega_2) = \lambda^{(\omega_1+\omega_2-2x)^2/(4\omega_1\omega_2)}\Upsilon(x|\omega_1,\omega_2) \tag{25.11}$$

while the shift relations read

$$\begin{aligned}\Upsilon(x+\omega_1|\omega_1,\omega_2) &= \omega_2^{2x/\omega_2-1}\gamma(x/\omega_2)\Upsilon(x|\omega_1,\omega_2)\\ \Upsilon(x+\omega_2|\omega_1,\omega_2) &= \omega_1^{2x/\omega_1-1}\gamma(x/\omega_1)\Upsilon(x|\omega_1,\omega_2)\end{aligned} \tag{25.12}$$

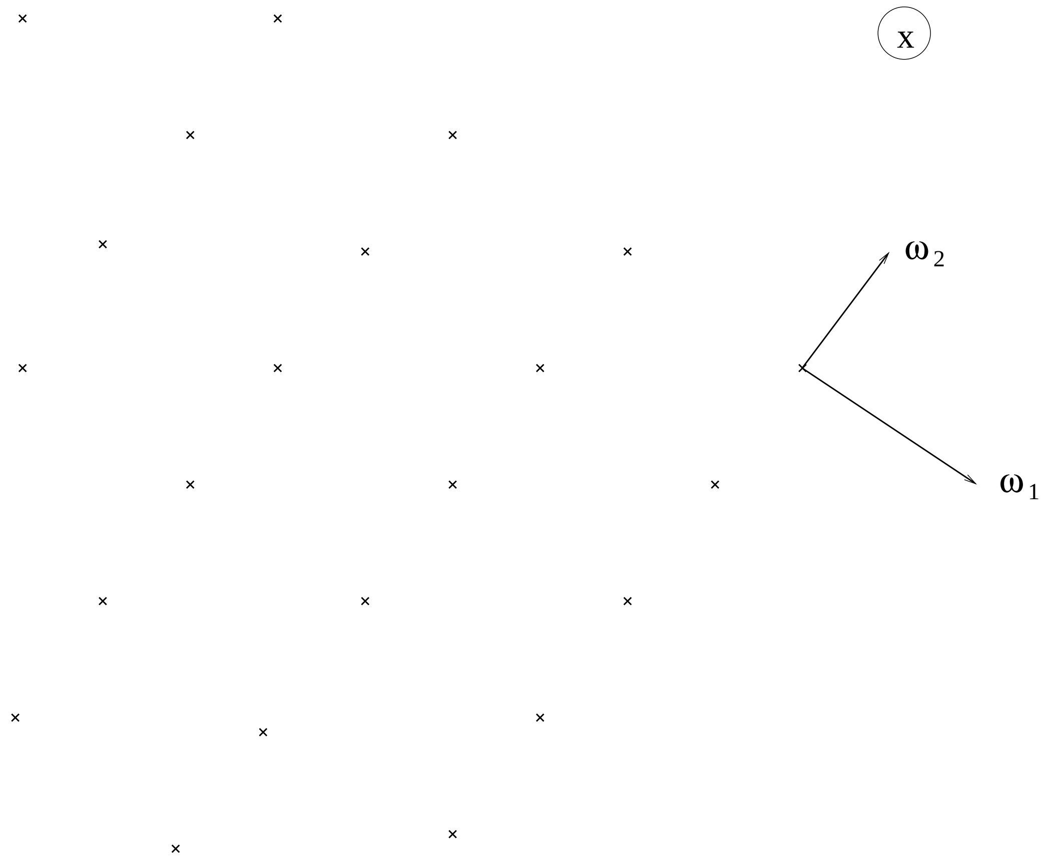


Figure 26: Poles of the double gamma function $\Gamma_2(x|\omega_1, \omega_2)$.

where as usual $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. Apparently Υ normalized in the way that

$$\Upsilon((\omega_1+\omega_2)/2|\omega_1, \omega_2) = 1 \tag{25.13}$$

It's also relevant to define [?]

$$\Upsilon_b(x) = \Upsilon(x|b, b^{-1}) \tag{25.14}$$

so that

$$\Upsilon(x|\omega_1, \omega_2) = (\omega_1\omega_2)^{(\omega_1+\omega_2-2x)^2/(8\omega_1\omega_2)}\Upsilon_b((\omega_1\omega_2)^{-1/2}\, x) \tag{25.15}$$

with $b = (\omega_1/\omega_2)^{1/2}$.

Stirling formula which controls the $|x| \to \infty$ asymptotic of the double-gamma function. To make it explicit we, following Barnes, introduce the double Bernoulli numbers

$$B_{2,n}(b, b^{-1}) = \sum_{k=0}^{n} \binom{n}{k} B_k B_{n-k} b^{2k-n}$$

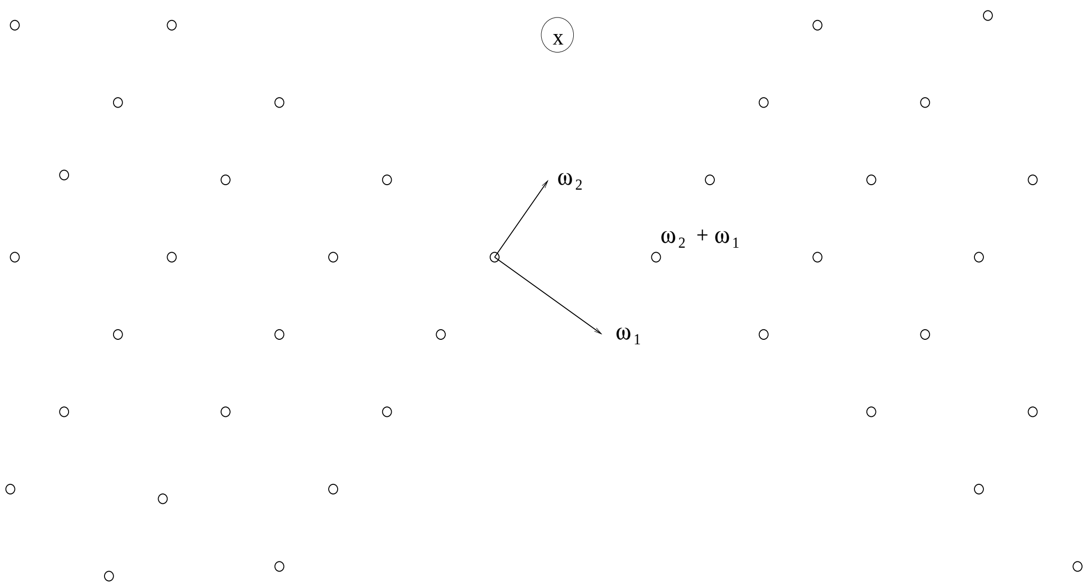


Figure 27: Position of poles (zeros?) of the function $\Upsilon(x|\omega_1,\omega_2)$.

(which can be summarized by a symbolic formula $B_{2,n}(b,b^{-1}) = (Bb+Bb^{-1})^n$) and the double Bernoulli polynomials

$$B_{2,n}(x|b,b^{-1}) = \sum_{k=0}^{n} \binom{n}{k} x^k B_{2,n-k}(b,b^{-1})$$

(symbolically $B_{2,n}(x|b,b^{-1}) = (Bb+Bb^{-1}+x)^n$). The Stirling formula reads

$$\log\Gamma_2(x|b,b^{-1}) \sim \left(\frac{b^2+b^{-2}}{24} - \frac{(x-Q/2)^2}{2}x^2\right)\log x + \frac{3}{4}x^2 - \frac{Q}{2}x + \sum_{k=1}^{\infty}\frac{(-)^k B_{2,k+2}(b,b^{-1})}{(k+2)(k+1)k}x^{-k}$$

6. Complementarity. Consider the following product

$$H(x|\omega_1,\omega_2) = \Upsilon(x|\omega_1,\omega_2)\Upsilon(x-\omega_1|e^{i\pi}\omega_1,\omega_2) \tag{25.16}$$

Here we suppose that $\tau = \omega_2/\omega_1$ has positive imaginary part, so that the rotation $\omega_1 \to e^{i\pi}\omega_1$ goes safely avoiding the negative real axis of τ. It is straightforward to verify that this product is scale invariant

$$H(\lambda x|\lambda\omega_1,\lambda\omega_2) = H(x|\omega_1,\omega_2) \tag{25.17}$$

Function $H(x|\omega_1,\omega_2)$ is an entire function of x with the regular lattice of zeros $x = m\omega_1+n\omega_2$, $m,n\in\mathbf{Z}$. Together with the shift relations

$$\begin{aligned} H(x+\omega_1|\omega_1,\omega_2) &= H(x|\omega_1,\omega_2) \\ H(x+\omega_2|\omega_1,\omega_2) &= -e^{-2i\pi x/\omega_1}H(x|\omega_1,\omega_2) \end{aligned} \tag{25.18}$$

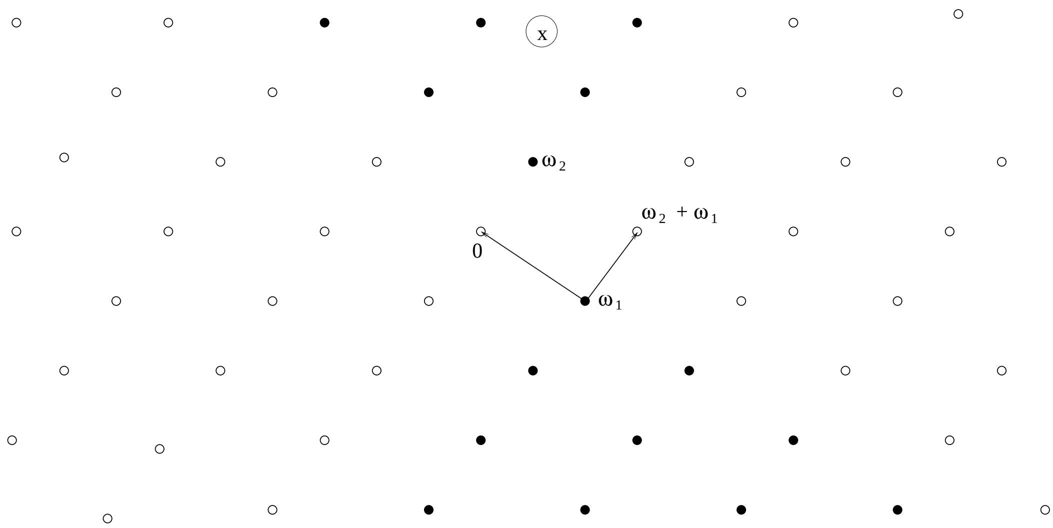


Figure 28: Zeros of the product $\Upsilon(x|\omega_1,\omega_2)\Upsilon(x-\omega_1|e^{i\pi}\omega_1,\omega_2)$ in the x-plane. Open circles are these of the first multiplier and filled ones are those of the second. Together they form the regular lattice of zeros of the theta function. Arrows show the "periods" $-\omega_1$ and ω_2 of $\Upsilon(x-\omega_1|e^{i\pi}\omega_1,\omega_2)$.

which follow from that for Υ, this requires $H(x|\omega_1,\omega_2)$ to have the form

$$H(x|\omega_1,\omega_2) = H_0 e^{i\pi x/\omega_1}\theta_1\left(\frac{\pi x}{\omega_1}|h\right) \tag{25.19}$$

where H_0 is some x-independent constant,

$$h = \exp(i\pi\tau) = \exp(i\pi\omega_2/\omega_1) \tag{25.20}$$

and θ_1 is the standard θ-function

$$\theta_1(u|h) = i\sum_{n=-\infty}^{\infty}(-)^n h^{(n-1/2)^2}e^{i(2n-1)u} \tag{25.21}$$

Normalization (25.13) entails $H((\omega_1+\omega_2)/2|\omega_1,\omega_2) = 1$ and allows to determine H_0. Finally

$$H(x|\omega_1,\omega_2) = -ie^{i\pi x/\omega_1}\frac{\theta_1(\pi x/\omega_1|h)}{h^{1/4}\theta_3(0|h)} \tag{25.22}$$

where

$$\theta_3(u|h) = \sum_{n=-\infty}^{\infty} h^{n^2}e^{2inu} \tag{25.23}$$

Implementing the scaling relations (25.11) and (25.17) with $\lambda = e^{-i\pi/2}$ we arrive at

$$H(x|b,b^{-1}) = e^{-i\pi(b+b^{-1}-2x)^2/8}\Upsilon_b(x)\Upsilon_{ib}(-ix+ib) = e^{i\pi x b^{-1}}\frac{-i\theta_1(\pi x b^{-1}|h)}{h^{1/4}\theta_3(0|h)}$$

This is equivalent to relation (**??**)

26. Pole sturcture

$$\operatorname*{res}_{\sum a_i = Q-nb} \mathcal{G}_{a_1,\ldots,a_N}(x_1,\ldots,x_N) = \mathcal{G}^{(n)}_{a_1,\ldots,a_N}(x_1,\ldots,x_N)\big|_{\sum \alpha_i = Q-nb} \tag{26.1}$$

$$\mathcal{G}^{(n)}_{\alpha_1,\alpha_2,\alpha_3}(x_1,x_2,x_3)\big|_{\sum \alpha_i = Q-nb} = |x_{12}|^{2\gamma_3}\,|x_{23}|^{2\gamma_1}\,|x_{31}|^{2\gamma_2}\, I_n(\alpha_1,\alpha_2,\alpha_3) \tag{26.2}$$

$$I_n(\alpha_1,\alpha_2,\alpha_3) = \left(\frac{-\pi\mu}{\gamma(-b^2)}\right)^n \frac{\prod_{j=1}^n \gamma(-jb^2)}{\prod_{k=0}^{n-1}\left[\gamma(2\alpha_1 b + kb^2)\gamma(2\alpha_2 b + kb^2)\gamma(2\alpha_3 b + kb^2)\right]}$$

27. Continuous OPE

Conjecture

$$V_{a_1}(x)V_{a_2}(0) = \int_{\uparrow}\frac{dp}{4\pi}\,(x\bar{x})^{\Delta_p-\Delta_1-\Delta_2}\,C^{p}_{a_1a_2}\,[V_p(0)]$$

where now

$$C^{a_1}_{a_2a_3} = D^{-1}(a_1)C_{a_1a_2a_3} = C_{Q-a_1a_2a_3}$$

Discrete terms

$$V_g(x)V_a(0) = \int_{\uparrow}'\frac{dp}{4\pi i}C^{(\mathrm{L})p}_{g,a}(x\bar{x})^{\Delta^{(\mathrm{L})}_p-\Delta^{(\mathrm{L})}_g-\Delta^{(\mathrm{L})}_a}\,[V_p(0)] \tag{27.1}$$

where $\uparrow$ passes through $Q/2$ along the imaginary axis and the prime indicates the deformations nessessery for the analytic continuation from the "basic domain" (**??**). The singularities of the structure constant

$$C^{(\mathrm{L})p}_{g,a} = \frac{(\pi\mu\gamma(b^2)b^{2-2b^2})^{(p-a-g)}\Upsilon_b(b)\Upsilon_b(2g)\Upsilon_b(2a)\Upsilon_b(2Q-2p)}{\Upsilon_b(p+a-g)\Upsilon_b(a+g+p-Q)\Upsilon_b(a+g-p)\Upsilon_b(p+g-a)} \tag{27.2}$$

are determined by zeros of the four Υ_b-functions in the denominator. An example of their location is shown in fig.29, where we suppose that a and g are both real, positive?? and less then $Q/2$. This pattern corresponds to the "basic domain", i.e., $a+g > Q/2$. The "right" zeros of all the four multipliers in the denominator are to the right and all "left" ones are to the left from the integration contour $\uparrow$, which in this case remains a straight line going vertically through $Q/2$. The strings of zeros are shifted slightly from the real axis to better destinguish zeros coming from different factors. The uppermost and the next string from

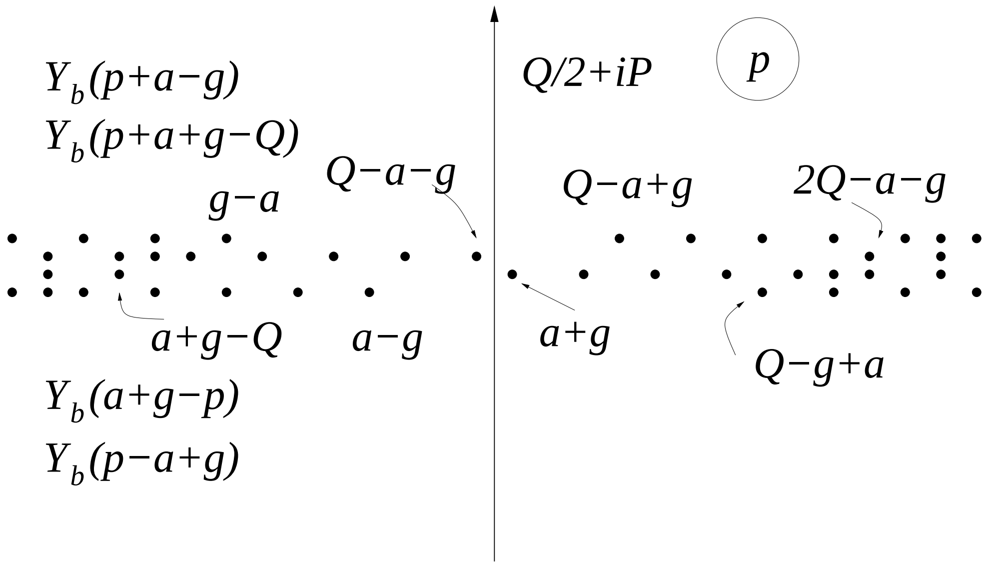


Figure 29: Location of the poles of the sturcture constant while $a + g > Q/2$???

the above are due to the factors $\Upsilon_b(p + a - g)$ and $\Upsilon_b(a + g + p - Q)$ respectively. Then lie the zeros of $\Upsilon_b(a + g - p)$ and the lowest string belongs to the multiplier $\Upsilon_b(p + g - a)$. Then if e.g. the parameter g decreases and $a + g$ becomes less then $Q/2$ the two poles at $a+g$ and $Q - a - g$ cross the vertical line $\mathrm{Re}\, p = Q/2$ (called ofthen the Seiberg bound **??**). Analyticity requires the integration contour to be deformed accordingly (fig.31). The effect of this deformation can be separated as the so called discrete terms

$$V_g(x)V_a(0) = \frac{1}{2}(x\bar{x})^{-2ag}[V_{a+g}(0)] + \frac{1}{2}(x\bar{x})^{-2ag}R_{\mathrm{L}}(a + g)[V_{Q-a-g}(0)] \\ + \int_{\uparrow} \frac{dp}{4\pi i} C^{(\mathrm{L})p}_{g,a}(x\bar{x})^{\Delta^{(\mathrm{L})}_p - \Delta^{(\mathrm{L})}_g - \Delta^{(\mathrm{L})}_a}[V_p(0)] \tag{27.3}$$

as it is shown in fig.**??**, where the two poles $a + g$ and $Q - a - g$ are picked up explicitly and the corresoponding residues are evaluated. Notice, that the two discrete terms in (27.3) are in fact identical due to the reflection relation (**??**). This is in fact a consequence of the complete symmetry of the integral (27.1) under the reflection $p \to Q - p$ and therefore holds for all "mirror images" w.r.t. this symmetry. Below we'll use this feature to keep only one of each pair of images, say that with $\mathrm{Re}\, p < Q/2$ and then supply the answer with the factor of 2. Further change of the parameters may force more poles to cross the contour and there will be more discrete terms in the right hand side of (27.3).

Another important remark is in order here. In the derivation of (27.3) above we implied that $\mathrm{Re}(a+g) < Q/2$. A quick reconsinderation of the opposite case $\mathrm{Re}(a+g) > Q/2$ shows that we have to pick up instead the poles $p = Q - a + g$ and $p = a - g$ and replaces eq.(27.3)

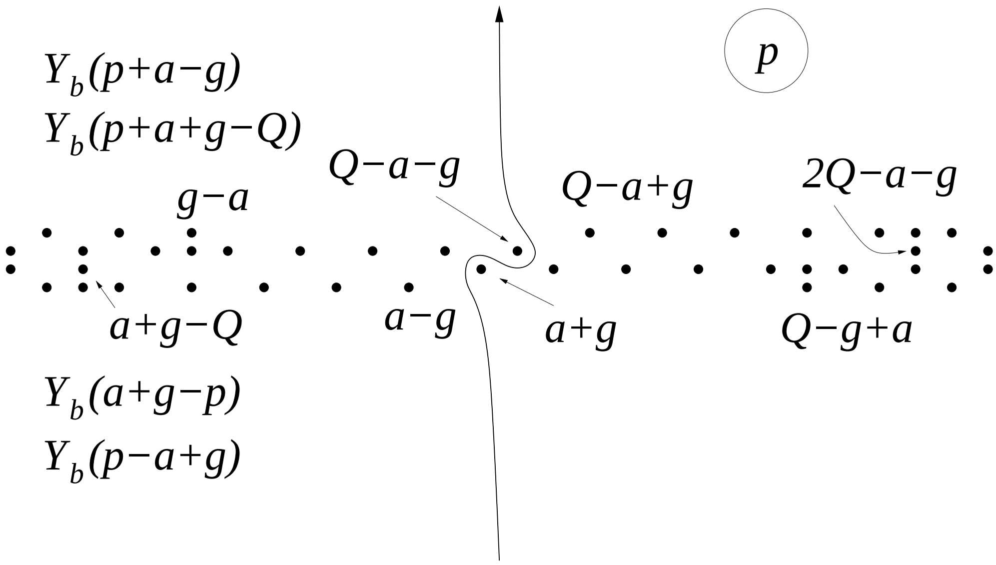


Figure 30: The contour deformation due to the analytic continuation of the OPE (27.1) away from the basic domain.

by

$$V_g(x)V_a(0) = (x\bar{x})^{-2(Q-a)g} R_{\rm L}(a)[V_{Q-a+g}(0)] + \int_{\uparrow} \frac{dp}{4\pi i} C^{({\rm L})p}_{g,a} (x\bar{x})^{\Delta^{({\rm L})}_p - \Delta^{({\rm L})}_g - \Delta^{({\rm L})}_a} [V_p(0)] \quad (27.4)$$

In general, only operators V_a with $\mathrm{Re}\, a < Q/2$ appear as the discrete terms in the r.h.s of (27.1)) allows formally to render these terms to the right from the Seiberg bound. This, however, wouldn't touch e.g., the exponent in the prefactor $(x\bar{x})^{-2(Q-a)g}$ and by no means implies any reflection symmetry $a \to Q - a$ of the discrete terms.

Our purpose is to study (27.1) at g close to certain degenerate value $g \to a_{m,n} = Q/2 - \lambda_{m,n}$. It is seen immediately that the structure constant (27.2) contains an overall multiplier $\Upsilon_b(2g)$ vanishing in this limit. Hence, the singularities arising from the divergencies of the integral are very important. To give an idea of what happens in general we consider first the simplest possible case $g \to a_{1,1} = 0$. The corresponding degenerate field $V_{1,1}$ is just the identity operator while the logarithmic primary $V'_{1,1}$ coincides with the basic Liouville field ϕ. In the limit $g \to 0$ the integral term in both equations (27.3) and (27.4) disappears and we arrive at pure $V_a(0)$ (as it of course should be for the identity operator at the place of V_g at the left hand side). This is the simplest, trivial case of the discrete degenerate OPE (similarly to (**??**) in GMM)

$$V_{m,n}(x)V_a(0) = \sum_{r,s}^{(m,n)} (x\bar{x})^{\lambda_{r,s}(Q-2a-\lambda_{r,s}) - \Delta^{({\rm L})}_{m,n}} C^{({\rm L})}_{r,s}(a) \left[V_{a+\lambda_{r,s}}\right] \quad (27.5)$$

which hold for the fields $V_{m,n}$ due to the decoupling (**??**) of the singular vectors.

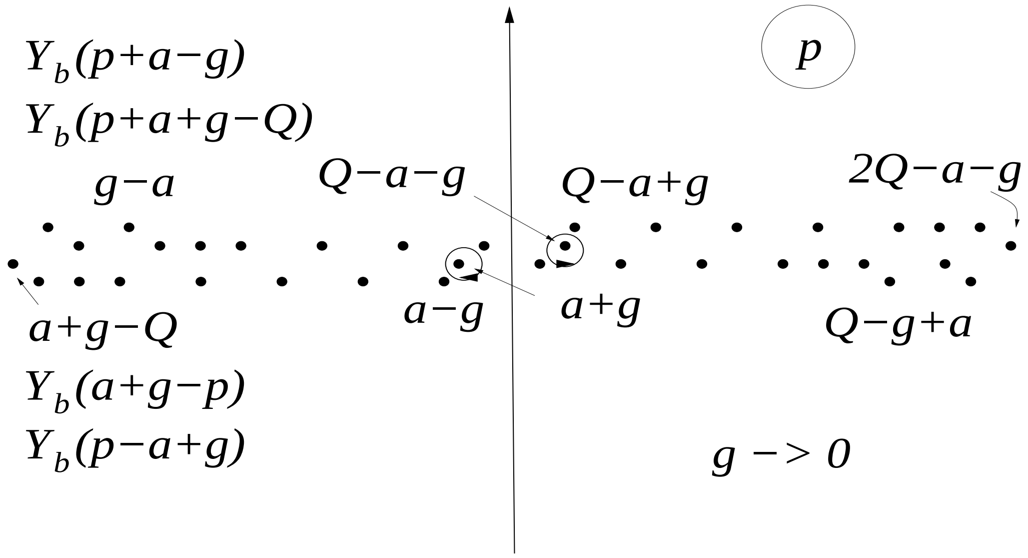


Figure 31: "Discrete terms" due to the poles at $p = a + g$ and $p = Q - a - g$ are picked up explicitly. These contributions are singular at $g \to 0$ due to close poles at $p = a - g$ (resp. at $p = Q - a + g$) which pinch the integration contour.

In particular, if $m, n = 1, 2$ we have

$$V_{-b/2}(x)V_a(0) = C_-(a)\left[V_{a-b/2}\right] + C_+(a)\left[V_{a+b/2}\right]$$
$$V_{-b^{-1}/2}(x)V_a(0) = \tilde{C}_-(a)\left[V_{a-b^{-1}/2}\right] + \tilde{C}_+(a)\left[V_{a+b^{-1}/2}\right]$$

with

$$\begin{aligned} C_+(a) &= \tilde{C}_+(a) = 1 \\ C_-(a) &= -\frac{\pi\mu}{\gamma(-b^2)}\frac{\gamma(2ab - b^2 - 1)}{\gamma(2ab)} \\ \tilde{C}_-(a) &= -\frac{\pi\tilde{\mu}}{\gamma(-b^{-2})}\frac{\gamma(2ab^{-1} - b^{-2} - 1)}{\gamma(2ab^{-1})} \end{aligned} \tag{27.6}$$

where

$$\left(\tilde{\mu}\gamma(b^{-2})\right)^b = \left(\mu\gamma(b^2)\right)^{1/b}$$

28. Classical Limit of Heavy Degenerate Fields

In the classical limit the field $V_{-1/2b}$ becomes "heavy". In general, in the limit $b^2 \to 0$ we set for heavy exponentials

$$a_i = \frac{\eta_i}{b}, \qquad \Delta_{a_i} \to \frac{\eta_i(1-\eta_i)}{b^2} = \frac{r_i}{4b^2}. \tag{28.1}$$

hence, in this case we have

$$\eta = -\frac{1}{2}\,, \qquad r = -3\,. \tag{28.2}$$

28.1. Metric

For $r=-3$ we have the local solutions

$$\psi_1(z) \simeq z^{-1/2}\,, \qquad \psi_2(z) \simeq z^{3/2}\,. \tag{28.3}$$

The local solution then is

$$e^{\sigma(z,\bar z)}\, dzd\bar z \sim \frac{z\bar z\, dzd\bar z}{(1+|z|^4)^2} = \frac{1}{4}\,\frac{dwd\bar w}{(1+w\bar w)^2}\,, \tag{28.4}$$

where $w = z^2$. This is two-fold cover of a sphere, with the branchings at $w = 0,\ \infty$.

28.2. Differential equation

We set

$$\langle\, V_{-1/2b}(z,\bar z)\, V_{a_1}(z_1,\bar z_1)\cdots V_{a_n}(z_n,\bar z_n)\,\rangle \sim \exp\left\{-\frac{S(z,z_i)}{4b^2}\right\} \tag{28.5}$$

The equation (23.13) becomes the "Hamilton-Jacobi" equation

$$\frac{1}{4}\left(\frac{\partial S}{\partial z}\right)^2 + \sum_{i=1}^{n}\left[\frac{r_i}{(z-z_i)^2} + \frac{1}{z-z_i}\,\frac{\partial S}{\partial z_i}\right] = 0\,. \tag{28.6}$$

Appendix

A. Quantum Equations of motion

Basic OPE:

$$\partial_z\varphi(z,\bar z)\, V_a(w,\bar w) = \frac{2a}{z-w}\, V_a(w,\bar w) + O(\mu) \tag{A.1}$$

Hence

$$\partial_{\bar z}\partial_z\varphi(z,\bar z) = -(\pi\mu\, b)\, V_b(z,\bar z)$$

We find that

$$4\,\partial^2\varphi + (4\pi\mu\, b)\, e^{2b\,\varphi} \simeq 0$$

is a redundant field.

Next, consider $\partial_{\bar z}(\partial_z\varphi)^2$. since

$$(\partial_z\varphi)^2(z)\, V_b(w,\bar w) = \left[\frac{-b^2}{(z-w)^2} + \frac{1}{z-w}\,\frac{\partial}{\partial w}\right]\, V_b(w,\bar w) + ...$$

we find

$$\partial_{\bar{z}}(\partial_z\varphi)^2 = \pi\mu\,(1+b^2)\,\partial_z V_b(z,\bar{z})\,.$$

Here I used

$$\partial_{\bar{z}}\frac{1}{z} = \pi\,\delta^{(2)}(z)\,, \qquad \partial_{\bar{z}}\frac{1}{z^2} = -\pi\,\partial_z\delta^{(2)}(z-w)\,.$$

B. Special functions

B1. Hypergeometric function

.

Integrals:

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,(1-zt)^{-a}\,dt\,. \tag{28.7}$$

Relations:

$$F(a,b,c;z) = (1-z)^{c-a-b}\,F(c-a,c-b,c;z)\,, \tag{28.8}$$

$$F(a,b,c;z) = (1-z)^{-a}\,F\left(a,c-b,c;\frac{z}{z-1}\right)\,. \tag{28.9}$$

Differential equation

$$z(1-z)\,u_{zz} + [c-(1+a+b)z]\;u_z - ab\,u = 0\,. \tag{28.10}$$

Solutions
We define three bases.
Canonical near $z=0$:

$$\begin{aligned} f_1(z) &= F(a,b,c;z) \\ f_2(z) &= z^{1-c}\,F(1+a-c,1+b-c,2-c;z) \end{aligned} \tag{28.11}$$

with Wronskian $W[f_1,f_2] = (1-c)$.
Canonical near $z=1$:

$$\begin{aligned} g_1(z) &= F(a,b,1+a+b-c;1-z) \\ g_2(z) &= (1-z)^{c-a-b}\,F(c-a,c-b,1+c-a-b;1-z) \end{aligned} \tag{28.12}$$

with Wronskian $W[g_1,g_2] = (a+b-c)$.
Canonical near $z=\infty$:

$$\begin{aligned} h_1(z) &= (-z)^{-a}\,F(a,1+a-c,1+a-b;1/z) \\ h_2(z) &= (-z)\,F(b,1+b-c,1+b-a;1/z) \end{aligned} \tag{28.13}$$

with Wronskian $W[h_1,h_2] = (b-a)$.

Transformations

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \mathbf{L} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \qquad \det \mathbf{L} = \frac{1-c}{a+b-c} \tag{28.14}$$

$$\begin{aligned} L_{11} &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, & L_{12} &= \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \\ L_{21} &= \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}, & L_{22} &= \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(1+b-c)\Gamma(1+a-c)} \end{aligned} \tag{28.15}$$

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \mathbf{K} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \qquad \det \mathbf{K} = \frac{1-c}{a-b} \tag{28.16}$$

$$\begin{aligned} K_{11} &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}, & K_{12} &= \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} \\ K_{21} &= \frac{\Gamma(2-c)\Gamma(b-a)}{\Gamma(1-a)\Gamma(1+b-c)}, & K_{22} &= \frac{\Gamma(2-c)\Gamma(a-b)}{\Gamma(1-b)\Gamma(1+a-c)} \end{aligned} \tag{28.17}$$